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ON A SELF-SIMILAR MEASURE ON A SELF-SIMILAR CANTOR SET

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Abstract. We compare a self-similar measure on a self-similar Cantor set with a quasi-self-similar measure on a deranged Cantor set. Further we study some properties of a self-similar measure on a self-similar Cantor set.

1. Introduction

Recently the multifractal spectrum by a self-similar measure of a self-similar Cantor set was studied([11, 13]) for the investigation of its geometrical properties. We([2, 5]) studied a deranged Cantor set which is the most generalized Cantor set which has a local structure of a perturbed Cantor set([1, 3, 4, 5, 6]), which is also a generalized form of self-similar Cantor set. In this paper, we compare the self-similar measure with a quasi-self-similar measure which also gives a spectrum of a deranged Cantor set([7, 10]). Recently we found the relation between a subset composing a spectrum by a self-similar measure of a self-similar Cantor set and a distribution set of the self-similar Cantor set([8, 9]). On the basis of the relation, we introduce an easy closed form of computing dimensions of a subset of the same local dimension of a self-similar measure on a self-similar Cantor set and give an example. Further we discuss some properties of the function of local dimension of self-similar measure at a point in a self-similar Cantor set, which plays an important role in the transformed dimension theory([7, 10]).

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2. Preliminaries

We recall the definition of a deranged Cantor set([2]). Let $X_\phi = [0,1]$. We obtain the left subinterval $X_{\mathfrak{i},1}$ and the right subinterval $X_{\mathfrak{i},2}$ of $X_\mathfrak{i}$ by deleting a middle open subinterval of $X_\mathfrak{i}$ inductively for each $\mathfrak{i} \in \{1,2\}^n$, where $n = 0,1,2,\ldots$. Let $E_n = \cup_{\mathfrak{i}\in\{1,2\}^n} X_\mathfrak{i}$. Then E_n is a decreasing sequence of closed sets. For each n, we set $|X_{\mathfrak{i},1}|/|X_\mathfrak{i}| = c_{\mathfrak{i},1}$ and $|X_{\mathfrak{i},2}|/|X_\mathfrak{i}| = c_{\mathfrak{i},2}$ for all $\mathfrak{i} \in \{1,2\}^n$, where $n = 0,1,2,\cdots$ where $|X|$ denotes the length of X. We assume that the contraction ratios $c_\mathfrak{i}$ and gap ratios $1-(c_{\mathfrak{i},1}+c_{\mathfrak{i},2})$ are uniformly bounded away from 0. We call $F = \cap_{n=0}^\infty E_n$ a deranged Cantor set([2]). We note that a deranged Cantor set satisfying $c_{\mathfrak{i},1} = a_{n+1}$ and $c_{\mathfrak{i},2} = b_{n+1}$ for all $\mathfrak{i} \in \{1,2\}^n$, for each $n = 0,1,2,\cdots$ is called a perturbed Cantor set([1]). Further a perturbed Cantor set with $a_{n+1} = a$ and $b_{n+1} = b$ for all $n = 0,1,2,\cdots$ is called a self-similar Cantor set([11]).

For $\mathfrak{i} \in \{1,2\}^n$, $X_\mathfrak{i}$ denotes a fundamental interval of the n-stage of construction of a deranged Cantor set. Let $\mathbb{R}$ be the set of all real numbers and $\mathbb{N}$ be the set of all natural numbers. For $y \in \mathbb{R}$, we([2]) define *a quasi-self-similar measure* μ_y on a deranged Cantor set F to be a Borel probability measure induced by

$$\mu_y(X_\mathfrak{i}) = p_{i_1} p_{i_1,i_2} \cdots p_{i_1,i_2,\cdots,i_n}$$

where

$$p_{i_1,\cdots,i_k} = \frac{c^y_{i_1,\cdots,i_{k-1},i_k}}{c^y_{i_1,\cdots,i_{k-1},1} + c^y_{i_1,\cdots,i_{k-1},2}}$$

for each $1 \le k \le n$ and $\mathfrak{i} = i_1,\cdots,i_n$. Then clearly we see that $p_{i_1,\cdots,i_{k-1},2} = 1 - p_{i_1,\cdots,i_{k-1},1}$.

REMARK 2.1. In a perturbed Cantor set F, for $y \in \mathbb{R}$ we find $p_{i_1,\cdots,i_{k-1},1} = p_k = \frac{a_k^y}{a_k^y + b_k^y}$ for each $k \in \mathbb{N}$. Further *the quasi-self-similar measure* μ_y on F is a Borel probability measure induced by

$$\mu_y(X_{\mathfrak{i}}) = r^{(1)}_{i_1} r^{(2)}_{i_2} \cdots r^{(n)}_{i_n} \qquad \text{where} \quad r^{(k)}_{i_k} = \begin{cases} p_k & \text{for } i_k = 1 \\ 1 - p_k & \text{for } i_k = 2 \end{cases},$$

$\mathfrak{i} = i_1, \cdots, i_k, \cdots, i_n$ and $1 \leq k \leq n$. We note that μ_y is just a self-similar measure if F is a self-similar Cantor set. We write μ_y as γ_p where $p = \frac{a^y}{a^y+b^y}$.

For $x \in F$, we write $X_n(x)$ for the n-th level set $X_{i_1 \cdots i_n}$ that contains x. We also note that if $x \in F$, then there is $\sigma \in \{1,2\}^N$ such that $\bigcap_{n=0}^{\infty} X_{\sigma|n} = \{x\}$ (Here $\sigma|n = i_1, i_2, \cdots, i_n$ where $\sigma = i_1, i_2, \cdots, i_n, i_{n+1}, \cdots$). Hereafter, we use $\sigma \in \{1,2\}^N$ and $x \in F$ as the same identity freely.
In a self-similar Cantor set F, we can consider a generalized expansion of x from σ, that is if $\sigma = i_1, i_2, \cdots, i_k, i_{k+1}, \cdots$ then the expansion of x is $0.j_1, j_2, \cdots, j_k, j_{k+1}, \cdots$ where $j_k = 0$ if $i_k = 1$ and $j_k = 2$ if $i_k = 2$. We denote $n_0(x|k)$ the number of times the digit 0 occurs in the first k places of the generalized expansion of x([12]).
For $r \in [0,1]$, we define lower(upper) distribution set $\underline{F}(r)(\overline{F}(r))$ containing the digit 0 in proportion r by

$$\underline{F}(r) = \{x \in F : \liminf_{k \to \infty} \frac{n_0(x|k)}{k} = r\},$$

$$\overline{F}(r) = \{x \in F : \limsup_{k \to \infty} \frac{n_0(x|k)}{k} = r\}.$$

We write $\underline{F}(r) \cap \overline{F}(r)$ as $F(r)$.

The *lower* and *upper local dimension* of a finite measure μ at $x \in \mathbb{R}$ are defined([11]) by

$$\underline{\dim}_{loc}\mu(x) = \liminf_{r \to 0} \frac{\log \mu(B(x,r))}{\log r}$$

$$\overline{\dim}_{loc}\mu(x) = \limsup_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}$$

where $B(x,r)$ is the closed ball with center $x \in \mathbb{R}$ and radius $r > 0$.

If $\underline{\dim}_{loc}\mu(x) = \overline{\dim}_{loc}\mu(x)$, we call it *the local dimension of* μ *at* x and write it as $\dim_{loc}\mu(x)$. These local dimensions express the power law behaviour of $\mu(B(x,r))$ for some $r > 0$.

For $\alpha \geq 0$ define

$$\begin{aligned} E^y_\alpha &= \{x \in \mathbb{R} | \dim_{loc} \mu_y(x) = \alpha\} \\ &= \{x \in \mathbb{R} | \lim_{r\to 0} \frac{\log \mu_y(B(x,r))}{\log r} = \alpha\} \end{aligned}$$

Also we write $\underline{E}^y_\alpha$ ($\overline{E}^y_\alpha$) for the set of points at which the lower(upper) local dimension of μ_y on F is exactly α, so that

$$\underline{E}^y_\alpha = \{x : \liminf_{r\to 0} \frac{\log \mu_y(B(x,r))}{\log r} = \alpha\},$$

$$\overline{E}^y_\alpha = \{x : \limsup_{r\to 0} \frac{\log \mu_y(B(x,r))}{\log r} = \alpha\}.$$

From now on, $\dim(E)$ denotes the Hausdorff dimension of $E \in \mathbb{R}$ and $\mathrm{Dim}(E)$ denotes the packing dimension of E. In this paper, we assume that $0 \log 0 = 0$ for convenience.

3. Main results

Consider a self-similar Cantor set F with two contraction ratios a and b. Let $y \in \mathbb{R}$ and consider $p^q a^{\beta^y(q)} + (1-p)^q b^{\beta^y(q)} = 1$ and $p = \frac{a^y}{a^y+b^y}$. For $\alpha \geq 0$, the Legendre transform $f^y(\alpha)$ of beta function β^y is defined by

$$f^y(\alpha) = \inf_{-\infty<q<\infty} \{\beta^y(q) + \alpha q\}.$$

It will be helpful for us to study the relation between the set $X_n(x)$ and the closed ball $B(x, r)$,that is, $|X_n(x)|$ is comparable with r.

LEMMA 3.1. *Given a Borel probability measure μ on a deranged Cantor set F, for all $x \in F$,*

$$\liminf_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} = \liminf_{n \to \infty} \frac{\log \mu(X_n(x))}{\log |X_n(x)|}$$

and

$$\limsup_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} = \limsup_{n \to \infty} \frac{\log \mu(X_n(x))}{\log |X_n(x)|}.$$

Proof. It is obvious from the fact that the contraction ratios are uniformly bounded away from 0. □

THEOREM 3.2. *Let $y \in \mathbb{R}$ and $p = \frac{a^y}{a^y + b^y}$. Consider a self-similar measure $\gamma_p (= \mu_y)$ on a self-similar Cantor set F and let $r \in [0, 1]$ and $g(r,p) = \frac{r \log p + (1-r) \log(1-p)}{r \log a + (1-r) \log b}$. Then $\dim(E^y_\alpha) = \mathrm{Dim}(E^y_\alpha) = g(r,r)$ where $\alpha = g(r,p)$.*

Proof. From (11.30) (11.35) and (11.50) in [11], we see that the dimensions of E^y_α is $f^y(\alpha) = \alpha q + \beta^y(q)$ where q and $\beta^y(q)$ satisfies the two equations such that $p^q a^{\beta^y(q)} + (1-p)^q b^{\beta^y(q)} = 1$ and

$$\alpha = \frac{p^q a^{\beta^y(q)} \log p + (1-p)^q b^{\beta^y(q)} \log(1-p)}{p^q a^{\beta^y(q)} \log a + (1-p)^q b^{\beta^y(q)} \log b}.$$

Putting $p^q a^{\beta^y(q)} = r$ for q and $\beta^y(q)$ satisfying the above two equations, we see that r satisfies $\alpha = g(r,p)$. We easily see that $g(r,r) = \alpha q + \beta^y(q)$. □

REMARK 3.1. From the proof in the above theorem, we get our result that $\dim(E^y_\alpha) = \mathrm{Dim}(E^y_\alpha) = g(r,r)$ where $\alpha = g(r,p)$. However this result was hinted from the relation between a distribution set and a subset E^y_α of same local dimension of a self-similar measure([8]).

REMARK 3.2. The calculation from our result in the above theorem that $\dim(E^y_\alpha) = \mathrm{Dim}(E^y_\alpha) = g(r,r)$ where $\alpha = g(r,p)$ is much easier to compute than that of Olsen([11, 13]). That is, it is so hard to find the values q and $\beta^y(q)$ satisfying the two equations such that $p^q a^{\beta^y(q)} + (1-p)^q b^{\beta^y(q)} = 1$ and

$$\alpha = \frac{p^q a^{\beta^y(q)} \log p + (1-p)^q b^{\beta^y(q)} \log(1-p)}{p^q a^{\beta^y(q)} \log a + (1-p)^q b^{\beta^y(q)} \log b}.$$

After finding such two values q and $\beta^y(q)$, we get $\dim(E^y_\alpha) = \mathrm{Dim}(E^y_\alpha) = f^y(\alpha) = \alpha q + \beta^y(q)$.

REMARK 3.3. In the above theorem, when we consider E^y_α, the range of α is $[\frac{\log p}{\log a}, \frac{\log(1-p)}{\log b}]$ or $[\frac{\log(1-p)}{\log b}, \frac{\log p}{\log a}]$ which has non-empty interior if $\frac{\log p}{\log a} \neq \frac{\log(1-p)}{\log b}$.

We give an example to show how much our calculation is easier than that of Olsen.

EXAMPLE 3.1. Consider a self-similar Cantor set with $a = \frac{1}{2}$ and $b = \frac{1}{4}$. Then there is a solution y such that $\frac{1}{2} = \frac{a^y}{a^y + b^y}$. That is $p = \frac{1}{2}$. In fact $y = 0$. Now we find the dimensions of $E^0_{\frac{3}{4}}$. Our calculation is easy. That is we find r such that $\frac{3}{4} = \frac{r \log \frac{1}{2} + (1-r)\log\frac{1}{2}}{r \log \frac{1}{2} + (1-r)\log \frac{1}{4}} = g(r, \frac{3}{4})$. In fact $r = \frac{2}{3}$. Now we easily find $g(\frac{2}{3}, \frac{2}{3}) = \frac{\log 4 - \log 27}{-\log 16}$. So the dimensions of $E^0_{\frac{3}{4}}$ are $\frac{\log 4 - \log 27}{-\log 16}$. That of Olsen is so complicated. Sometimes it is almost impossible to find algebraically the values q and $\beta^y(q)$ satisfying the two equations such that $p^q a^{\beta^y(q)} + (1-p)^q b^{\beta^y(q)} = 1$ and

$$\alpha = \frac{p^q a^{\beta^y(q)} \log p + (1-p)^q b^{\beta^y(q)} \log(1-p)}{p^q a^{\beta^y(q)} \log a + (1-p)^q b^{\beta^y(q)} \log b}.$$

In this case we adjusted the numbers to solve it possible even though it is also complicated. We solve $\frac{3}{4} = \frac{\frac{1}{2}^q \frac{1}{2}^{\beta^y(q)} \log \frac{1}{2} + (1-\frac{1}{2})^q \frac{1}{4}^{\beta^y(q)} \log(1-\frac{1}{2})}{\frac{1}{2}^q \frac{1}{2}^{\beta^y(q)} \log \frac{1}{2} + (1-\frac{1}{2})^q \frac{1}{4}^{\beta^y(q)} \log \frac{1}{4}}$ and find $\beta^y(q) = 1$. From $\frac{1}{2}^q\frac{1}{2}^{\beta^y(q)} + (1-\frac{1}{2})^q\frac{1}{4}^{\beta^y(q)} = 1$, we see that $q = \frac{\log\frac{4}{3}}{\log\frac{1}{2}}$. So we have $f^0(\frac{3}{4}) = \frac{3}{4}\frac{\log\frac{4}{3}}{\log\frac{1}{2}} + 1 = \frac{\log 4 - \log 27}{-\log 16}$.

Now we discuss the continuity of the lower(upper) local dimension function $\underline{\dim}_{loc}\mu_y(x)(\overline{\dim}_{loc}\mu_y(x))$ of μ_y at $x \in F$ and $y \in \mathbb{R}$.

Theorem 3.3. *Fix $x \in F$ where F is a self-similar Cantor set. Then $\underline{\dim}_{loc}\mu_y(x)$ is a continuous function for $y \in \mathbb{R}$. Similarly $\overline{\dim}_{loc}\mu_y(x)$ is a continuous function for $y \in \mathbb{R}$.*

Proof. Fix $x \in F$. Let $\delta_n(y) = \frac{\sum_{k=1}^n \log(a^y+b^y)}{\log|X_n(x)|}$ for $y \in \mathbb{R}$. We note that $\underline{\dim}_{loc}\mu_y(x) = y - \limsup_{n\to\infty}\delta_n(y)$ from Lemma 3.1. Assume that $B_1 = \min\{a,b\}$ and $B_2 = \max\{a,b\}$. Clearly $0 < B_1 \le a, b \le B_2 < 1$ for all $k \in \mathbb{N}$. Consider $h(z) = \frac{a^z+b^z}{a^y+b^y}$ for fixed y. From the mean value theorem we see that $h(z) - h(y) = h'(w)(z-y)$ for some w between z and y. Then

$$|\frac{a^z+b^z}{a^y+b^y} - 1| \le \frac{|\log B_1|}{B_1}|z-y|$$

for all $k \in \mathbb{N}$. Hence

$$|\delta_n(z) - \delta_n(y)| \le \frac{K|z-y|}{|\log B_2|}$$

for all $n \in \mathbb{N}$ where $0 < K < \infty$ which is from B_1 and independent of n. Putting $\frac{K}{|\log B_2|} = C$, we have $|\delta_n(z) - \delta_n(y)| \le C|z-y|$ all $n \in \mathbb{N}$. Writing $\delta(y) = \limsup_{n\to\infty}\delta_n(y)$ for every $y \in \mathbb{R}$, we only need to show that $\delta(y)$ is continuous for $y \in \mathbb{R}$. Fix $y \in \mathbb{R}$ and suppose that $\lim_{z\to y}\delta(z) \ne \delta(y)$. Then there is $\epsilon > 0$ and a sequence $\{t_m\}$ of real numbers such that $t_m \to y$ satisfying $\delta(t_m) > \delta(y)+\epsilon$ or $\delta(t_m) < \delta(y)-\epsilon$. Consider m satisfying $C|t_m - y| < \frac{\epsilon}{3}$. Then $|\delta_n(t_m) - \delta_n(y)| < \frac{\epsilon}{3}$ for all $n \in \mathbb{N}$.

Suppose that $\delta(t_m) > \delta(y) + \epsilon$. There is a sequence $\{m_k\}$ of natural numbers such that $\delta_{m_k}(t_m) \to \delta(t_m)$ and $|\delta_{m_k}(t_m) - \delta_{m_k}(y)| < \frac{\epsilon}{3}$ for all m_k. We have a contradiction since $\limsup_{k\to\infty} \delta_{m_k}(y) \geq \delta(y) + \frac{2\epsilon}{3}$.

Now assume that $\delta(t_m) < \delta(y) - \epsilon$. There is a natural number N_m such that $\delta_n(t_m) < \delta(y) - \epsilon$ for all $n \geq N_m$ and $|\delta_n(t_m) - \delta_n(y)| < \frac{\epsilon}{3}$ for such n. We have a contradiction since $\limsup_{n\to\infty} \delta_n(y) \leq \delta(y) - \frac{2\epsilon}{3}$. It follows that $\underline{\dim}_{loc}\mu_y(x)$ is a continuous function for y. Dually $\overline{\dim}_{loc}\mu_y(x)$ is a continuous function for y. □

THEOREM 3.4. *Let F be a self-similar Cantor set. Fix $y(\neq s) \in \mathbb{R}$ where $a^s + b^s = 1$. Then $\underline{\dim}_{loc}\mu_y(x)$ is a nowhere continuous function for $x \in F$. Similarly $\overline{\dim}_{loc}\mu_y(x)$ is a nowhere continuous function for $x \in F$.*

Proof. We note that each $x \in F$ is a limit point of F and the distribution set $F(r)$ is dense in F for each $r \in [0,1]$([12]). Fix $y(\neq s) \in \mathbb{R}$ where $a^s + b^s = 1$. Then $p = \frac{a^y}{a^y+b^y}$. For $z \in F(r)$, $\dim_{loc}\mu_y(z) = g(r,p)$. So $\{\underline{\dim}_{loc}\mu_y(z) : z \in B(x,u), u > 0\} = [\frac{\log p}{\log a}, \frac{\log(1-p)}{\log b}]$ or $[\frac{\log(1-p)}{\log b}, \frac{\log p}{\log a}]$, since $\{\underline{\dim}_{loc}\mu_y(z) : z \in B(x,u), u > 0\}$ contains $\{\dim_{loc}\mu_y(z) : z \in B(x,u), u > 0$ and $z \in F(r)$ for some $r \in [0,1]\} = [\frac{\log p}{\log a}, \frac{\log(1-p)}{\log b}]$ or $[\frac{\log(1-p)}{\log b}, \frac{\log p}{\log a}]$. It follows easily since $[\frac{\log p}{\log a}, \frac{\log(1-p)}{\log b}]$ or $[\frac{\log(1-p)}{\log b}, \frac{\log p}{\log a}]$ has non-empty interior if $y(\neq s) \in \mathbb{R}$ where $a^s + b^s = 1$. It holds dually for the case of $\overline{\dim}_{loc}\mu_y(x)$. □

REMARK 3.4. Note that the lower(upper) distribution set $\underline{F}(r)(\overline{F}(r))$ is dense in F for each $r \in [0,1]$ since the distribution set $F(r)$ is dense in F for each $r \in [0,1]$([12]). If $y \neq s$ where $a^s + b^s = 1$, $\{\underline{\dim}_{loc}\mu_y(z) : z \in B(x,u), u > 0\} = [\frac{\log p}{\log a}, \frac{\log(1-p)}{\log b}]$ or $[\frac{\log(1-p)}{\log b}, \frac{\log p}{\log a}]$ since $\underline{F}(r) = \underline{E}^y_\alpha$ where $\alpha = g(r,p)$ and $p = \frac{a^y}{a^y+b^y}$ with $0 < p < a^s$ and $\overline{F}(r) = \underline{E}^y_\alpha$ where $\alpha = g(r,p)$ with $a^s < p < 1$([8]).

Remark 3.5. We see that some variation of $\underline{\dim}_{loc}\mu_y(x)(\overline{\dim}_{loc}\mu_y(x))$ is a continuous function for $y \in \mathbb{R}$ for fixed $x \in F$ where F is a deranged Cantor set([6, 7, 10]), which plays an important role in their transformed dimension theories that give better estimation of dimensions of E_α^y.

Remark 3.6. ([8]) We see that $\underline{E}_\alpha^s = F = \overline{E}_\alpha^s$ if F is a self-similar Cantor set and $a^s + b^s = 1$. Further in this case the range of α is $[\frac{\log p}{\log a}, \frac{\log(1-p)}{\log b}] = [\frac{\log(1-p)}{\log b}, \frac{\log p}{\log a}] = \{s\}$. We also note that $\underline{\dim}_{loc}\mu_y(x)$ and $\overline{\dim}_{loc}\mu_y(x)$ are constant functions for $x \in F$ in this case. As in the above Theorem we used to assume in multifractal theory that $\frac{\log p}{\log a} \neq \frac{\log(1-p)}{\log b}$ to avoid the degenerate case .

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