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HYPERBOLICITY AND SUSTAINABILITY OF ORBITS

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ABSTRACT. Let $F : \mathbb{C}^k \to \mathbb{C}^k$ be a dynamical system and let $\{x_n\}_{n\geq 0}$ denote an orbit of F. We study the relation between $\{x_n\}$ and pseudoorbits $\{y_n\}, y_0 = x_0$. Here $y_{n+1} = F(y_n) + s_n$. In general y_n might diverge away from x_n. Our main problem is whether there exists arbitrarily small t_n so that if $\tilde{y}_{n+1} = F(\tilde{y}_n) + s_n + t_n$, then $\tilde{y}_n$ remains close to x_n. This leads naturally to the concept of sustainable orbits, and their existence seems to be closely related to the concept of hyperbolicity, although they are not in general equivalent.

1. Introduction

Let $F : \mathbb{C}^k \to \mathbb{C}^k$ be a dynamical system and let $\{x_n\}_{n\geq 0}$ denote an orbit of F. We study the relation between $\{x_n\}$ and pseudoorbits $\{y_n\}, y_0 = x_0$. Here $y_{n+1} = F(y_n)+s_n$. In general y_n might diverge away from x_n. Our main problem is whether there exists arbitrarily small t_n so that if $\tilde{y}_{n+1} = F(\tilde{y}_n)+s_n+t_n$, then $\tilde{y}_n$ remains close to x_n. This leads naturally to the concept of sustainable orbits, and their existence seems to be closely related to the concept of hyperbolicity, although they are not in general equivalent. The concept of *sustainability* is introduced in the next section, and may be viewed as the tangent space analogue of this situation. Thus we are considering the infinitesimal version of such perturbations. We say that an orbit is sustainable if errors of size δ can be corrected by (smaller) corrections of size ϵ.

Our main results are

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THEOREM 1.1. (See Theorem 3.1) *Let $F : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational function of degree at least 2. Then F is hyperbolic if and only if F is sustainable.*

THEOREM 1.2. (See Theorem 4.1) *Let F be a generalized Hénon map. Then F is hyperbolic if and only if F is sustainable.*

In the next section we define the concept of sustainability. Then in Section 3 we prove Theorem 1.1 and in Section 4, we prove Theorem 1.2.

2. Sustainability

The concept of sustainable orbits was introduced in ([3]). We recall the definition. Let M be a Hermitian complex manifold, and let $F : M \to M$ be a holomorphic map. Let us fix an orbit $\{x_n\}_{n\geq 0}$. We say that (s_n) is a *sequence of vectors over* $\{x_n\}$ if $s_n \in T_{x_n}(M)$ for each n. If (s_n) and (t_n) are sequences of vectors over the orbit $\{x_n\}$, we define a new sequence of vectors (ξ_n) by setting $\xi_0 = 0$ and

$$\xi_{n+1} = F'(x_n)\xi_n + s_{n+1} + t_{n+1}.$$

Loosely speaking, the orbit $\{x_n\}$ is sustainable if for every bounded sequence (s_n) there is an arbitrarily small "correction" $(t_n) = T(s_n)$ such that the resulting sequence (ξ_n) is bounded. More precisely, we say that the orbit $\{x_n\}$ is *weakly sustainable* if there is a $\delta > 0$ such that for any $0 < \epsilon < \delta$ there is an integer $N = N(x_0, \epsilon, \delta)$ such that for every sequence of vectors (s_n) over $\{x_n\}$ with $s_j = 0$ for $1 \leq j \leq N$ and $|s_n| \leq \delta$ for all n, there exists a sequence (t_n) over $\{x_n\}$ such that $|t_n| \leq \epsilon$, and the resulting sequence (ξ_n) satisfies $|\xi_n| \leq 1$.

We also need the following condition.

(@) $t = T(s), t' = T(s')$ and $s_n = s'_n, n \leq N + m \Rightarrow t_n = t'_n, n \leq m$.

DEFINITION 2.1. We say that the orbit $\{x_n\}$ is sustainable if there is a $\delta > 0$ so that for every $0 < \epsilon < \delta$ there is an integer $N = N(x_0, \epsilon, \delta)$ and a corresponding map T satisfying (@) such that $|\xi_n| \leq 1$ for all $n \geq 0$.

If the constants δ, ϵ, N can be chosen independently of the point $x_0 \in K$, we say that the map F is uniformly weakly sustainable respectively uniformly sustainable on K. If $K = M$ we say that F is uniformly (weakly) sustainable.

3. Rational maps on $\mathbb{P}^1$

Let $F : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational function of degree at least 2. See ([2]) for facts from the theory of complex dynamics in one variable. Our first main result is the following:

THEOREM 3.1. *Let $F : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational function of degree at least 2.*

(i) *If F is hyperbolic, then F is uniformly sustainable.*

(ii) *If every orbit in the Julia set is weakly sustainable, then F is hyperbolic.*

We recall that the Fatou set of a rational map F is the largest open set in the Riemann sphere on which the sequence of iterates F^n is a normal family. The Julia set is the complement of the Fatou set. The map F is said to be hyperbolic if there exist constants $c > 0, \lambda > 1$ so that $|(F^n)'(z)| \geq c\lambda^n$ for all $n \geq 1$ and for all $z \in J$.

Part (ii) of the Theorem can be proved by the machinery developed in ([3]) if we replace the condition "weakly sustainable" by the stronger condition "sustainable". Here we present a simple direct proof. The more abstract approach given in the next section gives yet another proof of the weaker version of (ii).

Proof. We first prove (i). Assume that F is hyperbolic. Let J be the Julia set of F. Then $J \neq \mathbb{P}^1$ and the Fatou set, $\mathbb{P}^1 \setminus J$ consists of finitely many attracting basins. We can choose coordinates so that ∞ is an attracting periodic point. Then there exists a neighborhood $U = U(J)$ and constants $C > 1, \lambda > 1 > \mu > 0$ so that if $F^n(z) \in U$, then

$$(*) \quad |(F^n)'(z)| \geq \frac{\lambda^n}{C}$$

and if z is not in U, $|(F^n)'(z)| \leq C\mu^n$. We can assume we use the Euclidean metric on U. This only amounts to a finite scaling in the constants. The basic idea is to define t_n to cancel s_{n+N} where the map is expanding and to set $t_n = 0$ where the map is contracting. This is worked out in the next Lemma.

LEMMA 3.2. *Let $N > 1$. Let $\{z_n\}_{n\geq 0}$ be any orbit and let $\{s_n\}_{n\geq 1}$ be any sequence of tangent vectors, $s_1 = \cdots = s_N = 0$, $|s| = \max |s_n| < \infty$. Then there is a sequence $t = t(s) = \{t_n\}_{n\geq 1}$, t_n depends only on s_{n+N}, such that*

(a) $|t| \leq \frac{C|s|}{\lambda^N}$,

(b) $|\xi_n| \leq |s| \left[\frac{C}{\lambda - 1} + \frac{C}{1-\mu} \right]$.

Proof of the lemma. Suppose at first that $z_n \in U$ for all $n \geq 0$. Then we define the sequence $\{t_n\}$ by

$$t_n = -\frac{s_{n+N}}{F'(z_{n+N-1}) \cdots F'(z_n)}.$$

Hence (a) follows from (*).

We get:

$$\begin{aligned} k \leq N \quad &\Rightarrow \\ \xi_k \quad &= \quad -\sum_{j=1}^{k} \frac{s_{N+j}}{F'(z_{N+j-1}) \cdots F'(z_k)}. \end{aligned}$$

By direct calculation we get for $k > N$:

$$\xi_k = -\sum_{j=\max\{1,k+1-N\}}^{k} \frac{s_{N+j}}{F'(z_{N+j-1}) \cdots F'(z_k)}.$$

Hence we get the estimate

$$\begin{aligned} |\xi_k| \quad &\leq \quad |s| \sum_{j=1}^{N} \frac{C}{\lambda^j} \\ &\leq \quad \frac{C|s|}{\lambda - 1}. \end{aligned}$$

Hence (b) holds in this case.

Next, we assume that $z_n \in U, n < m$, but z_m is not in U. In this case the orbit (z_n) converges to one of finitely many attracting periodic orbits. In this case we define

$$\begin{aligned} t_n \quad &= \quad -\frac{s_{n+N}}{F'(z_{n+N}) \cdots F'(z_{n+1})}, \text{ if } n+N < m \\ t_n \quad &= \quad 0, \text{ if } \quad n+N \geq m. \end{aligned}$$

The calculations are similar and we omit them.

The Lemma follows. □

To finish the proof of part (i) of the Theorem, it suffices to choose $\delta < 1$ so small that

$$\delta \left[\frac{C}{\lambda - 1} + \frac{C}{1-\mu} \right] < 1$$

and choose N so large that

$$\frac{C\delta}{\lambda^N} < \epsilon.$$

These constants are independent of the orbit. Hence part (i) follows.

Next we prove part (ii) of the Theorem.

Assume that every orbit in the Julia set is weakly sustainable. Let c be a critical point. Suppose first that $\{F^n(c)\}$ clusters on J.

Case 1: The critical point c is in the Fatou set.
Since there is no wandering Fatou components ([2]), the iterates $F^n(c)$ belongs to a periodic Fatou set $\Omega \cup \cdots \cup F^k(\Omega), F^k(\Omega) = \Omega$ for all large n. Since $\{F^n(c)\}$ clusters at the Julia set, this Fatou component must be a periodic parabolic basin ([2]). Hence there is a periodic point p for which $F^k(p) = p$ for some k and $|(F^k)'(p)| = 1$. We can assume that the orbit of p is bounded away from ∞ and that the metric is Euclidean there.

Set $p = z_0, z_n = F^n(z_0), \ell_n = F'(z_n), |\ell_0 \cdots \ell_{n-1}| = 1$. Let

$$T = \sup_{j=0,\ldots,n-1} |\ell_0 \cdots \ell_j| \geq 1.$$

Let δ be as in the definition of weakly sustainable for z_0. Let $\epsilon := \frac{\delta}{2kT}$ and suppose N is an integer as in the definition.

We calculate ξ_n.

$$\begin{aligned}
\xi_0 &= 0, \\
\xi_1 &= \ell_0\xi_0 + \tau_1, \\
&\vdots \\
\xi_n &= \ell_{n-1}\cdots\ell_1\tau_1 + \ell_{n-1}\cdots\ell_2\tau_2 + \ell_{n-1}\tau_{n-1} + \tau_n.
\end{aligned}$$

We can write $\xi_n = \xi_n^1 + \xi_n^2$ where

$$\begin{aligned}
\xi_n^1 &= \ell_{n-1}\cdots\ell_1 s_1 + \ell_{n-1}\cdots\ell_2 s_2 + \ell_{n-1}s_{n-1} + s_n, \\
\xi_n^2 &= \ell_{n-1}\cdots\ell_1 t_1 + \ell_{n-1}\cdots\ell_2 t_2 + \ell_{n-1}t_{n-1} + t_n.
\end{aligned}$$

Since $|t| \leq \epsilon$, we get the following estimate on ξ_n^2 :

$$|\xi_n^2| \leq nT\epsilon \leq \frac{n\delta}{2k}.$$

We will select s_n in the following way:

$$\begin{aligned} s_1 &= \cdots = s_N = 0, \\ s_{N+1} &= \delta, \\ |s_n| &= \delta, \frac{s_n}{\ell_n \cdots \ell_{N+2}} \in \mathbb{R}^+, \ n \geq N+2. \end{aligned}$$

Next we estimate ξ^1_{N+nk}, keeping only terms when the product of the derivatives is of modulus 1:

$$\begin{aligned} |\xi^1_{N+nk}| &= |\ell_{N+nk-1} \cdots \ell_1 s_1 + \ell_{N+nk-1} \cdots \ell_2 s_2 \\ &\quad + \cdots + \ell_{N+nk-1} s_{N+nk-1} + s_{N+nk}| \\ &= |\ell_{N+nk-1} \cdots \ell_1 s_1| + \cdots + |s_{N+nk}| \\ &\geq n\delta. \end{aligned}$$

This gives us an estimate on ξ_{N+nk},

$$|\xi_{N+nk}| \geq |\xi^1_{N+nk}| - |\xi^2_{N+nk}| \geq n\delta - \frac{(N+nk)\delta}{2k} \geq \frac{n\delta}{2} - \frac{N\delta}{2k}$$

which shows that the sequence ξ_n fails to be bounded, a contradiction. So case (i) is impossible.

Case 2: The critical point c belongs to the Julia set.

Let $z_0 = c, \{z_n\}$ denote the orbit of the critical point c. Then z_n is not a critical point for any large enough n. We first make some estimates for the orbit of the critical point. Afterwards we will use the fact that almost all orbits in the Julia set are dense in the Julia set ([2]) and hence will follow the critical orbit arbitrarily well. We will define at first a sequence $\tilde{s}_n, |\tilde{s}_n| = 1$ to maximize the disturbance along the critical orbit by the following implicit equations:

$$\begin{aligned} \tilde{\xi}_0 &:= 0 \\ \tilde{s}_1 &:= 1 \\ \tilde{\xi}_1 &= F'(z_0)\tilde{\xi}_0 + \tilde{s}_1 = 1 \\ &\vdots \\ \tilde{\xi}_{n+1} &= F'(z_n)\tilde{\xi}_n + \tilde{s}_{n+1} \\ |\tilde{\xi}_{n+1}| &= |F'(z_n)||\tilde{\xi}_n| + 1. \end{aligned}$$

Hence,

LEMMA 3.3.

$$\begin{aligned}\tilde{\xi}_{n+1} &= F'(z_n)\cdots F'(z_1)\tilde{s}_1 + \cdots + F'(z_n)\tilde{s}_n + \tilde{s}_{n+1},\\ |\tilde{\xi}_{n+1}| &= |F'(z_n)||F'(z_{n-1})|\cdots|F'(z_1)|\\ &\quad + |F'(z_n)||F'(z_{n-1})|\cdots|F'(z_2)| + \cdots + |F'(z_n)| + 1.\end{aligned}$$

We divide into two cases:

Case A: The sequence $\{|\tilde{\xi}_n|\}$ is unbounded.

Let $\{w_n\}_{n\geq 0}$ be an orbit in the Julia set which is dense in the Julia set and does not contain any critical point. Let $0 < \delta < 1$ be as in the definition of weakly sustainable orbit, set $\epsilon = \delta/2$ and let $N > 1$ be as in the definition of weakly sustainable orbit for w_0.

Fix an integer $m > 1$ so that $|\tilde{\xi}_{m+1}| > 4/\delta$.

Let $\{w_{\ell_j}\} \to c$. Fix j large enough that $\ell_j > N$. We define a sequence s_n as follows:

$$\begin{aligned} n < \ell_j : &\quad s_n = 0,\\ \ell_j \leq n \leq \ell_j + m : &\quad s_n = \delta\tilde{s}_{n-\ell_j},\\ n > \ell_j + m : &\quad s_n = 0.\end{aligned}$$

Since $\{w_n\}$ is weakly sustainable, there is a sequence $\{t_n\}_{n\geq 1}$, $|t_n| \leq \epsilon = \delta/2$ so that $|\xi_n| \leq 1$ for all n where $\xi_0 = 0, \xi_{n+1} = F'(w_n)\xi_n + s_{n+1} + t_{n+1}$. In particular, $|\xi_{\ell_j}| \leq 1$.

$$\begin{aligned}&\xi_{\ell_j+m+1}\\ =\ &\left[F'(w_{\ell_j+m})\cdots F'(w_{\ell_j})\right]\xi_{\ell_j}\\ &+ \left[F'(w_{\ell_j+m})\cdots F'(w_{\ell_j+1})\tilde{s}_1 + \cdots + F'(w_{\ell_j+m})\tilde{s}_m + \tilde{s}_{m+1}\right]\delta\\ &+ \left[F'(w_{\ell_j+m})\cdots F'(w_{\ell_j+1})t_{\ell_j+1} + \cdots\right.\\ &\left. + F'(w_{\ell_j+m})t_{\ell_j+m-1} + t_{\ell_j+m+1}\right].\end{aligned}$$

If we let $j \to \infty$, the first term converges to 0 because $F'(w_{\ell_j}) \to F'(c) = 0$. The second term converges to $\tilde{\xi}_{m+1}\delta$ and the third term is bounded by $(|\tilde{\xi}_{m+1}| + O(|w_{\ell_j} - c|))\epsilon \leq (|\tilde{\xi}_{m+1}| + O(|w_{\ell_j} - c|))\delta/2$. Since $|\xi_{\ell_j+m+1}| \leq 1$ and $|\tilde{\xi}_{m+1}|\delta/2 > 2$ we get a contradiction for large j.

We have shown that no critical orbit can cluster on the Julia set. Hence ([2]) the map F is hyperbolic.

Case B: The sequence $\{|\tilde{\xi}_n|\}$ is bounded.

LEMMA 3.4. *For any $n > N$ there exists an $n > m(n) \geq n - N$ so that $|F'(z_n)| \cdots |F'(z_{m(n)+1})| < 1/2$.*

Proof. Obvious. □

We use the previous Lemma repeatedly to obtain:

LEMMA 3.5. *For any $n = n_1 > N$, there exist $n_2, \ldots, n_r$ with $n_{j+1} = m(n_j)$ and $n_r \leq N$ so that $|F'(z_{n_j})| \cdots |F'(z_{n_{j+1}+1})| < 1/2$.*

LEMMA 3.6. *There is a $\tau > 0$ so that if $n \leq N$, $z \in J$ and if $0 < \sigma < \tau$, then $F^n(\Delta(z, \sigma)) \subset \Delta(F^n(z), |(F^n)'(z)|\sigma + \sigma/4)$.*

Proof. Let L denote the maximum of the double derivates of any function $F^n, 1 \leq n \leq N$ at all points of distance at most 1 from the Julia set. We then get if $\sigma < 1, z \in J$ and $|w - z| < \sigma, 1 \leq n \leq N$:

$$\begin{aligned} |F^n(w) - F^n(z)| &\leq \max_{z \in \Delta(z,\sigma)} |(F^n)'(z)|\sigma \\ &\leq \left[|(F^n)'(z)| + L\sigma\right]\sigma. \end{aligned}$$

Next just choose τ so small that $L\tau < 1/4$. □

We can then prove that the critical point is in the Fatou set, a contradiction since we assumed that c is in the Julia set:

LEMMA 3.7. *The critical point c is in the Fatou set.*

Proof. It follows from the previous Lemma that if $\Delta(c, \sigma)$ is a small enough disc, the images $F^{n_j}(\Delta(c, \sigma))$ are contained in discs $\Delta(F^{n_j}(c), \sigma_j)$ where the $\sigma_j \to 0$ geometrically. It follows that the disc $\Delta(c, \sigma)$ is contained in the Fatou set. □

4. Higher dimension

In this section we prove Theorem 1.2. We refer the reader to ([4]) for basic terminology for Hénon maps. Recall briefly that K^+ is the set of points with forward bounded orbits, K^- is the set of points with backward bounded orbits, $J^\pm = \partial K^\pm$ and $J = J^+ \cap J^-$. The set J contains the support J^* of the unique measure of maximal entropy.

THEOREM 4.1. *Let F be a generalized Hénon map.*
(i) *If F is hyperbolic, then F is uniformly sustainable.*
(ii) *If every orbit in J is sustainable, then F is hyperbolic.*

Part (i) of the Theorem is proved in ([4], Theorem 4.10). Furthermore we know that if F is sustainable on J^*, then F is hyperbolic on J^* ([3], Theorem 4.1). Hence we only need to show that $J \setminus J^*$ is empty in this case. This was done in ([4], Theorem 4.9) under the additional hypothesis that $|J(F)| < 1$. Hence our main contribution here is to remove the hypothesis that the volume is contracting.

First we make some general estimates about maps with sustainable orbits.

Let $\{x_n\}_{n\geq 0}$ be an orbit of a holomorphic map $F : M \to M$. (The following remarks are also valid for real smooth maps.) Let $X_n \subset T_{x_n}M$ be a hyperplane. We assume that $F'(X_n) \subset X_{n+1}$. For every X_n we let v_n be a unit vector perpendicular to X_n. For $n > m$, we write $(F^{n-m})'(v_m) = a_{m,n}v_n + w$ where w denotes some vector in X_n. We have for $n > m > \ell$ that $a_{\ell,m}a_{m,n} = a_{\ell,n}$. We can after rotation also assume that all the $a_{m,n} \geq 0$. The set $\{\{x_n, X_n\}_{n\geq 0}\}$ is closed in the product topology. We set $a_n = a_{0,n}$.

THEOREM 4.2. ([3], Theorem 3.17) *Suppose that the orbit of y is relatively compact and sustainable. Then the map F is uniformly sustainable on $\omega(y)$, the cluster set of the orbit of y.*

We let $\delta, N(\epsilon, \delta)$ denote uniform constants of sustainability for $\omega(y)$. Let $x_0 \in \omega(y)$ and suppose that X_0 is a hyperplane in T_{x_0}.

PROPOSITION 4.3. *There are two possibilities for $\{(x_n, X_n)\}_{n\geq 0}$.*
(1) *Fix m. Then $\lim_{n\to\infty} a_{m,n} = \infty$.*
(2) *Fix m. Then $\lim_{n\to\infty} a_{m,n} = 0$.*

The Proposition will be an easy consequence of the next two Lemmas.

LEMMA 4.4. *If $a_{m,n} < \frac{\delta}{2}$ for some $n > m$, then $a_\ell \leq \min\{2a_m, \frac{4}{\delta}a_n\} = \frac{4}{\delta}a_n \ \forall\, \ell > n$.*

Proof of the lemma. Let $C = \sup_{p\in\omega(y)} \|F'(p)\|$. Choose $\epsilon, \delta > \epsilon > 0$ so small that $\sum_{j=0}^{n-m-1} C^j\epsilon < \frac{\delta}{4}$. Let $N = N(\epsilon, \delta)$. Since $x_0 \in \omega(y)$ we can extend the orbit backwards to some $\{x_n\}_{n\geq -N} \subset \omega(y)$. We can also find hyperplanes $X_n, -N \leq n < 0$ using backwards induction so that $F'(X_n) \subset X_{n+1}$ always. Let $v_n \perp X_n$ and extend the definition of $a_{m,n}$ as well.

Next let $s_{-N+1} = \cdots = s_{n-1} = 0, s_k = \delta v_k, k \geq n$, and let $\{t_k\}_{k\geq -N+1}$, $\{\xi_k\}_{k\geq -N}$ be as in the definition of sustainability for the

orbit $\{x_n\}_{n\geq -N}$. Then $|\xi_m| \leq 1$ and

$$\xi_n = (F^{n-m})'(x_m)(\xi_0) + \sum_{j=1}^{n-m} (F^{n-m-j})'(x_{m+j})(t_j) + \delta v_n.$$

We write $\xi_k = \alpha_k v_k + \xi_k'$ where $|\alpha_k| \leq 1$ and $\xi_k' \in X_k, k \geq 0$.

$$\xi_n = (F^{n-m})'(x_m)(\alpha_m v_m) + (F^{n-m})'(x_m)(\xi_m') + \sum_{j=1}^{n-m} (F^{n-m-j})'(x_{m+j})(t_j) + \delta v_n.$$

Hence

$$\alpha_n = \delta + b_n, |b_n| \leq |\alpha_m|\frac{\delta}{2} + \sum_{j=0}^{n-m-1} C^j \epsilon < \frac{3}{4}\delta.$$

We get for $k \geq 1$,

$$\begin{aligned} \alpha_{n+k} &= a_{n,n+k}(\delta + b_n) + a_{n+1,n+k}(\delta + b_{n+1}) \\ &\quad + \cdots + a_{n+k-1,n+k}(\delta + b_{n+k-1}) + (\delta + b_{n+k}), \end{aligned}$$

where b_{n+j} are contributions from t when $j \geq 1$ and hence $|b_{n+j}| \leq \epsilon < \frac{\delta}{4}$ for $j \geq 1$. Hence

$$\begin{aligned} |\xi_{n+k}| &\geq |\alpha_{n+k}| \\ &\geq a_{n,n+k}\frac{\delta}{4}. \end{aligned}$$

Since $|\xi_{n+k}| \leq 1$, it follows that $a_{n,n+k} \leq \frac{4}{\delta}$. Hence $a_{n+k} = a_n a_{n,n+k} \leq \frac{4}{\delta} a_n$. □

LEMMA 4.5. *Let $0 < \eta < K$. Then there must exist some $n > 0$ for which $a_n \notin [\eta, K]$.*

Proof of the lemma. Suppose that $a_n \in [\eta, K]$ for all $n \geq 1$. Then, if $n > m \geq 1$,

$$\frac{\eta}{K} \leq a_{m,n} = \frac{a_{0,n}}{a_{0,m}} \leq \frac{K}{\eta}.$$

Let $\delta, \epsilon = \frac{\delta\eta^2}{2K^2}, N$ be as in the definition of sustainability. Next, we define $s_n = 0, 1 \leq n \leq N, s_n = \delta v_n, n > N$. Let $\{t_n\}_{n\geq 1}$ be the ϵ-corrections. We denote by w_n any vector in the space X_n. We can write $t_n = t_n' v_n + w_n$, $|t_n'| < \epsilon$.

If $n > N$,

$$\begin{aligned}\xi_n &= [(F^{n-N-1})'(x_{N+1})s_{N+1} + (F^{n-N-2})'(x_{N+2})s_{N+2} \\ &+ \cdots + s_n] + [(F^{n-1})'(x_1)t_1 \\ &+ (F^{n-2})'(x_2)t_2 + \cdots + t_n] \\ &= w_n + [a_{N+1,n}\delta + a_{N+2,n}\delta + \cdots + \delta]v_n \\ &+ [a_{1,n}t_1' + \cdots + t_n']v_n.\end{aligned}$$

Hence

$$\begin{aligned}|\xi_n| &\geq [a_{N+1,n}\delta + a_{N+2,n}\delta + \cdots + \delta] + [a_{1,n}t_1' + \cdots + t_n'] \\ &\geq (n-N)\frac{\eta}{K}\delta - n\frac{K}{\eta}\epsilon \\ &\geq \frac{\eta}{K}\delta\left[(n-N) - \frac{n}{2}\right] \\ &\to \infty \text{ as } n \to \infty.\end{aligned}$$

□

Proof of the Proposition. There are two cases, by Lemma 4.5.

Case (i) $\overline{\lim}\, a_n = \infty$.

By Lemma 4.4 it follows that whenever $m > n$, then $a_m \geq \frac{\delta}{2}a_n$. Hence $a_n \to \infty$.

Case (ii) $\underline{\lim}\, a_n = 0$.

Choose $r > 0, r < \frac{\delta}{2}$. Pick n so that $a_n < r$. Then $a_n < \frac{\delta}{2}$. Hence by Lemma 4.4, if $m > n$, then $a_m \leq \frac{4}{\delta}a_n < \frac{4r}{\delta}$. It follows that $a_n \to 0$. □

DEFINITION 4.6. We say that $\{x_n, X_n\}$ is an attractor if $a_n \to 0$ and that $\{x_n, X_n\}$ is a repellor if $a_n \to \infty$.

LEMMA 4.7. *If $\{x_n, X_n\}_{n\geq 0}$ is an attractor, $x_0 \in \omega(y)$, then all $\{y_n, Y_n\}_{n\geq 0}$, $y_0 \in \omega(y)$, close enough, are also attractors.*

Proof. By the Proposition $a_n \to 0$. Hence by continuity, for all $\{y_n, Y_n\}_{n\geq 0}$ nearby, the corresponding $a_n < \frac{\delta}{2}$ for some n. Hence by Lemma 4.4 these a_n form a bounded sequence, hence by the Proposition they must converge to 0. □

It follows from ([4], Section 4) that we only need to show that F is saddle hyperbolic on any $\omega(y), y \in J$. If so, the proof of Theorem 4.1 can be completed following the steps from ([4]).

Proof of Theorem 4.1. Since Henon maps have no critical points, $\{x_n, X_n\}_{n\geq 0}$ is uniquely determined by (x_0, X_0). Let $y \in J$ and set $L = \omega(y)$. We divide L into three pieces:

$$\begin{aligned} L^{ss} &= \{x_0 \in L;\ (x_0, X_0) \text{ is an attractor } \forall\ X_0\}, \\ L^{uu} &= \{x_0 \in L;\ (x_0, X_0) \text{ is a repellor } \forall\ X_0\}, \\ L^{su} &= L \setminus (L^{ss} \cup L^{uu}). \end{aligned}$$

We observe at first that these sets are invariant. Also, the sets L^{ss} and $L^{ss} \cup L^{su}$ are open in L. Hence, L^{uu} is a closed invariant subset. But then it follows by a compactness argument that F is uniformly expanding on L^{uu}. This implies that L^{uu} consists of finitely many repelling periodic orbits, hence L^{uu} belongs to the interior of K^-. This contradicts that $J \subset \partial K^-$. Hence L^{uu} is empty. Next suppose that $x_0 \in L^{ss}$. Then by compactness there exists some n so that for any hyperplane X_0 the corresponding $a_n < 1/2$. This implies that the map is contracting. Therefore, the constant Jacobian of the Hénon map is strictly less than one. Hence we are back in the case covered in ([4]), so the theorem follows in that case.

We are left with the situation where $L = L^{su}$. Suppose that X_0 is a repellor for $x_0 \in L$. Then for any small angle ζ, if u is a unit tangent vector at x_0 with angle at least ζ with X_0, then (F^n) is expanding on u and the expansion is uniform. It follows that if there is another repellor at x_0, then (F^n) is expanding in all directions at x_0. But this is only possible for points in L^{uu}. Therefore we know that for each point x_0 in L there is a unique hyperplane X_0 which is a repellor. Since the condition of being an attractor is open, it follows that the line field $Z = \{X_0\}$ varies continuously along L.

We next introduce a notion of Z sustainability. We simply mean that the map F is sustainable when we restrict both the vectors s_n and t_n to belong to the invariant line field Z. This concept was already introduced in ([3]) and applied to skew products $(P(z), Q(z, w))$ which have a natural invariant line field consisting of lines parallel to the w axis. In fact the proof of ([3], Lemma 5.3) carries over to the line field Z to show that F is uniformly Z sustainable.

Once we have uniform Z sustainability we can easily prove a version of Lemmas 4.4 and 4.5 to obtain a version of Proposition 4.3 showing the following:

LEMMA 4.8. *If $x_0 \in L$ and v_{x_0} is a unit tangent vector along Z, then there are two possibilities*:

(i) $\|(F^n)'(x_0)(v_{x_0})\| \to \infty$,
(ii) $\|(F^n)'(x_0)(v_{x_0})\| \to 0$.

Let L_u consist of points $x_0 \in L$ for which case (i) occurs and let L_s be the points of case (ii). Then L_s is open in L and L_u is closed. The sets are both invariant. If we restrict to L_u we see that (F^n) is uniformly expanding on L_u. But this is impossible for the same reason that applied to show that L^{uu} is empty. Hence we see that $L = L_s$. But this shows that Z is a stable field for F. Now it is straightforward to use the fact that Z is a line field of repellors to show that there also exists a continuous invariant unstable field on L. Hence we have shown that F is saddle hyperbolic on L. The rest of the proof is the same as in ([4]). □

Next we make some remarks on holomorphic endomorphisms on $\mathbb{P}^n$.

Suppose $F : \mathbb{P}^n \to \mathbb{P}^n$ is a holomorphic map of degree $d \geq 2$. Let J denote the support of the unique invariant measure μ of maximal entropy. Then J is completely invariant.

Assume that F is sustainable. Recall ([1]) that

$$J = \overline{\{p \in J, p \text{ is a repelling periodic point}\}}.$$

By ([3], Theorem 3.17]) F is uniformly sustainable on J. Let $\delta, N(\epsilon, \delta)$ be uniform constants.

THEOREM 4.9. *If $F : \mathbb{P}^k \to \mathbb{P}^k$ is a holomorphic map of degree $d \geq 2$ and all orbits in J are sustainable, then F is uniformly expanding on J.*

Proof. For every repelling periodic point p_0 in J and any hypersurface $X_{p_0} \subset T_{p_0}$ let $[p_0, X_{p_0}]$ denote the sequence $\{p_n, (F^n)'(p_0)(X_{p_0})\}$. Let $[p_0]$ denote the union $[p_0] = \cup\{[p_0, X_{p_0}]\}$. Next we set $S = \cup[p_0]$ where we take the union over all repelling periodic points contained in J. Next we let $T = \overline{S}$ denote the closure.

Using Proposition 4.3 we divide T into two complementary subsets, T_u and T_s, where T_u are those elements $x_0 \in J$ for which all $\{x_n, X_n\}$ are repellors. Then T_s is open in T. However none of the $[p_0]$ obtained as above from repelling periodic orbits are contained in T_s. Since those are dense in T it follows that in fact T_s is empty. But then it follows that (F^n) is uniformly expanding on J. □

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