APPROXIMATION OF NON-HOLOMORPHIC MAPS, AND POLETSKY THEORY OF DISCS

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ABSTRACT. We prove an approximation result, and we get a new proof of the main result in [7]. I hope that this new proof may be a step towards a generalization of the Poletsky theory of discs to the case of almost complex manifolds.

1. A general question (to be made more precise)

The general question, vaguely stated, is: *Is every map from the unit disc into a complex manifold, with small $\bar{\partial}$, close to a genuine holomorphic map?*

In the study of Poletsky discs, and having possible generalizations in mind (see [8] and Section 7 below), the following question arises naturally.

Let $\mathcal{M}$ be a complex manifold, equipped with some metric, and let Ω be a relatively compact region in $\mathcal{M}$. For every $\epsilon > 0$, does there exist $\delta > 0$ such that if u is a map from the unit disc Δ (in $\mathbb{C}$) into Ω, with $|\bar{\partial}u| \leq \delta$, then there exists a homomorphic map $h : \Delta \to \Omega$ such that (abusing notations) $|h - u| < \epsilon$?

For the application, it is absolutely essential that no bound be given on ∂u, and that no shrinking of the disc be allowed.

If we added the hypothesis $|\partial u| \leq M$, then the existence (but not an estimate!) of δ, depending on Ω, ϵ and M, would result immediately from a normal family argument, arguing by contradiction.

The answer to the above question is negative in general, as was shown to me by L. Lempert. His example is natural and embarrassingly simple. Take $\mathcal{M}$ any compact Riemann surface of genus ≥ 2, and $\Omega = \mathcal{M}$. See

Section 8. However the answer is positive for $\mathcal{M} = \mathbb{P}_n(\mathbb{C})$, as was shown to me by J. E. Fornaess ($n = 1$) and D. Chakrabarti.

At any rate, the general question has to be re-formulated. The result presented in Section 2 is hopefully only a step towards a better result. It is good enough for the first intended application, but one should in particular try to remove the non natural hypothesis of empty triple intersections made below.

2. A partial result (a restricted version of the question)

In Lempert's example the difficulty for solving the approximation problem arises even locally (at the boundary of the disc). In the intended application to the theory of discs there is no such local obstruction, because locally the maps to be approximated by holomorphic ones take value in a given coordinate patch.

This leads us to consider the following approximation problem in which the local difficulty disappears and for which we are simply left with a gluing problem.

NOTATIONS. As before Δ denotes the open unit disc in $\mathbb{C}$ and $\overline{\Delta}$ will be the closed unit disc. We fix a complex manifold $\mathcal{M}$ of complex dimension n, $\Omega_1, \ldots, \Omega_R$ some open sets in $\mathcal{M}$ with $K_1, \ldots, K_R$ compact subsets of $\Omega_1, \ldots, \Omega_R$ respectively. We assume that each Ω_j is biholomorphic to some open set in $\mathbb{C}^n$. Finally we fix $U_1, \ldots, U_R$ open sets in $\mathbb{C}$ such that $\overline{\Delta} \subset \bigcup_{j=1}^{R} U_j$.

We assume that *all triple intersections are empty* i.e. $U_j \cap U_k \cap U_\ell = \phi$ if j, k, ℓ are all distinct.

DEFINITION. A map φ from Δ into $\mathcal{M}$ will be called *restricted* if for every $j \in \{1, \ldots, R\}$, $\varphi(U_j \cap \Delta) \subset K_j$.

Of course that notion of "restricted" is relative to the above choices, and in order that there exists any restricted map one must have $K_j \cap K_k \neq \phi$ if $(U_j \cap U_k) \cap \Delta \neq \phi$, that we will therefore assume.

We equip $\mathcal{M}$ with some Riemannian metric. It allows to give meaning to $|\overline{\partial}\varphi| \leq \delta$ if φ is a map from Δ into $\mathcal{M}$. And if f and g are maps from Δ into $\mathcal{M}$ we set

$$d(f,g) = \sup_{\zeta \in \Delta} \text{dist}(f(\zeta), g(\zeta)).$$

The choice of the metric is inconsequential.

PROPOSITION 1. *With the above notations: For every $\epsilon > 0$, there exists $\delta > 0$ such that if u is a restricted map from Δ into $\mathcal{M}$ satisfying $|\bar{\partial}u| \leq \delta$, there exists a holomorphic map h from Δ into $\mathcal{M}$ such that $d(u, h) \leq \epsilon$, and $u(0) = h(0)$.*

3. The Cartan "Lemma" with bounds

A crucial ingredient will be the following "Lemma" proved in [2].

LEMMA [2]. *Let $(V_j)_{j=1}^N$ be a covering of the closed unit disc by open subsets of $\mathbb{C}$. For each $(j, k) \in \{1, \dots, N\}^2$ let g_{jk} be a holomorphic $(n \times n)$ matrix bounded and with bounded inverse defined on $(V_j \cap V_k) \cap \Delta$, with the usual cocycle conditions: $g_{jj} = \mathbb{1}$, $g_{jk} = g_{kj}^{-1}$, $g_{jk}g_{k\ell}g_{\ell j} = \mathbb{1}$. Then there exist bounded holomorphic matrices g_j with bounded inverses defined on $V_j \cap \Delta$ $(j = 1, \dots, N)$ such that $g_{jk} = g_k^{-1}g_j$ on $(V_j \cap V_k) \cap \Delta$, with bounds for the g_j's and their inverses depending only on the covering and on the sup norm of the g_{jk}'s and of their inverses.*

Having in mind possible generalizations (to the theory of discs) it may be good to point out that an older and simpler result suffices for establishing Proposition 2 below. Roughly speaking it consists in dealing with a special case of Proposition 1 where $U_j \cap U_k = \phi$ if both j and $k \neq 1$, and with the major requirement that the restriction of the map u (not h) to $U_1 \cap \Delta$ is a fixed holomorphic map from $U_1 \cap \Delta$ into $\Omega_1 \subset \mathcal{M}$, that extends continuously to $U_1 \cap \overline{\Delta}$.

Then, following the proofs below, one has to find holomorphic matrices g_j for a fixed cocycle of holomorphic matrices g_{jk} continuous up to the boundary. This is essentially given by Proposition 2, page 48 in [1] (but with no bounds on the g_j's just in terms of bounds on the g_{jk}'s). It is enough to adapt slightly the proof in [1], after adding a mild requirement on the U_j's in order to allow the approximation of invertible holomorphic matrices given on $\bigcup_{j \geq 2}(U_j \cap U_1) \cap \Delta$ by invertible holomorphic matrices defined on $\mathbb{C}$. Some other features of the proofs also simplify.

4. Proof of Proposition 1 (reduction to a nonlinear Cousin problem)

From now on F_j will denote a fixed biholomorphism from Ω_j onto an open set $F_j(\Omega_j) \subset \mathbb{C}^n$ $(j = 1, \ldots, R)$. Shrinking the Ω_j's if needed with can assume that the derivatives F'_j and $(F_j^{-1})'$ have operator norms bounded by some constant A.

A restricted map u from Δ into $\mathcal{M}$ corresponds to R maps $u_1, \ldots, u_R$ respectively from $U_j \cap \Delta$ into $F_j(K_j) \subset \mathbb{C}^n$ such that on $(U_j \cap U_k) \cap \Delta$, $u_k = F_k \circ F_j^{-1} \circ u_j$.

The map $F_k \circ F_j^{-1}$ is defined on a neighborhood of $F_j(K_j \cap K_k)$ (non empty if $(U_j \cap U_k) \cap \Delta \neq \phi$).

Given a restricted map u from Δ into $\mathcal{M}$, with sufficiently small $\overline{\partial}$ we want to approximate it by a holomorphic map h from Δ into $\mathcal{M}$. It amounts to finding R holomorphic maps $h_1, \ldots, h_R$ respectively from $U_j \cap \Delta$ into $\mathbb{C}^n$ $(j = 1, \ldots, R)$

(a) satisfying $h_k = F_k \circ F_j^{-1} \circ h_j$ on $(U_j \cap U_k) \cap \Delta$,

(b) and such that $h_j - u_j$ is small, as desired.

We first drop requirement (a).

If $u_j : U_j \cap \Delta \to \mathbb{C}^n$ is such that $|\overline{\partial} u_j| \leq \delta$, there exists $w_j^o : U_j \cap \Delta \to \mathbb{C}^n$ such that $u_j + w_j^o$ is holomorphic and $|w_j^o| \leq K\delta$ (for some appropriate constant K). But we do *not* have

$$u_k + w_k^o = F_j \circ F_j^{-1} \circ (u_j + w_j^o).$$

We only have

$$|(u_k + w_k^o) - F_k \circ F_j^{-1}(u_j + w_j^o)| \leq (1 + A^2)K\delta = K_1\delta.$$

We wish to perturb slightly the holomorphic maps $u_j + w_j^o$ in order to get holomorphic maps $u_j + w_j^o + v_j$ defined on $U_j \cap \Delta$ such that on $(U_j \cap U_k) \cap \Delta$ $(u_k + w_k^o + v_k^o) = F_k \circ F_j^{-1} \circ (u_j + w_j^o + v_j)$.

Changing notations, replacing $u_j + w_j^o$ by u_j it is clear that Proposition 1 follows from the following Proposition 1' which is entirely about maps in $\mathbb{C}^n$ (the manifold $\mathcal{M}$ not longer appears).

We shall start the next section by saying which notations we keep and by introducing some other notations in order to make it possible to read Proposition 1' independently of what precedes.

5. Proposition 1′ (nonlinear Cousin problem)

As before $U_1, \ldots, U_R$ are open sets in $\mathbb{C}$ that cover $\overline{\Delta}$ and with empty triple intersections.

For $1 \leq j < k \leq R$ we shall introduce subsets of $\mathbb{C}^n$ ω'_{jk} and ω_{jk} which correspond respectively in the previous setting to $F_j(\Omega_j \cap \Omega_k)$ and to the image under F_j of the intersection of given neighborhoods of K_j and K_k. So for $1 \leq j < k \leq R$, let ω_{jk} and ω'_{jk} be open sets in $\mathbb{C}^n$ with $\omega_{jk} \subset \omega'_{jk}$.

Let F_{jk} (corresponding to $F_k \circ F_j^{-1}$ earlier) be a holomorphic immersion from ω'_{jk} into $\mathbb{C}^n$. Note that we define F_{jk} and ω_{jk} only for $j < k$.

PROPOSITION 1′. *With the above notations: For every $\epsilon > 0$ there exists $\delta > 0$ such that if for every $j \in \{1, \ldots, R\}$, u_j is a holomorphic map from $U_j \cap \Delta$ into $\mathbb{C}^n$ such that for $1 \leq j < k \leq R$, u_j $u_j[(U_j \cap U_k) \cap \Delta] \subset \omega_{jk}$ and $|u_k - F_{jk} \circ u_j| \leq \delta$, then there exist holomorphic maps $v_1, \ldots, v_R$ respectively from U_j into $\mathbb{C}^n$ such that*

$$\begin{cases} |v_j| \leq \epsilon \\ u_k + v_k = F_{jk}(u_j + v_j) \end{cases} \qquad on \ (U_j \cap U_k) \cap \Delta.$$

Moreover if $p_1 \in U_1 \cap \Delta$, we can impose $v_1(p_1) = 0$.

For F_{jk} being the identity map this would be the standard additive Cousin Problem.

6. Proof of Proposition 1′

In order to solve $u_k + v_k = F_{jk}(u_j + v_j)$, one uses linearization. For $z \in (U_j \cap U_k) \cap \Delta$ and $t \in \mathbb{C}^n$

$$F_{jk}(u_j(z) + t) = F_{jk}(u_j(z)) + [F'_{jk}(u_j(z))]t + \mathcal{O}(|t|^2).$$

The linearized problem consists in solving, for $j < k$:

$$(*) \qquad [F'_{jk}(u_j(z))]v_j(z) - v_k(z) = -F_{jk}(u_j(z)) + u_k(z),$$

where the v_j's have to be found, and the right hand side is given. Naturally $F'_{jk}(u_j(z))$ denotes the derivative (linear tangent map) of F_{jk} at the point $u_j(z)$.

The Cartan Lemma with bounds allows us to write $F'_{jk}(u_j(z)) = g_k^{-1}(z)g_j(z)$ $(j < k)$, where each g_j is a holomorphic matrix defined on U_j whose norm and whose norm of the inverse are bounded by a constant

depending only on the U_j's, and on the supremum of the norms of the F'_{jk}'s and of their inverses on ω_{jk}. So, and this is crucial, this constant does not depend on the u_j's (as long as $u_j[(U_j \cap U_k) \cap \Delta] \subset \omega_{jk}$). (*) can then be rewritten:

$$(**) \qquad g_j(z)v_j(z) - g_k(z)v_k(z) = g_k(z)[-F_{jk}(u_j(z)) + u_k(z)].$$

By the standard additive Cousin problem (with bound, adapting Theorem 1.4.5 in [6] in a very simple setting) there exists a continuous linear operator T which to each family $\alpha = (\alpha_{jk})_{1 \leq j < k \leq R}$ of n-tuples of bounded holomorphic functions $\alpha_{jk} \in [H^\infty((U_j \cap U_k) \cap \Delta)]^n$ associates a collection $T(\alpha) = T_1(\alpha), \dots, T_R(\alpha)$ of n-tuples of bounded holomorphic functions $T_j(\alpha) \in [H^\infty(U_j \cap \Delta)]^n$ such that, if $j < k$, on $(U_j \cap U_k) \cap \Delta$: $\alpha_{jk} = T_j(\alpha) - T_k(\alpha)$. For α as above let $\beta = (\beta_{jk})_{j<k}$ be defined by $\beta_{jk} = g_k \alpha_{jk}$ and set

$$S_j(\alpha) = g_j^{-1} T_j(\beta).$$

Then, for $j < k$, $g_j S_j(\alpha) - g_k S_k(\alpha) = g_k \alpha_{jk}$, therefore

$$(***) \qquad [F'_{jk}(u_j(z))] \, S_j(\alpha) - S_k(\alpha) = \alpha_{jk}.$$

For $u_1, \dots, u_R$ given let Φ be the map from $\mathcal{H} = \bigoplus_{j<k} \left(H^\infty[(U_j \cap U_k) \cap \Delta]\right)^n$ into itself defined by

$$(\Phi(\alpha))_{jk} = \alpha_{jk} - [F_{jk}(u_j + S_j(\alpha)) - u_k - S_k(\alpha)].$$

We claim that given $\epsilon > 0$, if all the $|F_{jk}(u_j) - u_k|$ are small enough (less than some δ) Φ has a fixed point α with $|S(\alpha)| \leq \epsilon$. The corresponding family

$$(v_1, \dots, v_R) = \big(S_1(\alpha), \dots, S_R(\alpha)\big)$$

solves therefore $F_{jk}(u_j + v_j) = u_k + v_k$, with bounds as desired.

To end the proof, we have to prove the claim and it is done by a standard fixed point argument (for contractions). On each $\left(H^\infty[(U_j \cap U_k) \cap \Delta]\right)^n$ ($j < k$) we consider the norm

$$\|f\|_{jk} = \max_m \sup |f_m|, \quad f = (f_1, \dots, f_n).$$

On $\mathcal{H} = \bigoplus_{j<k} \left(H^\infty[(U_j \cap U_k) \cap \Delta]\right)^n$, we can consider the norm defined by: if $\alpha = (\alpha_{jk})$, $\|\alpha\| = \max \|\alpha_{jk}\|_{jk}$. The closed ball of radius ρ in $\mathcal{H}$ will be denoted by B_ρ.

(***) says that $\Phi'(0) = 0$.

On a small ball around 0 of radius independent of the u_j's (as long as $u_j[(U_j \cap U_k) \cap \Delta] \subset \omega_{jk}$), the $\mathcal{C}^2$ norm of Φ is bounded independently of

the u_j's (no differentiation of the u_j's is involved). So if $\rho > 0$ is small enough, not depending on the u_j's $\|\Phi(\alpha) - \Phi(\beta)\| < \frac{\|\alpha - \beta\|}{2}$ for every α and $\beta \in B_\rho$. Choose furthermore $\rho \leq \frac{\epsilon}{\|S\|}$ ($\|S\|$ the operator norm of S), we have $|S(\alpha)| \leq \epsilon$ for every $\alpha \in B_\rho$.

Having thus chosen ρ (not depending on $u_1, \ldots, u_r$) we finally restrict the choice of the u_j's in order to have $\Phi(B_\rho) \subset B_\rho$. Since $\Phi(0) = (-F_{jk}u_j + u_k)_{jk}$ and $\|\Phi(\alpha) - \Phi(0)\| \leq \frac{\|\alpha\|}{2} \leq \frac{\rho}{2}$, it is enough to require that for $j < k$

$$\|F_{jk}u_j - u_k\| < \frac{\rho}{2}.$$

This concludes the proof of Proposition $1'$.

7. Application to the Poletsky theory of discs

By disc (resp. holomorphic disc) in a complex manifold $\mathcal{M}$, we mean a smooth (resp. holomorphic) map from a neighborhood of the closed unit disc $\overline{\Delta}$ in $\mathbb{C}$, into $\mathcal{M}$. Abusively, for such a disc φ, we will write $\varphi : \overline{\Delta} \to \mathcal{M}$.

THEOREM. *Let f be an upper semi-continuous function on a complex manifold $\mathcal{M}$. For $p \in \mathcal{M}$ let*

$$\widehat{f}(p) = \operatorname{Inf} \int_0^{2\pi} f \circ \varphi(e^{i\theta}) \frac{d\theta}{2\pi},$$

with the infimum taken over the family of all holomorphic discs φ with $\varphi(0) = p$.

Then the function $\widehat{f}$ is plurisubharmonic.

This theorem, proved in [8], is a generalization of the ground-breaking work of Poletsky [7] (the case $\mathcal{M} = \mathbb{C}^n$), and of previous work of Lárusson and Sigurdsson [3]. See also [4] and [5].

We wish to give a new proof of the theorem. Although the proof still uses a fair amount of complex analysis (especially the existence of Stein neighborhoods for embedded discs), it avoids the ingredient which was new in [8], and which was the unpleasant construction of some Stein neighborhoods of some kind of incomplete Hartogs figure. It is our hope that part of this new proof may constitute a step towards a generalization to the case of almost complex manifolds. Although (assuming an approximation result such as Proposition 1) it has been sketched in [8], it seems more convenient to write here a complete proof, with the only

exception that we will not repeat the proof that the function f is upper semi-continuous ([3], or [8] Lemma 1).

By using approximation from above by continuous functions, it is enough to prove the theorem for (real valued) functions f that are continuous, and bounded from below.

Take $p \in \mathcal{M}$ and φ a holomorphic disc with center at p ($\varphi(0) = p$). We have to show that

$$\widehat{f}(\varphi(0)) \leq \int_0^{2\pi} \widehat{f}(\varphi(e^{i\theta})) \frac{d\theta}{2\pi}.$$

We can restrict our attention to "small" embedded discs, lying entirely in a coordinate patch.

Since the function $\widehat{f}$ is upper semi-continuous, using approximation from above (as already done in [7] (step 1)), it is enough to show that for any continuous function v, on $\mathbb{R}/2\pi\mathbb{Z}$, satisfying $v(\theta) > \widehat{f} \circ \varphi(e^{i\theta})$

$$\widehat{f}(\varphi(0)) \leq \int_0^{2\pi} v(\theta) \frac{d\theta}{2\pi}.$$

By definition of $\widehat{f}$ at the point $\varphi(e^{i\theta})$, and since $v > f \circ \varphi$, for each $\theta \in [0, 2\pi]$, there exists a holomorphic disc $\psi_0 : \overline{\Delta} \to \mathcal{M}$, such that $\psi_\theta(0) = \varphi(e^{i\theta})$ and

$$v(\theta) > \int_0^{2\pi} f(\psi_\theta(e^{i\nu})) \frac{d\nu}{2\pi}.$$

In order to deal only with embedded discs, we add one dimension. We denote by Π the canonical projection from $\mathcal{M} \times \mathbb{C}$ onto $\mathcal{M}$. We set $\widetilde{f} = f \circ \Pi$, and let $\widetilde{\varphi}$ and $\widetilde{\psi}_\theta$ be the maps from a neighborhood of $\overline{\Delta}$ into $\mathcal{M} \times \mathbb{C}$, defined by:

$$\widetilde{\varphi}(\zeta) = (\varphi(\zeta), 0),$$

$$\widetilde{\psi}_\theta(z) = (\psi_\theta(z), z).$$

Note that $\widetilde{\varphi}(e^{i\theta}) = \widetilde{\psi}_\theta(0)$. Since $\widetilde{\psi}_\theta$ is an embedding, there is a neighborhood Ω_θ of $\widetilde{\psi}_\theta(\overline{\Delta})$ in $\mathcal{M} \times \mathbb{C}$, and a biholomorphism χ_θ from that neighborhood onto a bounded open set $\chi_\theta(\Omega_\theta) \subset \mathbb{C}^{n+1}$ ($n = \dim \mathcal{M}$). See [9], [3], or [8] Lemma 0.

For $\theta' \cong \theta$, instead of using the disc $\widetilde{\psi}_{\theta'}$, we can use the disc $\widetilde{\psi}_{\theta'}^\theta$ obtained from $\widetilde{\psi}_\theta$ by appropriate translation in the coordinates provided by χ_θ. Precisely, set

$$\widetilde{\psi}_{\theta'}^\theta(z) = \chi_\theta^{-1}\big(\chi_\theta \circ \widetilde{\psi}_\theta(z) + \chi_\theta \circ \widetilde{\varphi}(e^{i\theta'}) - \chi_\theta \circ \widetilde{\varphi}(e^{i\theta})\big).$$

For $\theta' \cong \theta$ one still has

$$v(\theta) > \int_0^{2\pi} \widetilde{f}(\widetilde{\psi}_{\theta'}^{\theta}(e^{i\nu}))\frac{d\nu}{2\pi}.$$

For fixed θ, $(\theta', z) \mapsto \widetilde{\psi}_{\theta'}^{\theta}(z)$ is holomorphic in z ($|z| < a$, with $|a| > 1$) and smooth in θ' ($\theta' \cong \theta$).

Fix $\tau > 0$. By compactness one can find disjoint intervals $J_j \subset \mathbb{R}/2\pi\mathbb{Z}$ ($j = 1, \ldots, k$), and $\theta_j \in J_j$ such that:

$$\int_{[0,2\pi]-\cup J_j} (|v(\theta)| + |f \circ \varphi(e^{i\theta})\frac{d\theta}{2\pi}|) < \tau,$$

$\widetilde{\varphi}(J_j) \subset \Omega_{\theta_j}$, and for every $\theta' \in J_j$, $\widetilde{\psi}_{\theta'}^{\theta_j}(\bar{\Delta}) \subset \Omega_{\theta_j}$, and

$$v(\theta') > \int_0^{2\pi} \widetilde{f}(\widetilde{\psi}_{\theta'}^{\theta_j}(e^{i\nu}))\frac{d\nu}{2\pi}.$$

For $d > 0$ we set

$$U_2^j = \{z \in \mathbb{C}; \operatorname{dist}(z, J_j) < 2d\},$$
$$U_1 = \{z \in \mathbb{C}; \operatorname{dist}(z, \cup J_j) > d\}.$$

We take d small enough in order that the sets U_2^j are disjoint ($j = 1, \ldots, k$), and such that $\widetilde{\varphi}(U_2^j \cap \Delta) \subset \Omega_{\theta_j}$ (a coordinate patch).

Every smooth function, or $\mathbb{C}^{n+1}$ valued map, that is defined on some arc of the unit circle can be extended to a function with small support containing this arc, and whose $\bar{\partial}$ vanishes to infinite order along the unit circle. It can be done with holomorphic dependence on a parameter, and with bounds.

We apply it to the arcs J_j and the functions

$$\zeta = e^{i\theta} \mapsto \chi_{\theta_j}\widetilde{\psi}_{\theta}^{\theta_j}(z),$$

where z is treated as a holomorphic parameter. Then, we pull back by $\chi_{\theta_j}^{-1}$. Although here the notations are rather unpleasant, the goal reached is clear. Taking advantage of the fact that locally (in θ) one works in coordinate charts. The families of holomorphic discs given on the J_j's (identified with arcs on the unit circle), can be extended to a family of holomorphic discs $z \mapsto \Phi(\zeta, z)$ parameterized by ζ varying in a neighborhood of $\bar{\Delta}$, with the following properties:

(i) The holomorphic discs $z \mapsto \Phi(\zeta, z)$ say in a fixed compact set in $\mathcal{M} \times \mathbb{C}$,

(ii) $\Phi(\zeta, 0) \equiv \widetilde{\varphi}(\zeta)$,

(iii) off an arbitrarily small neighborhood of the J_j's, in particular for $\zeta \in U_1$, the discs are simply constant discs $\Phi(\zeta, z) = \widetilde{\varphi}(\zeta)$,

(iv) for $\theta \in J_j$, $\Phi(e^{i\theta}, z) = \widetilde{\psi}_\theta^{\theta_j}(z)$,

(v) $\frac{\partial \Phi}{\partial \overline{\zeta}} = 0$, for $|\zeta| = 1$,

(vi) $\Phi(U_2^j \times \overline{\Delta}) \subset \Omega_{\theta_j}$,

(vii) $\int_0^{2\pi} v(\theta) \frac{d\theta}{2\pi} \geq \int \int \widetilde{f} \circ \Phi(e^{i\theta}, e^{i\nu}) \frac{d\theta}{2\pi} \frac{d\nu}{2\pi} - \tau$.

The following step is a key trick due to Poletsky (similar arguments were used by J. Globevnik and B. Stensones). Note that for every $N \in \mathbb{N}$,

$$\int \int \widetilde{f} \circ \Phi(e^{i\theta}, e^{i\nu}) \frac{d\theta}{2\pi} \frac{d\nu}{2\pi} = \int \int \widetilde{f} \circ \Phi(e^{i\theta}, e^{i(\nu + N\theta)}) \frac{d\theta}{2\pi} \frac{d\nu}{2\pi}.$$

There exists γ (depending on N) such that

$$\int \widetilde{f} \circ \Phi(e^{i\theta}, e^{i(\gamma + N\theta)}) \frac{d\theta}{2\pi} \leq \int \int \widetilde{f} \circ \Phi(e^{i\theta}, e^{i\nu}) \frac{d\theta}{2\pi} \frac{d\nu}{2\pi}.$$

This leads us to the consideration of the non holomorphic discs $\widetilde{\varphi}_N$ in $\mathcal{M} \times \mathbb{C}$ defined by

$$\widetilde{\varphi}_N(\zeta) = \Phi(\zeta, e^{i\gamma} \zeta^N).$$

Note that $\overline{\partial} \widetilde{\varphi}_N$ tends to 0 as N tends to ∞, due to the holomorphy in z, and to conditions (ii) (and the holomorphicity of $\widetilde{\varphi}$) and (v). (Note also that $\partial \varphi_N$ blows up!)

All triple intersections of U_1 and of the U_2^j's are empty. So Proposition 1 allows us to approximate $\widetilde{\varphi}_N$ by a genuine holomorphic disc h_N in $\mathcal{M} \times \mathbb{C}$, with $h_N(0) = \widetilde{\varphi}(0) = (\varphi(0), 0)$.

We assumed that the given function f is continuous. For N large enough we will have

$$\int \widetilde{f} \circ h_N(e^{i\theta}) \frac{d\theta}{2\pi} \leq \int \widetilde{f} \circ \widetilde{\varphi}_N(e^{i\theta}) \frac{d\theta}{2\pi} + \tau.$$

Since, by the very definition of $\widehat{f}$

$$\widehat{f}(\varphi(0)) \leq \int f \circ (\Pi \circ h_N)(e^{i\theta}) \frac{d\theta}{2\pi} = \int \widetilde{f} \circ h_N \frac{d\theta}{2\pi},$$

it follows from the above inequalities that

$$\widehat{f}(\varphi(0)) \leq \int v(\theta) \frac{d\theta}{2\pi} + 2\tau.$$

Since $\tau > 0$ is arbitrary, this establishes the Theorem.

8. Lempert's example

The answer to the general question raised in 1 is negative.

PROPOSITION 2. *Let $\mathcal{M}$ be a compact Riemann surface of genus ≥ 2, endowed with some metric. There exist $\epsilon > 0$ such that for every $\delta > 0$ there exists a smooth map $\rho : \Delta \to \mathcal{M}$ such that $|\bar{\partial}\rho| \leq \delta$, but such that for every holomorphic map $\lambda : \Delta \to \mathcal{M}$ $\mathrm{Sup}_{z \in \Delta}\mathrm{dist}(\rho(z), \lambda(z)) \geq \epsilon$.*

So, uniform approximation of maps with small $\bar{\partial}$ is not possible.

Proof. Let P be a covering map of $\mathcal{M}$, by the unit disk. Let d denote the distance function on $\mathcal{M}$, and let d_0 denote the Poincaré metric on Δ. There exists $\epsilon > 0$ such that if f and g are continuous maps from Δ into $\mathcal{M}$, and $\mathrm{Sup}_{z \in \Delta}d(f(z), g(z)) \leq \epsilon$, then f and g can be lifted to continuous maps $\tilde{f}$ and $\tilde{g}$ ($f = P \circ \tilde{f}$, $g = P \circ \tilde{g}$), with $\mathrm{Sup}_{z \in \Delta}d_0(\tilde{f}(z), \tilde{g}(z)) \leq 1$ (1 playing no special role).

Let B be a function defined on a neighborhood of $\overline{\Delta}$ in $\mathbb{C}$ with the following properties:

(i) $|B(e^{i\theta})| \equiv 1$,

(ii) $|B| < 1$ on Δ,

(iii) B is holomorphic on a neighborhood of the unit circle,

(iv) for every $k \in \mathbb{N}$, the restriction of $z^k B$ to the unit circle does not extend holomorphically to the unit disk.

For B one can take, near the unit circle, the square root of a Blaschke product with 2 simple zeroes, and cut off. Note that (i) and (iv) imply that for every holomorphic map $h : \Delta \to \Delta$, $\mathrm{Sup}_{z \in \Delta}d_0(h(z), z^k B(z)) = +\infty$.

By a normal family argument, there exists α_k, $0 < \alpha_k < 1$, such that for every holomorphic map $h : \Delta \to \Delta$, $\mathrm{Sup}_{z \in \Delta}d_0(h(z), (1 - \alpha_k)z^k B(z)) > 1$.

Take $\rho_k : \Delta \to \mathcal{M}$ defined by

$$\rho_k(z) = P \circ ((1 - \alpha_k)z^k B(z)).$$

Then $\bar{\partial}\rho_k$ tend to 0 uniformly on $\overline{\Delta}$, as $k \to \infty$ (due to condition (iii)). But, for every holomorphic map $\lambda : \Delta \to \mathcal{M}$, $\mathrm{Sup}_{z \in \Delta}d(\rho_k(z), \lambda(z)) > \epsilon$. Otherwise, lift ρ_k to the map $(1 - \alpha_k)z^k B(z)$. By our choice of ϵ, we could lift λ to a map $\tilde{\lambda}$ such that $\mathrm{Sup}_{z \in \Delta}d_0((1 - \alpha_k)z^k B(z), \tilde{\lambda}(z)) \leq 1$, in contradiction with the choice of α_k. Proposition 2 is proved.

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