SOBOLEV-TYPE EMBEDDING THEOREMS FOR HARMONIC AND HOLOMORPHIC SOBOLEV SPACES

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ABSTRACT. In this paper we consider Sobolev-type embedding theorems for harmonic and holomorphic Sobolev spaces on a bounded domain with C^2 boundary.

1. Introduction and statement of results

Let D be a bounded domain in $\mathbb{R}^N$ with C^2 boundary. For $x \in D$ let $\delta_D(x)$ denote the distance from x to ∂D. For $0 < p, q < \infty$ let $\|h\|_{p,q}$ be the L^p-norms with respect the weighted measures $dV_q(x) = \delta_D(x)^{q-1} dV(x)$. For a function h we define a functional $\| \cdot \|_{m+\sigma,p,q}$, where m is a non-negative integer, $0 < p, q < \infty$, and $0 \leq \sigma \leq 1$, as follows:

$$\|h\|_{m,p,q} := \left\{ \sum_{j=0}^{m} \int_D |\nabla^j h|^p dV_q \right\}^{1/p} \quad \text{if} \quad \sigma = 0,$$

$$\|h\|_{m+\sigma,p,q} := \left\{ \|h\|_{m,p,q}^p + \int_D |\nabla^{m+1}h|^p \delta_D^{(1-\sigma)p} dV_q \right\}^{1/p}.$$

We define harmonic Sobolev spaces $\mathcal{H}^{m+\sigma,p,q}(D)$ by

$$\mathcal{H}^{m+\sigma,p,q}(D) = \{h \text{ harmonic on } D : \|h\|_{m+\sigma,p,q} < \infty\}.$$

THEOREM 1.1. *Let D be a bounded domain in $\mathbb{R}^N$ with C^2 boundary. Let $1 \le p < \infty, q_i > 0, s_i \ge 1 (i = 1, 2)$ with $q_1 - q_0 = (s_1 - s_0)p$. Then we have*

$$\mathcal{H}^{s_0, p, q_0}(D) = \mathcal{H}^{s_1, p, q_1}(D).$$

Let D be a bounded domain in $\mathbb{C}^n$ with C^2 boundary. Let m be a non-negative integer, $0 < p, \ q < \infty$, and $0 \le \sigma \le 1$. We define holomorphic Sobolev spaces $A^{m+\sigma, p, q}(D)$ by

$$A^{m+\sigma, p, q}(D) = \{f \text{ holomorphic on } D : \|f\|_{m+\sigma, p, q} < \infty\}.$$

THEOREM 1.2. *Let $0 < p_0 < p_1 < \infty, q_i > 0, s_i \ge 0 (i = 0, 1)$ with $(n + q_1)/p_1 - (n + q_0)/p_0 = s_1 - s_0$. Then we have*

$$A^{s_0, p_0, q_0}(D) \subset A^{s_1, p_1, q_1}(D).$$

REMARK 1.3. (i) According to Theorem 1.1, a derivative is compensated for by a factor δ_D, so that the norms $\| \cdot \|_{s_0, p, q_0}$ and $\| \cdot \|_{s_1, p, q_1}$ are equivalent if $q_1 - q_0$ equals the difference in the number of derivatives, which is $(s_1 - s_0)p$. Beatrous [2] proved this result for the holomorphic Sobolev spaces on a strongly pseudoconvex domain.

(ii) In Theorem 1.2, if $s_1 = 0$ and $q_0 = q_1 = 1$, then we have

$$W^{s_0, p_0}(D) \cap \mathcal{O}(D) \subset L^{p_1}(D) \cap \mathcal{O}(D), \quad \frac{1}{p_1} = \frac{1}{p_0} - \frac{s_0}{n+1}.$$

This is the Sobolev embedding theorem for holomorphic Sobolev spaces. In this holomorphic Sobolev embedding theorem the dimension $2n$ is replaced by $n + 1$; the complex tangent derivatives are counted for one half.

(iii) Theorem 1.2 has been proved by Beatrous-Burbea [3] for the unit ball and by Beatrous [2] for the strongly preudoconvex domain. The key point is the reproducing kernel with right estimate matching quasimetric on ∂D. As usual we study the behavior of holomorphic functions in terms of the basic invariant objects attached to the domain; the Bergman kernel and its metric, the Szegö kernel, and the Poisson-Szegö kernel, since all these would naturally be taken into account the simple geometric considerations. However in general domains much enough is known about these domain functions and so we must use a different approach. For Sobolev norm estimates we replace the role of the reproducing kernel by the mean value theorem and Hardy's inequalities.

In Section 4 we observe that the assumption of C^2-smoothness of the boundary of D is an essential condition for the Sobolev-type embedding of Theorem 1.2. We give a counter-example of a convex domain with

$C^{1,\lambda}$ smooth boundary for $0 < \lambda < 1$ which does not satisfy our embedding results. Here a $C^{1,\lambda}$-function means that first derivatives of the function are Lipschitz continuous of order λ. The counter-example shows that even a little loss of derivatives of the boundary is not permitted for the sharp embedding results.

2. Harmonic Sobolev spaces

Throughout this section h will denote a harmonic function on D.

LEMMA 2.1. *Let* $K \Subset D$. *Let* $\alpha \in \mathbb{Z}_+^N$ *and* $1 \leq< p < \infty, 0 < q < \infty$. *Then we have*

$$\sup_K |D^\alpha h| \leq C\|h\|_{p,q},$$

where the constant C *is independent of* h.

Proof. Let $x \in \partial K$. Since h is harmonic, it follows that (see [6])

$$(2.1) \qquad |D^\alpha h(x)| \lesssim \frac{1}{\delta_D(x)^{N/p+|\alpha|}} \left(\int_{B(x,\delta_D(x)/2)} |h(y)|^p dV(y) \right)^{1/p}.$$

We have

$$(2.2) \qquad \delta_D(x)/2 \leq \delta_D(y) \leq 2\delta_D(x) \quad \text{for} \quad y \in B(x,\delta_D(x)/2).$$

By (2.2), the right-hand side of (2.1) is bounded by

$$\text{dist}(K,\partial D)^{-N/p-|\alpha|-(q-1)/p}\|h\|_{p,q}.$$

By the maximum principle, we get the result. $\qquad\square$

LEMMA 2.2. *Let* $s \geq 0, 1 \leq p < \infty, 0 < q < \infty$, *and let* k *be a non-negative integer such that* $(k-s)p+q > 0$. *Then there is a positive constant* C, *depending on* s, p, q, *and* k, *such that*

$$\int_D |\nabla^{k+1}h|^p \delta_D^{(k+1-s)p} dV_q \leq C \int_D |\nabla^k h|^p \delta_D^{(k-s)p} dV_q.$$

Proof. We use a Whitney decomposition of D (see [1]). Let $\epsilon > 0$ be sufficiently small. With ϵ fixed, there exists a sequence $\{x_j\}$ in D, and a positive integer M, where M depends only on D, such that

$(2.3) \qquad B(x_j, \delta_D(x_j))$ are pairwise disjoint,

$(2.4) \qquad \cup_j B(x_j, \delta_D(x_j)/2) = D,$

$(2.5) \qquad$ each point of D lies in at most M of the sets $B(x_j, \delta_D(x_j))$.

Let $x \in B(x_j, \delta_D(x_j)/2)$. Then $B(x, \delta_D(x)/4) \subset B(x_j, \delta_D(x_j))$. Since $\nabla^k u$ is harmonic, it follows that

$$\delta_D(x)|\nabla\nabla^k h(x)| \lesssim \frac{1}{\delta_D(x)^N} \int_{B(x,\delta_D(x)/4)} |\nabla^k h(y)| dV(y).$$

By Hölder's inequality, we have

$$(2.6) \qquad \delta_D(x)^p |\nabla^{k+1} h(x)|^p \lesssim \frac{1}{\delta_D(x)^N} \int_{B(x,\delta_D(x)/4)} |\nabla^k h(y)|^p dV(y).$$

Since $\delta_D(x)/2 \leq \delta_D(y) \leq 2\delta_D(x)$ for $y \in B(x, \delta_D(x)/4)$, it follows from (2.6) that

$$|\nabla^{k+1} h(x)|^p \delta_D^{(k+1-s)p+q-1}$$
$$\lesssim \frac{1}{\delta_D(x)^N} \int_{B(x,\delta_D(x)/4)} |\nabla^k h(y)|^p \delta_D(y)^{(k-s)p} dV_q(y).$$

Thus it follows that

$$\int_{B(x_j,\delta_D(x_j)/2)} |\nabla^{k+1} h(x)|^p \delta_D^{(k+1-s)p} dV_q$$
$$(2.7) \qquad\qquad \lesssim \int_{B(x,\delta_D(x)/4)} |\nabla^k h(y)|^p \delta_D(y)^{(k-s)p} dV_q(y)$$
$$\lesssim \int_{B(x_j,\delta_D(x_j))} |\nabla^k h(y)|^p \delta_D(y)^{(k-s)p} dV_q(y).$$

By (2.4) and (2.5), and (2.7), we have

$$\int_D |\nabla^{k+1} h(x)|^p \delta_D^{(k+1-s)p} dV_q \lesssim \sum_j \int_{B(x_j,\delta_D(x_j)/2)} |\nabla^{k+1} h|^p \delta_D^{(k+1-s)p} dV_q$$
$$\lesssim \sum_j \int_{B(x_j,\delta_D(x_j))} |\nabla^k h|^p \delta_D^{(k-s)p} dV_q$$
$$\leq M \int_D |\nabla^k h|^p \delta_D^{(k-s)p} dV_q.$$

$\square$

LEMMA 2.3. *Let* $s, p, q,$ *and* k *be the same as in Lemma 2.2. There is a compact subset* K *of* D *such that*

$$\int_{D \backslash K} |\nabla^k h|^p \delta_D^{(k-s)p} dV_q \lesssim \int_{D \backslash K} |\nabla^{k+1} h|^p \delta_D^{(k+1-s)p} dV_q + \sup_K |\nabla^k h|^p.$$

Proof. Since ∂D is C^2, there is a C^1 vector field ν such that $\nu(y)$ is the outward unit normal vector at $y \in \partial D$. For $\delta > 0$ we let $D_\delta = \{y - t\nu(y) : y \in \partial D, t > \delta\}$. Then there exists a number $\delta_0 > 0$ such that the map $\Phi(y,t) = y - t\nu(y)$ is a C^1 diffeomorphism of $\partial D \times (0, \delta_0)$ onto $D \setminus \bar{D}_{\delta_0}$. Thus we have

$$\int_{D\setminus\bar{D}_{\delta_0}} |\nabla^k h|^p \delta_D^{(k-s)p} dV_q \sim \int_0^{\delta_0} \int_{\partial D} |\nabla^k h(y - t\nu(y))|^p d\sigma(y) t^{(k-s)p+q-1} dt.$$

Since

$$\nabla^k h(y - t\nu(y)) = -\int_t^{\delta_0} \frac{d}{d\tau} \nabla^k h(y - \tau\nu(y)) d\tau + \nabla^k h(y - \delta_0 \nu(y))$$

$$= \int_t^{\delta_0} \nabla^{k+1} h(y - \tau\nu(y)) \cdot \nu(y) d\tau + \nabla^k (y - \delta_0 \nu(y)),$$

it follows that

$$\int_{D\setminus\bar{D}_{\delta_0}} |\nabla^k h|^p \delta_D^{(k-s)p} dV_q$$

$$\lesssim \int_0^{\delta_0} \int_{\partial D} \left(\int_t^{\delta_0} |\nabla^{k+1} h(y - \tau\nu(y))| d\tau d \right)^p d\sigma(y) t^{(k-s)p+q-1} dt$$

$$+ \int_0^{\delta_0} \int_{\partial D} |\nabla^k h(y - \delta_0 \nu(y))|^p d\sigma(y) t^{(k-s)p+q-1} dt.$$

By Hardy's inequality, we have

$$\int_{\partial D} \int_0^{\delta_0} \left(\int_t^{\delta_0} |\nabla^{k+1} h(y - \tau\nu(y))| d\tau \right)^p t^{(k-s)p+q-1} dt d\sigma$$

$$\lesssim \int_{\partial D} \int_0^{\delta_0} |\nabla^{k+1} h(y - \tau\nu(y))|^p \tau^{p+(k-s)p+q-1} d\tau d\sigma.$$

Thus it follows that

$$\int_{D\setminus\bar{D}_{\delta_0}} |\nabla^k h|^p \delta_D^{(k-s)p} dV_q \lesssim \int_{D\setminus\bar{D}_{\delta_0}} |\nabla^{k+1} h|^p \delta_D^{(k+1-s)p} dV_q$$

$$+ \delta_0^{(k-s)p+q} \sup_{\partial D_{\delta_0}} |\nabla^k h|^p.$$

By the maximum principle, we get the result. $\square$

Proof of Theorem 1.1. Let $m_i = [s_i]$ and $\sigma_i = s_i - m_i$. By symmetry, it is enough to prove the inequality $\|h\|_{s_0,p,q_0} \lesssim \|h\|_{s_1,p,q_1}$ for

$h \in \mathcal{H}^{p,q_1,s_1}(D)$. By Lemmas 2.1, 2.2, and 2.3, we have

$$\|h\|_{s_i,p,q_i}^p \sim \|h\|_{p,q_i}^p + \int_D |\nabla^{m_i+1} h|^p \delta_D^{(1-\sigma_i)p} dV_{q_i}.$$

First we assume that $s_0 \leq s_1$. By Lemma 2.1 and Lemma 2.3, it follows that

$$
\begin{aligned}
\|h\|_{p,q_0}^p &\lesssim \int_D |\nabla^{m_1+1} h|^p \delta_D^{(m_1+1)p} dV_{q_0} + \|h\|_{p,q_1}^p \\
&\lesssim \int_D |\nabla^{m_1+1} h|^p \delta_D^{(1-\sigma_1)p} dV_{q_1} + \|h\|_{p,q_1}^p \\
&\sim \|h\|_{s_1,p,q_1}^p.
\end{aligned}
$$
(2.8)

By the similar method as (2.8), it follows that

$$(2.9) \quad \int_D |\nabla^{m_0+1} h|^p \delta_D^{(1-\sigma_0)p} dV_{q_0} \lesssim \int_D |\nabla^{m_1+1} h|^p \delta_D^{(1-\sigma_1)p} dV_{q_1} + \|h\|_{p,q_1}^p.$$

By (2.8) and (2.9), we have

$$\|h\|_{s_0,p,q_0} \lesssim \|h\|_{s_1,p,q_1} \quad \text{for} \quad s_0 \leq s_1.$$

Now we assume that $s_1 \leq s_0$. By Lemma 2.2, it follows that

$$\int_D |\nabla^{m_0+1} h|^p \delta_D^{(1-\sigma_0)p} dV_{q_0} \lesssim \int_D |\nabla^{m_1+1} h|^p \delta_D^{(1-\sigma_1)p} dV_{q_1}.$$

Clearly, $\|h\|_{p,q_0} \lesssim \|h\|_{p,q_1}$. Thus it follows that

$$\|h\|_{s_0,p,q_0} \lesssim \|h\|_{s_1,p,q_1} \quad \text{for} \quad s_1 \leq s_0.$$

$\square$

3. Holomorphic Sobolev spaces

THEOREM 3.1. *Let $0 < p, q < \infty$, and $s \geq 0$. Let $f \in A^{s,p,q}(D)$. Let m be a non-negative integer with $m \geq s$. Then we have*

$$(3.1) \qquad \sup\{\delta_D(z)^{m-(s-(n+q)/p)} |\nabla^m f(z)| : z \in D\} \lesssim \|f\|_{s,p,q}.$$

Proof. For $p_0 \in D$ sufficiently near ∂D, we translate and rotate the coordinate system so that $z(p_0) = 0$ and the Im z_1 axis is perpendicular to ∂D. Let $\mathcal{B}_\epsilon(p_0)$ denote the non-isotropic ball

$$\mathcal{B}_\epsilon(p_0) = \left\{ \frac{|z_1|^2}{(\epsilon \delta_D(p_0))^2} + \sum_2^n \frac{|z_j|^2}{\epsilon \delta_D(p_0)} < 1 \right\}.$$

Since ∂D is C^2, it follows that there is an $\epsilon_0 > 0$ such that for p_0 sufficiently near ∂D and $z \in \mathcal{B}_{\epsilon_0}(p_0)$ we have $z \in D$ and

$$(3.2) \qquad \frac{\delta_D(p_0)}{2} \leq \delta_D(z) \leq 2\delta_D(p_0)$$

(see [1]). Since the plurisubharmonicity of $|\nabla^m f|^p$ is invariant by the affinity

$$(z_1, z_2, \ldots, z_n) \rightarrow \left(\frac{z_1}{\epsilon_0 \delta_D(p_o)}, \frac{z_2}{\sqrt{\epsilon_0 \delta_D(p_0)}}, \ldots, \frac{z_n}{\sqrt{\epsilon_0 \delta_D(p_0)}} \right),$$

it follows that

$$(3.3) \qquad |\nabla^m f(p_0)|^p \lesssim \frac{1}{\mathrm{Vol}(\mathcal{B}_{\epsilon_0}(p_0))} \int_{\mathcal{B}_{\epsilon_0}(p_0)} |\nabla^m f(z)|^p dV(z).$$

By (3.2), the right-hand side of (3.3) is bounded by

$$(3.4) \qquad \frac{1}{\delta_D(p_0)^{n+q+(m-s)p}} \int_D |\nabla^m f| \delta_D^{(m-s)p} dV_q.$$

Thus we get the result of the case $m = s$.

Now if $m > s$, then by Lemma 2.2, the right-hand side of (3.3) is bounded by

$$\frac{1}{\delta_D(p_0)^{n+q+(m-s)p}} \int_D |\nabla^{[s]+1} f|^p \delta_D^{([s]+1-s)p} dV_q.$$

Thus we have

$$|\nabla^m f(p_0)| \lesssim \delta_D(p_0)^{-m+(s-(n+q)/p)} \|f\|_{s,p,q}.$$

$\square$

By Hardy-Littlewood lemma and Theorem 3.1, we get the following results.

COROLLARY 3.2. Let $0 < p, q < \infty$. Then we have
(i) $A^{s,p,q}(D) \subset \Lambda_{s-(n+q)/p}(D)$, if $s > (n+q)/p$.
(ii) $A^{s,p,q}(D) \subset BMOA(D)$, if $s = (n+q)/p$.

LEMMA 3.3. Let $0 < p_0 \leq p_1 < \infty, q_i > 0 (i = 1, 2)$ with $(n + q_0)/p_0 = (n + q_1)/p_1$. For $s \geq 0$ we have $A^{s,p_0,q_0}(D) \subset A^{s,p_1,q_1}(D)$ and the inclusion is continuous.

Proof. We will show that $\|f\|_{s,p_1,q_1} \lesssim \|f\|_{s,p_0,q_0}$ for $f \in A^{s,p_0,q_0}(D)$. Let $s = [s] + \sigma$. For $0 \le j \le [s]$ we have

$$
\begin{aligned}
(3.5) \quad \int_D |\nabla^j f|^{p_1} dV_{q_1} &= \int_D |\nabla^j f|^{p_0} |\nabla^j f|^{p_1-p_0} \delta_D^{q_0-1} \delta_D^{(n+q_0)(p_1/p_0-1)} dV \\
&\le \left(\int_D |\nabla^j f|^{p_0} dV_{q_0} \right) \left(\sup \delta_D^{(n+q_0)/p_0} |\nabla^j f| \right)^{p_1-p_0}.
\end{aligned}
$$

By (3.5), it follows that

$$
\begin{aligned}
(3.6) \quad \|\nabla^j f\|_{p_1,q_1} &\le \|\nabla^j f\|_{p_0,q_0}^{p_0/p_1} \left(\sup \delta_D^{(n+q_0)/p_0} |\nabla^j f| \right)^{1-p_0/p_1} \\
&\le \|\nabla^j f\|_{p_0,q_0} + \sup \delta_D^{(n+q_0)/p_0} |\nabla^j f|.
\end{aligned}
$$

By (3.1), the right-hand side of (3.6) is bounded by $\|\nabla^j f\|_{p_0,q_0}$. Thus it follows that

$$
(3.7) \qquad \|f\|_{[s],p_1,q_1} \lesssim \|f\|_{[s],p_0,q_0}.
$$

Now it follows that

$$
\begin{aligned}
(3.8) \quad \int_D |\nabla^{[s]+1} f|^{p_1} \delta_D^{(1-\sigma)p_1} dV_{q_1} &\lesssim \int_D |\nabla^{[s]+1} f|^{p_0} \delta_D^{(1-\sigma)p_0} dV_{q_0} \\
&\times \left(\sup \delta_D^{1-\sigma+(n+q_0)/p_0} |\nabla^{[s]+1} f| \right)^{p_1-p_0}.
\end{aligned}
$$

By (3.1), it follows that

$$
(3.9) \qquad \sup \delta_D^{1-\sigma+(n+q_0)/p_0} |\nabla^{[s]+1} f| \lesssim \|f\|_{s,p_0,q_0}.
$$

By (3.7), (3.8), and (3.9), we get the result.   □

Proof of Theorem 1.2. By Lemma 3.3, we have

$$
(3.10) \qquad A^{s_0,p_0,q_0}(D) \subset A^{s_0,p_1,q}(D),
$$

where $(n+q_0)/p_0 = (n+q)/p_1$. Since $q_1 - q = (s_1 - s_0)p_1$, by Theorem 1.1, we have

$$
(3.11) \qquad A^{s_0,p_1,q}(D) \subset A^{s_1,p_1,q_1}(D).
$$

By (3.10) and (3.11), we get the required result.   □

4. A counter-example

In this section we observe that the assumption of C^2-smoothness of the boundary of D is an essential condition for the sharp embedding of Theorem 1.2.

EXAMPLE 4.1. We consider the domain defined by

$$D = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^{1+\lambda} < 1\}, \quad \text{where} \quad 0 < \lambda < 1.$$

We can see that D is a bounded convex domain with $C^{1,\lambda}$ boundary, but it has no C^2 boundary. In [4] we have compared the growth rate of the functions in $A^{p,q}(D)$ between this domain D and the bounded domain with C^2 boundary.

Let $0 < p_0 < p_1 < \infty, q_i > 0, s_i \geq 0 (i = 0, 1)$ with $(1 + q_1)/p_1 - (1 + q_0)/p_0 = s_1 - s_0$. Let $f(z_1, z_2)$ be some branch of $(1 - z_1)^{-d}$ on $\bar{D}$, where $(1 + q_1 + 2/(1+\lambda))/p_1 - s_1 < d < (1 + q_0 + 2/(1+\lambda))/p_0 - s_0$. We prove that

- $f \in A^{s_0, p_0, q_0}(D)$,
- $f \notin A^{s_1, p_1, q_1}(D)$.

The two facts above imply that $A^{s_0, p_0, q_0}(D)$ cannot be embedded into $A^{s_1, p_1, q_1}(D)$.

Since D is a Lipschitz domain, we have

$$(4.1) \qquad 1 - |z_1|^2 - |z_2|^{1+\lambda} \sim \delta_D(z_1, z_2) \quad \text{for} \quad (z_1, z_2) \in D$$

(see Lemma 2 in [6], Section 3.2.1 of Chapter VI).

We have

$$|\nabla^{[s_i]+1} f| \sim \frac{1}{|1 - z_1|^{d+[s_i]+1}}.$$

Set $r(z_1) = (1 - |z_1|^2)^{1/(1+\lambda)}$. By (4.1), it follows that

$$\int_D |\nabla^{[s_0]+1} f|^{p_0} \delta_D^{(1-\sigma_0)p_0} dV_{q_0}$$

$$\sim \int_{|z_1|<1} \frac{dA(z_1)}{|1 - z_1|^{(d+[s_0]+1)p_0}}$$

$$\times \int_{|z_2|<r(z_1)} (1 - |z_1|^2 - |z_2|^{1+\lambda})^{(1-\sigma_0)p_0+q_0-1} dA(z_2).$$

We estimate the integral

$$I(z_1) = \int_{|z_2|<r(z_1)} (1 - |z_1|^2 - |z_2|^{1+\lambda})^{(1-\sigma_0)p_0+q_0-1} dA(z_2).$$

By the polar coordinates, we have

$$I(z_1) \sim \int_0^{r(z_1)} (1 - |z_1|^2 - r^{1+\lambda})^{(1-\sigma_0)p_0+q_0-1} r\, dr$$

$$\sim \int_0^{1-|z_1|^2} (1 - |z_1|^2 - s)^{(1-\sigma_0)p_0+q_0-1} s^{2/(1+\lambda)-1}\, ds$$

$$\sim (1 - |z_1|^2)^{2/(1+\lambda)+(1-\sigma_0)p_0+q_0-1}$$

$$\times \int_0^1 (1 - \tau)^{(1-\sigma_0)p_0+q_0-1} \tau^{2/(1+\lambda)-1}\, d\tau.$$

Note that

$$\int_0^1 (1 - \tau)^{(1-\sigma_0)p_0+q_0-1} \tau^{2/(1+\lambda)-1}\, d\tau = B\left(\frac{2}{1+\lambda}, (1 - \sigma_0)p_0 + q_0\right),$$

where $B(\cdot, \cdot)$ is the beta function. Hence we have

$$\int_D |\nabla^{[s_0]+1} f|^{p_0} \delta_D^{(1-\sigma_0)p_0}\, dV_{q_0}$$

$$\sim \int_{|z_1|<1} \frac{(1 - |z_1|^2)^{2/(1+\lambda)+(1-\sigma_0)p_0+q_0-1}}{|1 - z_1|^{(d+[s_0]+1)p_0}}\, dA(z_1)$$

$$= \lim_{r \to 1^-} \int_{|z_1|<1} \frac{(1 - |z_1|^2)^{2/(1+\lambda)+(1-\sigma_0)p_0+q_0-1}}{|1 - rz_1|^{(d+[s_0]+1)p_0}}\, dA(z_1) dA(z_1)$$

$$\sim \lim_{r \to 1^-} \frac{1}{(1 - r^2)^{(d+s_0)p_0-q_0-1-2/(1+\lambda)}} \lesssim 1,$$

since $(d + s_0)p_0 - q_0 - 1 - 2/(1+\lambda) < 0$ (see [5]). Hence $f \in A^{s_0,p_0,q_0}(D)$.

By the similar calculation as above, we get

$$\int_D |\nabla^{[s_1]+1} f|^{p_1} \delta_D^{(1-\sigma_0)p_1}\, dV_{q_1} \sim \lim_{r \to 1^-} \frac{1}{(1 - r^2)^{(d+s_1)p_1-q_1-1-2/(1+\lambda)}} = \infty,$$

since $(d + s_1)p_1 - q_1 - 1 - 2/(1+\lambda) > 0$. Hence $f \notin A^{s_1,p_1,q_1}(D)$.

References

[1] F. Beatrous, L^p *estimates for extensions of holomorphic functions*, Michigan Math. J. **32** (1985), 361–380.

[2] ———, *Estimates for derivatives of holomorphic functions in pseudoconvex domains*, Math. Z. **191** (1986), 91–116.

[3] F. Beatrous and J. Burbea, *Holomorphic Sobolev spaces on the ball*, Dissertationes Math. **256** (1989), 1–57.

[4] H. R. Cho and E. G. Kwon, *Growth rate of the functions in Bergman type spaces*, submitted.

[5] W. Rudin, *Function theory in the unit ball of* $\mathbb{C}^n$, Spinger-Verlag, New York, 1980.
[6] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton, N.J., 1970.

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