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HYERS–ULAM–RASSIAS STABILITY OF A FUNCTIONAL EQUATION IN THREE VARIABLES

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Abstract. In this paper, we solve the following functional equation

$$\begin{aligned} af\left(\frac{x+y+z}{b}\right) + af\left(\frac{x-y+z}{b}\right) + af\left(\frac{x+y-z}{b}\right) \\ + af\left(\frac{-x+y+z}{b}\right) = cf(x) + cf(y) + cf(z), \end{aligned}$$

and prove the Hyers-Ulam-Rassias stability of the functional equation as given above.

1. Introduction

Let X and Y be Banach spaces with norms $||\cdot||$ and $\|\cdot\|$, respectively. Hyers [4] showed that if $\epsilon > 0$ and $f : X \to Y$ such that

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon$$

for all $x, y \in X$, then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \le \epsilon$$

for all $x \in X$.

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Consider $f : X \to Y$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$. Assume that there exist constants $\epsilon \geq 0$ and $p \in [0, 1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(||x||^p + ||y||^p)$$

for all $x, y \in X$. Th.M. Rassias [6] showed that there exists a unique $\mathbb{R}$-linear mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{2-2^p}||x||^p$$

for all $x \in X$. Găvruta [3] generalized the Rassias' result.

A square norm on an inner product space satisfies the important parallelogram equality $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$. The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic function*. A Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [7] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [1] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. In [2], Czerwik proved the Hyers-Ulam-Rassias stability of the quadratic functional equation.

In [5], the authors solved the quadratic type functional equation

$$\begin{aligned} a^2 f(\frac{x+y+z}{a}) + a^2 f(\frac{x-y+z}{a}) + a^2 f(\frac{x+y-z}{a}) \\ + a^2 f(\frac{-x+y+z}{a}) = 4f(x) + 4f(y) + 4f(z), \end{aligned}$$

and proved the generalized Hyers-Ulam-Rassias stability of the quadratic type functional equation.

Throughout this paper, assume that a, b, c are positive real numbers, and that X and Y are real vector spaces.

In this paper, we solve the following functional equation

$$\begin{aligned} &af(\frac{x+y+z}{b}) + af(\frac{x-y+z}{b}) + af(\frac{x+y-z}{b}) \\ &\quad + af(\frac{-x+y+z}{b}) = cf(x) + cf(y) + cf(z) \end{aligned} \tag{1.i}$$

for all $x, y, z \in X$, and prove the generalized Hyers-Ulam-Rassias stability of the functional equation as given above.

2. Solutions of a functional equation in three variables

LEMMA 1. *If an even mapping $f : X \to Y$ satisfies* (1.i) *for all $x, y, z \in X$ and $f(0) = 0$, then f is quadratic.*

Proof. Note that $f(-x) = f(x)$ for all $x \in X$ since f is an even mapping. Putting $y = z = 0$ in (1.i), we have

$$4af(\frac{x}{b}) = cf(x) \tag{2.1}$$

for all $x \in X$. Using (2.1) and (1.i), we get

$$\begin{aligned} &f(x+y+z) + f(x-y+z) + f(x+y-z) + f(-x+y+z) \\ &\qquad = 4f(x) + 4f(y) + 4f(z) \end{aligned} \tag{2.2}$$

for all $x, y, z \in X$. Putting $z = 0$ in (2.2), we deduce $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ for all $x, y \in X$. So f is quadratic. □

LEMMA 2. *If an odd mapping $f : X \to Y$ satisfies* (1.i) *for all $x, y, z \in X$, then f is additive.*

Proof. Note that $f(0) = 0$ and $f(-x) = -f(x)$ for all $x \in X$ since f is an odd mapping. Putting $y = z = 0$ in (1.i), we have

$$2af(\frac{x}{b}) = cf(x) \tag{2.3}$$

for all $x \in X$. Using (2.3) and (1.i), we get

$$f(x+y+z)+f(x-y+z)+f(x+y-z)+f(-x+y+z) \\ = 2f(x)+2f(y)+2f(z) \tag{2.4}$$

for all $x, y, z \in X$. Putting $z = 0$ in (2.4), we deduce $f(x+y) = f(x)+f(y)$ for all $x, y \in X$. So f is additive. □

THEOREM 3. *If a mapping $f : X \to Y$ satisfies* (1.i) *for all $x, y, z \in X$ and $f(0) = 0$, then there exist an additive mapping $A : X \to Y$ and a quadratic mapping $Q : X \to Y$ such that*

$$f(x) = Q(x) + A(x)$$

for all $x \in X$.

Proof. Let $A(x) := \frac{f(x)-f(-x)}{2}$ for all $x \in X$. Then $A(-x) = -A(x)$ and A satisfies (1.i) for all $x, y, z \in X$. By Lemma 2, A is additive.

Let $Q(x) := \frac{f(x)+f(-x)}{2}$ for all $x \in X$. Then $Q(0) = 0$, $Q(-x) = Q(x)$ and Q satisfies (1.i) for all $x, y, z \in X$. By Lemma 1, Q is quadratic. Clearly, we have $f(x) = Q(x) + A(x)$ for all $x \in X$. □

3. Stability of a functional equation in three variables

Let $\mathbb{R}_+$ denote the set of nonnegative real numbers. Recall that a function $H : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is homogeneous of degree $p > 0$ if it satisfies $H(tu, tv, tw) = t^p H(u, v, w)$ for all nonnegative real numbers t, u, v, w.

Throughout this section, assume that X and Y are a real normed vector space with norm $|| \cdot ||$ and a real Banach space with norm $\| \cdot \|$, respectively, and that H is homogeneous of degree p. Given a mapping $f : X \to Y$, we set

$$\begin{aligned} Df(x,y,z) :=& af\left(\frac{x+y+z}{b}\right) + af\left(\frac{x-y+z}{b}\right) + af\left(\frac{x+y-z}{b}\right) \\ &+ af\left(\frac{-x+y+z}{b}\right) - cf(x) - cf(y) - cf(z) \end{aligned}$$

for all $x, y, z \in X$.

THEOREM 4. *Assume that* $\delta \geq 0$, $p \in (0, \infty) \setminus \{1\}$ *and* $\delta = 0$ *when* $p > 1$. *Let an odd mapping* $f : X \to Y$ *satisfy*

$$\|Df(x, y, z)\| \leq \delta + H(||x||, ||y||, ||z||) \tag{3.1}$$

for all $x, y, z \in X$. *Then there exists a unique additive mapping* $A : X \to Y$ *such that*

$$\|f(x) - A(x)\| \leq \frac{2}{c}\delta + \frac{1}{|2 - 2^p|}h(x) \tag{3.2}$$

for all $x \in X$, *where* $h(x) = \frac{1}{4}(H(||x||, ||x||, 0) + H(||2x||, 0, 0))$.

Proof. Note that $f(0) = 0$ and $f(-x) = -f(x)$ for all $x \in X$ since f is an odd mapping. Putting $y = z = 0$ in (3.1) and then replacing x by $2x$, we have

$$\|af(\frac{2x}{b}) - \frac{c}{2}f(2x)\| \leq \frac{1}{2}(\delta + H(||2x||, 0, 0)) \tag{3.3}$$

for all $x \in X$. Putting $y = x$ and $z = 0$ in (3.1), we have

$$\|af(\frac{2x}{b}) - cf(x)\| \leq \frac{1}{2}(\delta + H(||x||, ||x||, 0)) \tag{3.4}$$

for all $x \in X$. By (3.3) and (3.4), we have

$$\|f(2x) - 2f(x)\| \leq \frac{2}{c}\delta + h(x) \tag{3.5}$$

for all $x \in X$, where $h(x) = \frac{1}{c}(H(||x||, ||x||, 0) + H(||2x||, 0, 0))$.

We divide the remaining proof by two cases.

(I) The case $0 < p < 1$. By (3.5), we have

$$\|f(x) - \frac{f(2x)}{2}\| \leq \frac{1}{c}\delta + \frac{1}{2}h(x) \tag{3.6}$$

for all $x \in X$. Using (3.6), we have

$$\begin{aligned} \|\frac{f(2^n x)}{2^n} - \frac{f(2^{n+1}x)}{2^{n+1}}\| &= \frac{1}{2^n}\|f(2^n x) - \frac{f(2\cdot 2^n x)}{2}\| \\ &\le \frac{1}{2^n\ c}\delta + \frac{1}{2}\ 2^{(p-1)n}h(x) \end{aligned} \tag{3.7}$$

for all $x \in X$ and all positive integers n. By (3.7), we have

$$\|\frac{f(2^m x)}{2^m} - \frac{f(2^n x)}{2^n}\| \le \sum_{k=m}^{n-1}\frac{1}{2^k\ c}\delta + \sum_{k=m}^{n-1}\frac{1}{2}\ 2^{(p-1)k}h(x) \tag{3.8}$$

for all $x \in X$ and all positive integers m and n with $m < n$. This shows that the sequence $\{\frac{f(2^n x)}{2^n}\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{f(2^n x)}{2^n}\}$ converges for all $x \in X$. So we can define a mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty}\frac{f(2^n x)}{2^n}$$

for all $x \in X$. Since $f(-x) = -f(x)$ for all $x \in X$, we have $A(-x) = -A(x)$ for all $x \in X$. Also, we get

$$\begin{aligned} \|DA(x,y,z)\| &= \lim_{n\to\infty}\frac{1}{2^n}\|Df(2^n x, 2^n y, 2^n z)\| \\ &\le \lim_{n\to\infty}\frac{1}{2^n}\delta + 2^{(p-1)n}H(||x||,||y||,||z||) = 0 \end{aligned}$$

for all $x, y, z \in X$. By Lemma 2, it follows that A is additive. Putting $m = 0$ and letting $n \to \infty$ in (3.8), we get (3.2).

Now, let $A' : X \to Y$ be another additive mapping satisfying (3.2). Then we have

$$\begin{aligned} \|A(x) - A'(x)\| &= \frac{1}{2^n}\|A(2^n x) - A'(2^n x)\| \\ &\le \frac{1}{2^n}(\|A(2^n x) - f(2^n x)\| + \|A'(2^n x) - f(2^n x)\|) \\ &\le \frac{4}{2^n\ c}\delta + \frac{2}{2-2^p}2^{(p-1)n}h(x) \end{aligned} \tag{3.9}$$

for all $x \in X$ and all positive integers n. The right-hand side of (3.9) tends to zero as $n \to \infty$. So we can conclude that $A(x) = A'(x)$ for all $x \in X$. This proves the uniqueness of A.

(II) The case $p > 1$. Replacing x by $\frac{x}{2}$ in (3.5), we have

$$\|f(x) - 2f(\frac{x}{2})\| \leq \frac{1}{2^p}h(x) \tag{3.10}$$

for all $x \in X$. Using (3.10), we have

$$\|2^n f(\frac{x}{2^n}) - 2^{n+1}f(\frac{x}{2^{(n+1)}})\| \leq \frac{1}{2^p}2^{(1-p)n}h(x)$$

for all $x \in X$ and all positive integers n.

The rest of the proof is similar to the corresponding part of the case $0 < p < 1$. □

THEOREM 5. *Assume that $\delta \geq 0$, $p \in (0, \infty) \setminus \{2\}$ and $\delta = 0$ when $p > 2$. Let an even mapping $f : X \to Y$ satisfy (3.1) for all $x, y, z \in X$ and $f(0) = 0$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \leq \frac{1}{c}\delta + \frac{1}{|4 - 2^p|}h(x) \tag{3.11}$$

for all $x \in X$, where $h(x) = \frac{1}{2}H(||x||, ||x||, 0) + \frac{1}{4}H(||2x||, 0, 0)$.

Proof. Putting $y = x$ and $z = 0$ in (3.1), we have

$$\|af(\frac{2x}{b}) - cf(x)\| \leq \frac{1}{2}(\delta + H(||x||, ||x||, 0)) \tag{3.12}$$

for all $x \in X$. Putting $y = z = 0$ in (3.1) and then replacing x by $2x$, we have

$$\|af(\frac{2x}{b}) - \frac{c}{4}f(2x)\| \leq \frac{1}{4}(\delta + H(||2x||, 0, 0)) \tag{3.13}$$

for all $x \in X$. By (3.12) and (3.13), we have

$$\|f(2x) - 4f(x)\| \leq \frac{3}{c}\delta + h(x) \tag{3.14}$$

for all $x \in X$, where $h(x) = \frac{2}{c}H(||x||, ||x||, 0) + \frac{1}{c}H(||2x||, 0, 0)$

We divide the remaining proof by two cases.

(I) The case $0 < p < 2$. By (3.14), we have

$$\|f(x) - \frac{f(2x)}{4}\| \leq \frac{3}{4}\frac{}{c}\delta + \frac{1}{4}h(x) \tag{3.15}$$

for all $x \in X$. Using (3.15), we have

$$\begin{aligned} \|\frac{f(2^n x)}{4^n} - \frac{f(2^{n+1}x)}{4^{n+1}}\| &= \frac{1}{4^n}\|f(2^n x) - \frac{f(2 \cdot 2^n x)}{4}\| \\ &\leq \frac{3}{4}\frac{1}{c}\frac{1}{4^n}\delta + \frac{1}{4}2^{(p-2)n}h(x) \end{aligned} \tag{3.16}$$

for all $x \in X$ and all positive integers n. By (3.16), we have

$$\|\frac{f(2^m x)}{4^m} - \frac{f(2^n x)}{4^n}\| \leq \sum_{k=m}^{n-1} \frac{3}{4}\frac{1}{c}\frac{1}{4^k}\delta + \sum_{k=m}^{n-1} \frac{1}{4}2^{(p-2)k}h(x) \tag{3.17}$$

for all $x \in X$ and all nonnegative integers m and n with $m < n$. This shows that the sequence $\{\frac{f(2^n x)}{4^n}\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{f(2^n x)}{4^n}\}$ converges for all $x \in X$. So we can define a mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty} \frac{f(2^n x)}{4^n}$$

for all $x \in X$. We have $Q(0) = 0$, $Q(-x) = Q(x)$ and

$$\begin{aligned} \|DQ(x, y, z)\| &= \lim_{n\to\infty} \frac{1}{4^n}\|Df(2^n x, 2^n y, 2^n z)\| \\ &\leq \lim_{n\to\infty} (\frac{1}{4^n}\delta + 2^{(p-2)n}H(||x||, ||y||, ||z||)) = 0 \end{aligned}$$

for all $x, y, z \in X$. By Lemma 1, it follows that Q is quadratic. Putting $m = 0$ and letting $n \to \infty$ in (3.17), we get (3.11). The proof of the uniqueness of Q is similar to the proof of Theorem 4.

(II) The case $p > 2$. Replacing x by $\frac{x}{2}$ in (3.14), we have

$$\|f(x) - 4f(\frac{x}{2})\| \leq \frac{1}{2^p} h(x) \tag{3.18}$$

for all $x \in X$. Using (3.18), we have

$$\|4^n f(\frac{x}{2^n}) - 4^{n+1} f(\frac{x}{2^{(n+1)}})\| \leq \frac{1}{2^p}\ 2^{(2-p)n} h(x)$$

for all $x \in X$.

The rest of the proof is similar to the corresponding part of the case $p < 2$. □

THEOREM 6. *Let $\delta \geq 0$ and $p \in (0, \infty) \setminus \{1, 2\}$. Assume that $\delta = 0$ if $p > 1$ and $\|(4a - 3c)f(0)\| = 0$ if $p > 2$. If a mapping $f : X \to Y$ satisfy* (3.1) *for all $x, y, z \in X$, then there exist a unique quadratic mapping $Q : X \to Y$ and a unique additive mapping $A : X \to Y$ such that*

$$\begin{aligned} \|f(x) - f(0) - Q(x) - A(x)\| \leq & \frac{3}{c}\delta + \|(\frac{4a}{c} - 3)f(0)\| \\ & + \frac{1}{|4 - 2^p|} h_1(x) + \frac{1}{|2 - 2^p|} h_2(x), \end{aligned} \tag{3.19}$$

$$\begin{aligned} \|\frac{f(x) + f(-x)}{2} - f(0) - Q(x)\| \leq & \frac{1}{c}\delta + \|(\frac{4a}{c} - 3)f(0)\| \\ & + \frac{1}{|4 - 2^p|} h_1(x), \end{aligned} \tag{3.20}$$

$$\|\frac{f(x) - f(-x)}{2} - A(x)\| \leq \frac{2}{c}\delta + \frac{1}{|2 - 2^p|} h_2(x) \tag{3.21}$$

for all $x \in X$, where $h_1(x) = \frac{1}{2} H(||x||, ||x||, 0) + \frac{1}{4} H(||2x||, 0, 0)$ and $h_2(x) = \frac{1}{4}(H(||x||, ||x||, 0) + H(||2x||, 0, 0))$.

Proof. Let $q_1(x) := \frac{1}{2}(f(x) + f(-x))$ for all $x \in X$. Then $q_1(0) = f(0)$, $q_1(-x) = q_1(x)$ and

$$\|Dq_1(x, y, z)\| \leq \delta + H(||x||, ||y||, ||z||)$$

for all $x, y, z \in X$.

Let $q(x) := q_1(x) - q_1(0)$ for all $x \in X$. Then $q(0) = 0$, $q(-x) = q(x)$ and

$$\begin{aligned} \|Dq(x, y, z)\| &= \|Dq_1(x, y, z) - (4a - 3c)q_1(0)\| \\ &\leq \|Dq_1(x, y, z)\| + \|(4a - 3c)q_1(0)\| \\ &\leq \delta + \|(4a - 3c)f(0)\| + H(||x||, ||y||, ||z||) \end{aligned}$$

for all $x, y, z \in X$. By Theorem 5, there exists a unique quadratic mapping $Q : X \to Y$ satisfying (3.20).

Let $g(x) := \frac{1}{2}(f(x) - f(-x))$ for all $x \in X$. Then $g(-x) = -g(x)$ and

$$\|Dg(x, y, z)\| \leq \delta + H(||x||, ||y||, ||z||)$$

for all $x, y, z \in X$. By Theorem 4, there exists a unique additive mapping $A : X \to Y$ satisfying (3.21). Clearly, we have (3.19) for all $x \in X$. □

Define a function $H : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ by $H(a, b, c) = (a^p + b^p + c^p)\theta$ where $\theta \geq 0$ and $p \in (0, \infty)$. Then H is homogeneous of degree p. So we have the following corollary.

Corollary 7. *Let $\delta \geq 0$ and $p \in (0, \infty) \setminus \{1, 2\}$. Assume that $\delta = 0$ if $p > 1$ and $\|(4a - 3c)f(0)\| = 0$ if $p > 2$. If a mapping $f : X \to Y$ satisfy*

$$\|Df(x, y, z)\| \leq \delta + \theta(||x||^p + ||y||^p + ||z||^p)$$

for all $x, y, z \in X$, then there exist a unique quadratic mapping $Q : X \to Y$ and a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - f(0) - Q(x) - A(x)\| \le \frac{3}{c}\delta + \|(\frac{4a}{c} - 3)f(0)\| + (\frac{4+2^p}{4|4-2^p|} + \frac{2+2^p}{4|2-2^p|})\theta||x||^p,$$

$$\|\frac{f(x)+f(-x)}{2} - f(0) - Q(x)\| \le \frac{1}{c}\delta + \|(\frac{4a}{c} - 3)f(0)\| + \frac{4+2^p}{4|4-2^p|}\theta||x||^p,$$

$$\|\frac{f(x)-f(-x)}{2} - A(x)\| \le \frac{2}{c}\delta + \frac{2+2^p}{4|2-2^p|}\theta||x||^p$$

for all $x \in X$.

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