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A CHARACTERIZATION OF $\mathbf{C}^k \times (\mathbf{C}^*)^\ell$ FROM THE VIEWPOINT OF BIHOLOMORPHIC AUTOMORPHISM GROUPS

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ABSTRACT. We show that if a connected Stein manifold M of dimension n has the holomorphic automorphism group $\text{Aut}(M)$ isomorphic to $\text{Aut}(\mathbf{C}^k \times (\mathbf{C}^*)^{n-k})$ as topological groups, then M itself is biholomorphically equivalent to $\mathbf{C}^k \times (\mathbf{C}^*)^{n-k}$. Besides, a new approach to the study of $U(n)$-actions on complex manifolds of dimension n is given.

1. Introduction and results

This article is the outgrowth of the talk given by the first author at the Sixth International Conference on Several Complex Variables and Complex Geometry in Gyeong-Ju, Korea.

In the study of the holomorphic automorphism group $\text{Aut}(M)$ of a complex manifold M, it seems to be natural to direct our attention to not only the abstract group structure of $\text{Aut}(M)$ but also the topological group structure of $\text{Aut}(M)$ equipped with the compact-open topology. In fact, a well-known theorem of H. Cartan says that the topological group given as the holomorphic automorphism group of a bounded domain in $\mathbf{C}^n$ has the structure of a Lie group, and this result enables us to make various kinds of detailed studies of bounded domains in $\mathbf{C}^n$. On the other hand, in contrast to the case of bounded domains, the holomorphic automorphism group $\text{Aut}(\mathbf{C}^k \times (\mathbf{C}^*)^\ell)$ of the unbounded domain $\mathbf{C}^k \times (\mathbf{C}^*)^\ell$ is terribly big when $k + \ell \geq 2$, and can not have the

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structure of a Lie group. But, by looking at topological subgroups of $\mathrm{Aut}(\mathbf{C}^k \times (\mathbf{C}^*)^\ell)$ with Lie group structures, we can find a lead to apply the Lie group theory to the investigation of the problems related to the structure of $\mathrm{Aut}(\mathbf{C}^k \times (\mathbf{C}^*)^\ell)$.

In this article, we try to approach from this standpoint to the fundamental problem of what complex manifold has the holomorphic automorphism group isomorphic to $\mathrm{Aut}(\mathbf{C}^k \times (\mathbf{C}^*)^\ell)$ as topological groups. In fact, we can prove the following results. The details can be found in [11]:

MAIN THEOREM. *Let M be a connected Stein manifold of dimension n. Assume that $\mathrm{Aut}(M)$ is isomorphic to $\mathrm{Aut}(\mathbf{C}^k \times (\mathbf{C}^*)^{n-k})$ as topological groups for some integer k with $0 \leq k \leq n$. Then M is biholomorphically equivalent to $\mathbf{C}^k \times (\mathbf{C}^*)^{n-k}$.*

As a consequence of the above theorem, we can obtain the fundamental result on the topological group structure of $\mathrm{Aut}(\mathbf{C}^k \times (\mathbf{C}^*)^\ell)$:

COROLLARY. *If two pairs (k, ℓ) and (k', ℓ') of nonnegative integers do not coincide, then the topological groups $\mathrm{Aut}(\mathbf{C}^k \times (\mathbf{C}^*)^\ell)$ and $\mathrm{Aut}(\mathbf{C}^{k'} \times (\mathbf{C}^*)^{\ell'})$ are not isomorphic.*

It should be remarked that, as shown in Ahern-Rudin [1], the groups $\mathrm{Aut}(\mathbf{C}^n)$ and $\mathrm{Aut}(\mathbf{C}^m)$ are isomorphic as abstract groups precisely when $n = m$. Also, as a consequence of the study of $U(n)$-actions on complex manifolds of dimension n, Isaev-Kruzhilin [8] showed that exactly the same conclusion in the Main Theorem remains valid for the case of $k = n$ without assuming the Steinness of M.

Our method can be applied to the study of unitary group actions on complex manifolds. The following Theorems A and B give a different approach from Kaup [9] and Isaev-Kruzhilin [8] to the study of $U(n)$-actions on a complex manifold of dimension n.

THEOREM A. *Let M be a connected Stein manifold of dimension $n \geq 2$. Assume that $U(n)$ acts effectively on M as a Lie transformation group through ρ. Then M is biholomorphically equivalent to either B^n or $\mathbf{C}^n$, where B^n denotes the unit ball in $\mathbf{C}^n$.*

THEOREM B. *Let M be a connected Stein manifold of dimension $n \geq 2$. Assume that there are two injective continuous group homomorphisms ρ_1 and ρ_2 of $U(n)$ into $\mathrm{Aut}(M)$. Then there exists an element ψ of $\mathrm{Aut}(M)$ such that $\psi\rho_1(U(n))\psi^{-1} = \rho_2(U(n))$. More precisely, in this case one can choose an element Ψ of $\mathrm{Aut}(M)$ in such a way that*

$$\Psi\rho_1(u)\Psi^{-1} = \rho_2(u) \quad \text{or} \quad \Psi\rho_1(u)\Psi^{-1} = \rho_2(\bar{u}) \quad \text{for all } u \in U(n),$$

where $\bar{u}$ denotes the complex conjugate of a matrix u.

Our proof of the Main Theorem relies on the one hand on the theory of Reinhardt domains developed in Shimizu [16], [17] (cf. Kruzhilin [13]), on the other hand on the fundamental result on torus actions on complex manifolds due to Barrett-Bedford-Dadok [3].

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2. Basic concepts and notations

Let M be a complex manifold. An *automorphism of M* means a biholomorphic mapping of M onto itself. We denote by $\mathrm{Aut}(M)$ the topological group of all automorphisms of M equipped with the compact-open topology. Let G be a Lie group and consider a continuous group homomorphism $\rho : G \to \mathrm{Aut}(M)$ of the Lie group G into the topological group $\mathrm{Aut}(M)$. Then the mapping

$$G \times M \ni (g, p) \longmapsto (\rho(g))(p) \in M$$

is continuous. It follows from Akhiezer [2] that this mapping is actually of class C^ω, and therefore G acts on M as a Lie transformation group. In view of this, when a continuous group homomorphism $\rho : G \to \mathrm{Aut}(M)$ of G into $\mathrm{Aut}(M)$ is given, we say that *G acts on M as a Lie transformation group through ρ*. Also, the action of G on M is called *effective* if ρ is injective.

We denote by $U(k)$ the *unitary group of degree k*. Write $T^n = (U(1))^n$. The n-dimensional compact torus T^n acts as a group of automorphisms on $\mathbf{C}^n$ by the standard rule $\alpha \cdot z = (\alpha_1 z_1, \ldots, \alpha_n z_n)$ for $\alpha = (\alpha_1, \ldots, \alpha_n) \in T^n$ and $z = (z_1, \ldots, z_n) \in \mathbf{C}^n$. By definition, a *Reinhardt domain D* in $\mathbf{C}^n$ is a domain in $\mathbf{C}^n$ which is stable under the action of T^n. Each element α of T^n then induces an automorphism π_α of D given by $\pi_\alpha(z) = \alpha \cdot z$, and the mapping ρ_D sending α to π_α is an injective continuous group homomorphism of the torus T^n into the topological group $\mathrm{Aut}(D)$. The subgroup $\rho_D(T^n)$ of $\mathrm{Aut}(D)$ is denoted by $T(D)$.

3. Some lemmas and fundamental theorems

For later purpose, in this section we shall recall some lemmas and fundamental theorems. We refer the reader to [11] for the details.

Let f be a holomorphic function on a Reinhardt domain D in $\mathbf{C}^n$. Then f can be expanded uniquely into a "Laurent series"

$$f(z) = \sum_{\nu \in \mathbf{Z}^n} a_\nu z^\nu,$$

which converges absolutely and uniformly on any compact set in D, where $z = (z_1, \ldots, z_n)$, $\nu = (\nu_1, \ldots, \nu_n)$, and $z^\nu = z_1^{\nu_1} \cdots z_n^{\nu_n}$.

The following lemma is a consequence of the uniqueness of the Laurent series expansion:

LEMMA 1. *Let f be a holomorphic function on a Reinhardt domain D in $\mathbf{C}^n$. If f satisfies the condition that, for some $\nu_0 \in \mathbf{Z}^n$,*

$$f(\alpha \cdot z) = \alpha^{\nu_0} f(z) \quad \text{for all } \alpha \in T^n \text{ and all } z \in D,$$

then f has the form $f(z) = a_{\nu_0} z^{\nu_0}$.

We denote by $\Pi(\mathbf{C}^n)$ the group of all automorphisms of $\mathbf{C}^n$ of the form

$$\mathbf{C}^n \ni (z_1, \ldots, z_n) \longmapsto (\alpha_1 z_1, \ldots, \alpha_n z_n) \in \mathbf{C}^n,$$

where $(\alpha_1, \ldots, \alpha_n) \in (\mathbf{C}^*)^n$. For a Reinhardt domain D in $\mathbf{C}^n$, we denote by $\Pi(D)$ the subgroup of $\Pi(\mathbf{C}^n)$ consisting of all elements of $\Pi(\mathbf{C}^n)$ leaving D invariant. Identifying $\Pi(\mathbf{C}^n)$ with the multiplicative group $(\mathbf{C}^*)^n$, we see that, when $\Pi(D)$ is regarded as a topological subgroup of $\mathrm{Aut}(D)$, it is isomorphic to a closed Lie subgroup of $(\mathbf{C}^*)^n$. Using Lemma 1, we obtain the following characterization of $\Pi(D)$ as a subgroup of $\mathrm{Aut}(D)$:

LEMMA 2. *Let D be a Reinhardt domain in $\mathbf{C}^n$. Then $\Pi(D)$ is the centralizer $C_{\mathrm{Aut}(D)}(T(D))$ of $T(D)$ in $\mathrm{Aut}(D)$.*

As stated in the introduction, our proof of the Main Theorem is based on the following fact by Shimizu [17], which is shown implicitly in the process of determining the automorphism groups of bounded Reinhardt domains in $\mathbf{C}^n$, and also on the fundamental result on torus actions on complex manifolds due to Barrett-Bedford-Dadok [3]:

FUNDAMENTAL THEOREM 1 ([17]). *Let D be a bounded Reinhardt domain in $\mathbf{C}^n$ and suppose that*

$$D \cap \{z_i = 0\} \neq \emptyset, \quad 1 \leq i \leq m,$$
$$D \cap \{z_i = 0\} = \emptyset, \quad m+1 \leq i \leq n,$$

or that $D \subset \mathbf{C}^m \times (\mathbf{C}^*)^{n-m}$. If G is a connected compact subgroup of $\mathrm{Aut}(D)$ containing $T(D)$, then there exists a transformation

$$\varphi : \mathbf{C}^m \times (\mathbf{C}^*)^{n-m} \ni (z_1, \ldots, z_n) \longmapsto (w_1, \ldots, w_n) \in \mathbf{C}^m \times (\mathbf{C}^*)^{n-m},$$
$$\begin{cases} w_i = r_i z_{\sigma'(i)} (z'')^{\nu_i''}, & \text{if} \quad 1 \leq i \leq m, \\ w_i = r_i z_{\sigma''(i)}, & \text{if} \quad m+1 \leq i \leq n, \end{cases}$$

such that, for $\tilde{D} = \varphi(D)$ and $\tilde{G} = \varphi G \varphi^{-1} \subset \mathrm{Aut}(\tilde{D})$, one has

$$\tilde{G} = U(k_1) \times \cdots \times U(k_s) \times U(k_{s+1}) \times \cdots \times U(k_t),$$
$$k_1 + \cdots + k_s + k_{s+1} + \cdots + k_t = n,$$
$$k_1 + \cdots + k_s = m,$$
$$k_{s+1} = \cdots = k_t = 1,$$

where $r_1, \ldots, r_n$ are positive constants, σ' and σ'' are permutations of $\{1, \ldots, m\}$ and $\{m+1, \ldots, n\}$, respectively, z'' denotes the coordinates $(z_{m+1}, \ldots, z_n)$, and $\nu_1'', \ldots, \nu_m''$ are elements of $\mathbf{Z}^{n-m}$.

From this, we obtain the following corollary which will play an important role in our proof of the Main Theorem:

COROLLARY. *In the above theorem, if G is isomorphic to $U(k) \times (U(1))^{n-k}$ as topological groups and if $k \geq 2$, then we have $m \geq k$.*

FUNDAMENTAL THEOREM 2 ([3]). *Let M be a connected Stein manifold of dimension n. Assume that T^n acts effectively on M as a Lie transformation group through ρ. Then there exist a biholomorphic mapping F of M into $\mathbf{C}^n$ and a continuous group automorphism θ of the torus T^n such that*

$$F((\rho(\alpha))(p)) = \theta(\alpha) \cdot F(p) \quad \text{for all } \alpha \in T^n \text{ and all } p \in M.$$

Consequently, $D := F(M)$ is a Reinhardt domain in $\mathbf{C}^n$, and one has $F\rho(T^n)F^{-1} = T(D)$.

LEMMA 3. *In the above theorem, if $M = \mathbf{C}^k \times (\mathbf{C}^*)^{n-k}$, then we have $D = F(M) = \mathbf{C}^k \times (\mathbf{C}^*)^{n-k}$.*

LEMMA 4. *Let M be a connected Stein manifold of dimension n. If $N > n$, then there is no injective continuous group homomorphism of the torus T^N into the topological group* $\mathrm{Aut}(M)$.

This lemma can be shown by using the fact that the group $T(D)$ is a maximal torus in $\mathrm{Aut}(D)$, provided that D is a bounded Reinhardt domain in $\mathbf{C}^n$ [16; Section 4, Proposition 1].

4. Proofs of the theorems and the corollary

For the sake of simplicity, we write $X_{k,\ell} = \mathbf{C}^k \times (\mathbf{C}^*)^\ell$ and $\Omega_k = X_{k,n-k}$ in this section.

Proof of the Main Theorem. Let us assume that there exists a topological group isomorphism $\Phi : \mathrm{Aut}(\Omega_k) \to \mathrm{Aut}(M)$. Since Ω_k is a Reinhardt domain in $\mathbf{C}^n$, we have the injective continuous group homomorphism $\rho_{\Omega_k} : T^n \to \mathrm{Aut}(\Omega_k)$. Thus, we obtain an injective continuous group homomorphism $\Phi \circ \rho_{\Omega_k} : T^n \to \mathrm{Aut}(M)$. Hence, by Fundamental Theorem 2 there exists a biholomorphic mapping F of M into $\mathbf{C}^n$ such that $D := F(M)$ is a Reinhardt domain in $\mathbf{C}^n$ and $F(\Phi \circ \rho_{\Omega_k})(T^n)F^{-1} = T(D)$. Therefore we may assume that M is a Reinhardt domain D in $\mathbf{C}^n$ and we have a topological group isomorphism $\Phi : \mathrm{Aut}(\Omega_k) \to \mathrm{Aut}(D)$ such that $\Phi(T(\Omega_k)) = T(D)$. Now we will proceed in steps.

1) *D has the form $D = \Omega_h$ after a suitable permutation of coordinates.* First we wish to show that $(\mathbf{C}^*)^n \subset D$. Since $\Phi : \mathrm{Aut}(\Omega_k) \to \mathrm{Aut}(D)$ is a topological group isomorphism and since $\Phi(T(\Omega_k)) = T(D)$, we see that Φ gives rise to a topological group isomorphism $\Phi : C_{\mathrm{Aut}(\Omega_k)}(T(\Omega_k)) \to C_{\mathrm{Aut}(D)}(T(D))$. Moreover, by Lemma 2 we have

$$C_{\mathrm{Aut}(\Omega_k)}(T(\Omega_k)) = \Pi(\Omega_k) = \Pi(\mathbf{C}^n) \quad \text{and} \quad C_{\mathrm{Aut}(D)}(T(D)) = \Pi(D).$$

Thus $\Pi(D)$ is a $2n$-dimensional Lie subgroup of the connected Lie group $\Pi(\mathbf{C}^n) = (\mathbf{C}^*)^n$, and therefore $\Pi(D) = \Pi(\mathbf{C}^n)$. By taking a point z_0 in $D \cap (\mathbf{C}^*)^n$, this shows that

$$(\mathbf{C}^*)^n = \Pi(\mathbf{C}^n) \cdot z_0 = \Pi(D) \cdot z_0 \subset D,$$

as required. Since D is now a Stein subdomain of $\mathbf{C}^n$ containing $(\mathbf{C}^*)^n$, we see that D has the form $D = \Omega_h$ after a suitable permutation of coordinates (cf. [15; p. 46, Theorem 1.5]), completing the proof of the assertion 1).

In the case of $n = 1$, it is easy to prove the Main Theorem. Therefore, in what follows, we assume that $n \geq 2$. Under this assumption, we next prove the following:

2) *We have* $h \geq k$.

When $k = 0$, there is nothing to prove. To prove our assertion when $k \neq 0$, we divide the proof into the two cases of $k = 1$ and $k \geq 2$.

First consider the case of $k \geq 2$. Noting that $\mathrm{Aut}(\Omega_k)$ contains the subgroup $U(k) \times (U(1))^{n-k}$, we set $G = \Phi(U(k) \times (U(1))^{n-k})$, which is a connected compact subgroup of $\mathrm{Aut}(D)$ containing $T(D)$, because $U(k) \times (U(1))^{n-k} \supset T(\Omega_k)$ and $\Phi(T(\Omega_k)) = T(D)$. Take a bounded domain U in $\mathbf{C}^n$ contained in D and put

$$D_0 = \{g(z) \in D \,|\, g \in G,\ z \in U\} = \bigcup_{g \in G} g(U) = \bigcup_{z \in U} G \cdot z.$$

Then D_0 is a bounded Reinhardt domain in D and G can be regarded as a connected compact subgroup of the Lie group $\mathrm{Aut}(D_0)$ containing $T(D_0)$. Since G is isomorphic to $U(k) \times (U(1))^{n-k}$ and $k \geq 2$, we can apply the corollary to Fundamental Theorem 1 to D_0 and $G \subset \mathrm{Aut}(D_0)$. Therefore, after a suitable permutation of coordinates, we have for some $m \geq k$,

$$\emptyset \neq D_0 \cap \{z_i = 0\} \subset D \cap \{z_i = 0\}, \quad 1 \leq i \leq m.$$

This implies that $\Omega_m \subset D$; and consequently, we have $h \geq m \geq k$, as required.

Now consider the case of $k = 1$. The only thing which has to be proved now is that the topological groups $\mathrm{Aut}(\Omega_1)$ and $\mathrm{Aut}(\Omega_0)$ are not isomorphic. Suppose contrarily that we have an isomorphism $\Phi : \mathrm{Aut}(\Omega_1) \to \mathrm{Aut}(\Omega_0)$. Then, by Fundamental Theorem 2 and Lemma 3, we may assume that we have a topological group isomorphism $\Phi : \mathrm{Aut}(\Omega_1) \to \mathrm{Aut}(\Omega_0)$ such that $\Phi(T(\Omega_1)) = T(\Omega_0)$. For $s = 0, 1$, let us set

$$T'(\Omega_s) = \{(1, \alpha_2, \ldots, \alpha_n) \in T(\Omega_s) \,|\, \alpha_2, \ldots, \alpha_n \in U(1)\}.$$

Then $\Phi(T'(\Omega_1))$ is an $(n-1)$-dimensional subtorus of $T(\Omega_0)$; and hence, after a suitable change of coordinates by a transformation of the form

$$\begin{aligned} &\Omega_0 = (\mathbf{C}^*)^n \ni (z_1, \ldots, z_n) \longmapsto (w_1, \ldots, w_n) \in (\mathbf{C}^*)^n = \Omega_0, \\ &w_i = z^{\nu_i}, \quad 1 \leq i \leq n, \end{aligned}$$

where $\nu_1, \ldots, \nu_n$ are elements of $\mathbf{Z}^n$, we have $\Phi(T'(\Omega_1)) = T'(\Omega_0)$. This combined with the fact that $\Phi : \mathrm{Aut}(\Omega_1) \to \mathrm{Aut}(\Omega_0)$ is a group isomorphism yields that Φ maps the centralizer Z_1 of $T'(\Omega_1)$ in $\mathrm{Aut}(\Omega_1)$ onto the centralizer Z_0 of $T'(\Omega_0)$ in $\mathrm{Aut}(\Omega_0)$. Therefore, for the groups Z_0

and Z_1, their commutator groups $[Z_0, Z_0]$ and $[Z_1, Z_1]$ must be isomorphic. To derive a contradiction, we here assert that $[Z_0, Z_0]$ is an abelian group, while $[Z_1, Z_1]$ is not an abelian group. We verify this only in the case of $n = 2$, because the verification in the case of $n > 2$ is almost identical. First of all, we can show that Z_1 and Z_0 are the groups of all elements

$$g_1 \in \mathrm{Aut}(\Omega_1) = \mathrm{Aut}(\mathbf{C} \times \mathbf{C}^*) \quad \text{and} \quad g_0 \in \mathrm{Aut}(\Omega_0) = \mathrm{Aut}((\mathbf{C}^*)^2)$$

having the forms

$$(*) \qquad g_1(z) = (\alpha z_1 + \beta, f(z_1)z_2) \quad \text{and} \quad g_0(z) = (\alpha z_1, f(z_1)z_2)$$

respectively, where $\alpha \in \mathbf{C}^*$, $\beta \in \mathbf{C}$, and $f(z_1)$ is a nowhere vanishing holomorphic function that is defined on $\mathbf{C}$ for g_1 and on $\mathbf{C}^*$ for g_0. Take any two transformations $K_{\alpha,\beta,f}$ and $K_{\alpha',\beta',f'}$ of the form $(*)$ given by

$$K_{\alpha,\beta,f}(z) = (\alpha z_1 + \beta, f(z_1)z_2) \quad \text{and} \quad K_{\alpha',\beta',f'}(z) = (\alpha' z_1 + \beta', f'(z_1)z_2)$$

and write $[K_{\alpha,\beta,f}, K_{\alpha',\beta',f'}](z) = (K_1(z), K_2(z))$ in terms of the coordinates in $\mathbf{C}^2$, where $[\varphi, \psi] := \varphi^{-1} \circ \psi^{-1} \circ \varphi \circ \psi$ denotes the commutator of transformations φ and ψ. Then, by direct calculations we have

$$K_1(z) = (\alpha\alpha' z_1 + \alpha\beta' - \beta\alpha' + \beta - \beta')/\alpha\alpha',$$

$$K_2(z) = \frac{f(\alpha' z_1 + \beta')f'(z_1)z_2}{f((\alpha\alpha' z_1 + \alpha\beta' - \beta\alpha' + \beta - \beta')/\alpha\alpha')f'((\alpha\alpha' z_1 + \alpha\beta' + \beta - \beta')/\alpha')}.$$

In particular, considering the case of $(\beta, \beta') = (0, 0)$, we have

$$[K_{\alpha,0,f}, K_{\alpha',0,f'}](z) = (z_1, (f(\alpha' z_1)f'(z_1)z_2)/(f(z_1)f'(\alpha z_1))),$$

which implies that $[Z_0, Z_0]$ is abelian. On the other hand, consider three elements

$$P(z) = (\alpha z_1 + \beta, z_2), \; Q(z) = (z_1, z_2 \exp z_1), \; \text{and } R(z) = (\gamma z_1, z_2 \exp z_1)$$

in Z_1. Then, using the computation result above, we obtain

$$[P, Q](z) = (z_1, z_2 \exp\{(1 - \alpha)z_1 - \beta\}),$$

$$[P, R](z) = (\{\alpha\gamma z_1 + \beta(1 - \gamma)\}/\alpha\gamma, z_2 \exp\{(1 - \alpha)z_1 - (\beta/\gamma)\}),$$

and therefore $[[P, Q], [P, R]]$ is not the identity mapping whenever $\beta(\alpha - 1)(\gamma - 1) \neq 0$. This implies that $[Z_1, Z_1]$ is not abelian, and our assertion that the topological groups $\mathrm{Aut}(\Omega_1)$ and $\mathrm{Aut}(\Omega_0)$ are not isomorphic is shown.

Summarizing our results obtained so far, we have shown that *if M is a connected Stein manifold of dimension n and if the topological groups $\mathrm{Aut}(M)$ and $\mathrm{Aut}(\Omega_k)$ are isomorphic, then M is biholomorphically equivalent to some Ω_h with $h \geq k$.*

Finally, we shall complete the proof by showing the following:

3) *M is biholomorphically equivalent to Ω_k.* Suppose that $h \neq k$, and so $h > k$ by 2). For the connected Stein manifold Ω_k of dimension n, we know that the topological groups $\mathrm{Aut}(\Omega_k)$ and $\mathrm{Aut}(\Omega_h)$ are isomorphic. Then, by letting $M = \Omega_k$, an application of what we have shown just above yields that Ω_k is biholomorphically equivalent to Ω_p with $p \geq h$. Since $k < h \leq p$, this contradicts the fact that Ω_s and Ω_t are not homeomorphic when $s \neq t$. We thus conclude that $h = k$. □

Proof of the Corollary to the Main Theorem. If $k + \ell = k' + \ell'$, the topological groups $\mathrm{Aut}(X_{k,\ell})$ and $\mathrm{Aut}(X_{k',\ell'})$ are isomorphic precisely when $(k,\ell) = (k',\ell')$ by our Main Theorem.

Now, suppose that $k + \ell \neq k' + \ell'$, say, $k + \ell < k' + \ell'$, and write $n = k + \ell$, $n' = k' + \ell'$. If there exists a topological group isomorphism $\Phi : \mathrm{Aut}(X_{k',\ell'}) \to \mathrm{Aut}(X_{k,\ell})$, then we have an injective continuous group homomorphism $\Phi \circ \rho_{X_{k',\ell'}} : T^{n'} \to \mathrm{Aut}(X_{k,\ell})$. Since $X_{k,\ell}$ is a connected Stein manifold of dimension $n < n'$, this contradicts the fact in Lemma 4. Therefore the topological groups $\mathrm{Aut}(X_{k,\ell})$ and $\mathrm{Aut}(X_{k',\ell'})$ are not isomorphic. □

Proof of Theorem A. Choose a maximal torus T^n in $U(n)$.

Then, by Fundamental Theorem 2 there exists a biholomorphic mapping $F : M \to D$ of M onto a Reinhardt domain D in $\mathbf{C}^n$ such that $F\rho(T^n)F^{-1} = T(D)$. Set $G = F\rho(U(n))F^{-1}$ and take a bounded domain U in $\mathbf{C}^n$ contained in D. Then, $D_0 := \{g(z) \in D \mid g \in G,\ z \in U\}$ is a bounded Reinhardt domain in $\mathbf{C}^n$ contained in D and G can be regarded as a connected compact subgroup of the Lie group $\mathrm{Aut}(D_0)$ containing $T(D_0)$. Since G is isomorphic to $U(n)$ and $n \geq 2$, we can apply Fundamental Theorem 1 and its corollary to D_0 and $G \subset \mathrm{Aut}(D_0)$. Therefore there exists a transformation

$$\varphi : \mathbf{C}^n \ni (z_1, \ldots, z_n) \longmapsto (w_1, \ldots, w_n) \in \mathbf{C}^n$$
$$w_i = r_i z_{\sigma(i)}, \quad 1 \leq i \leq n,$$

such that, for $\tilde{D}_0 = \varphi(D_0)$ and $\tilde{G} = \varphi G \varphi^{-1} \subset \mathrm{Aut}(\tilde{D}_0)$, we have $\tilde{G} = U(n)$. Put $\tilde{D} = \varphi(D)$. Then, since $\tilde{D}_0$ is a non-empty subdomain of $\tilde{D}$, we see by the uniqueness theorem on holomorphic functions that $U(n) = \tilde{G} \subset \mathrm{Aut}(\tilde{D})$, or $g(\tilde{D}) = \tilde{D}$ for all $g \in U(n)$. Being a Stein manifold, $\tilde{D}$ is now to be of the form

$$\tilde{D} = \left\{ (z_1, \ldots, z_n) \in \mathbf{C}^n \;\middle|\; \sum_{i=1}^{n} |z_i|^2 < r \right\},$$

where $0 < r \leq +\infty$. This shows that $\tilde{D}$, and hence M is biholomorphically equivalent to either B^n or $\mathbf{C}^n$. □

Proof of Theorem B. By Theorem A, we may assume that $M = B^n$ or $M = \mathbf{C}^n$. When $M = B^n$, our assertion is a consequence of the conjugacy of maximal compact subgroups of the Lie group $\mathrm{Aut}(B^n)$. Therefore, in what follows, we consider the case where $M = \mathbf{C}^n$.

To prove our assertion, it suffices to prove that, for any injective continuous group homomorphism ρ of $U(n)$ into $\mathrm{Aut}(M)$, we have an element ψ of $\mathrm{Aut}(M)$ such that $\psi\rho(U(n))\psi^{-1} = U(n)$. Suppose that such ρ is given. Choose a maximal torus T^n in $U(n)$. Then, by Fundamental Theorem 2 and Lemma 3, there exists a biholomorphic mapping $F : M = \mathbf{C}^n \to D = \mathbf{C}^n$ such that $F\rho(T^n)F^{-1} = T(D)$. Set $G = F\rho(U(n))F^{-1}$. As in the proof of Theorem A, there exists a transformation

$$\begin{aligned} \varphi : \mathbf{C}^n \ni (z_1, \ldots, z_n) \quad &\longmapsto \quad (w_1, \ldots, w_n) \in \mathbf{C}^n \\ w_i = r_i z_{\sigma(i)}, &\quad 1 \leq i \leq n, \end{aligned}$$

such that, for $\tilde{D} = \varphi(D) = \mathbf{C}^n$ and $\tilde{G} = \varphi G \varphi^{-1} \subset \mathrm{Aut}(\tilde{D})$, we have $\tilde{G} = U(n)$. Therefore, putting $\psi = \varphi \circ F \in \mathrm{Aut}(M)$, we have

$$\psi\rho(U(n))\psi^{-1} = \varphi(F\rho(U(n))F^{-1})\varphi^{-1} = \varphi G \varphi^{-1} = \tilde{G} = U(n),$$

as desired.

Finally, notice that every continuous, and hence analytic, group automorphism of $U(n)$ is an inner automorphism up to the complex conjugation in $U(n)$. Indeed, this follows from the following fact: Both the groups $\mathrm{Aut}(U(1))$ and $\mathrm{Aut}(SU(n))/\mathrm{Int}(SU(n))$ $(n \geq 3)$ are the cyclic groups of order 2 generated by the complex conjugation $u \mapsto \bar{u}$ and $\mathrm{Aut}(SU(2)) = \mathrm{Int}(SU(2))$, where $\mathrm{Aut}(L)$ (resp. $\mathrm{Int}(L)$) denotes the group of all analytic automorphisms (resp. inner automorphisms) of a given Lie group L (cf. [6]). Then one can find an element $u_o \in U(n)$ such that

$$\psi\rho_1(u)\psi^{-1} = \rho_2(u_o u u_o^{-1}) \quad \text{or} \quad \psi\rho_1(u)\psi^{-1} = \rho_2(u_o \bar{u} u_o^{-1})$$

for all $u \in U(n)$. Thus, the element $\Psi := \rho_2(u_o^{-1})\psi \in \mathrm{Aut}(M)$ is a required one in Theorem B. □

5. A remark

As mentioned in the introduction, Isaev [7] and Krantz [12] obtained the following theorem, which is a special case of $k = n$ in our Main Theorem:

THEOREM I-K. *Let M be a connected Stein manifold of dimension n. Assume that* $\mathrm{Aut}(M)$ *is isomorphic to* $\mathrm{Aut}(\mathbf{C}^n)$ *as topological groups. Then M is biholomorphically equivalent to $\mathbf{C}^n$.*

Let us recall the key point of their proof of this theorem. Firstly, by using Fundamental Theorem 2 and Lemma 2, they also prove that M must be biholomorphically equivalent to $\mathbf{C}^h \times (\mathbf{C}^*)^{n-h}$ for some integer h with $0 \leq h \leq n$, as we did in the step 1) of the proof of the Main Theorem. Secondly, they verify that

(1) the topological group $\mathrm{Aut}(\mathbf{C}^n)$ is connected; while

(2) the topological group $\mathrm{Aut}(\mathbf{C}^h \times (\mathbf{C}^*)^{n-h})$ is disconnected, provided that $h \neq n$.

Consequently, since $\mathrm{Aut}(M)$ is now assumed to be isomorphic to $\mathrm{Aut}(\mathbf{C}^n)$ as topological groups, they conclude that M is in fact biholomorphically equivalent to $\mathbf{C}^n$, completing the proof of Theorem I-K.

Here it should be remarked the following: In the case where $0 \leq h, k \leq n-1$ and $h \neq k$, the assertion (2) above does not guarantee that $\mathrm{Aut}(\mathbf{C}^h \times (\mathbf{C}^*)^{n-h}))$ is not isomorphic to $\mathrm{Aut}(\mathbf{C}^k \times (\mathbf{C}^*)^{n-k}))$ as topological groups. So, it seems to be difficult to prove our Main Theorem with the same arguments as those in the proof of Theorem I-K.

In connection with this, we would like to ask the following two questions: For a given integer h with $0 \leq h \leq n$, we denote by C_h the cardinality of the set consisting of all connected components of the topological group $\mathrm{Aut}(\mathbf{C}^h \times (\mathbf{C}^*)^{n-h})$. For instance, we have $C_n = 1$ by the assertion (1) above.

QUESTION 1. *Is it possible to determine the cardinality C_h by means of the integer h?*

QUESTION 2. *Is it true that $C_h = C_k$ if and only if $\mathbf{C}^h \times (\mathbf{C}^*)^{n-h}$ is biholomorphically equivalent to $\mathbf{C}^k \times (\mathbf{C}^*)^{n-k}$?*

Of course, for general domains in $\mathbf{C}^n$, the answer to Question 2 is negative. In fact, there exists a family $\{D_t\}_{t \in \mathbf{R}}$ of bounded strictly pseudoconvex domains in $\mathbf{C}^n$ with smooth boundaries such that the automorphism group $\mathrm{Aut}(D_t)$ is the identity only for every $t \in \mathbf{R}$ and

D_s is not biholomorphically equivalent to D_t if $s \neq t$ (cf. [4], [5]). Also, for generalized complex ellipsoids

$$E(k,\alpha) = \left\{ z \in \mathbf{C}^n \,\middle|\, \sum_{i=1}^{k} |z_i|^2 + \left(\sum_{j=k+1}^{n} |z_j|^2 \right)^{\alpha} < 1 \right\}$$

in $\mathbf{C}^n$, where $k \in \mathbf{Z}$ with $1 \leq k \leq n$ and $0 < \alpha \in \mathbf{R}$, we know that the Lie group $\mathrm{Aut}(E(k,\alpha))$ is connected for every (k,α) and $E(k,\alpha)$ is not biholomorphically equivalent to $E(\ell,\beta)$ if $(k,\alpha) \neq (\ell,\beta)$ (cf. [10], [14]).

Anyway, it would be interesting to investigate these questions; however, these seem to be very difficult at this moment.

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