Large-System Analyses of Multiple-Antenna System Capacities

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Abstract: **Asymptotic theorems are very commonly used in probability. For systems whose performance depends on a set of n random parameters, asymptotic analyses for $n \to \infty$ are often used to simplify calculations and obtain results yielding useful hints at the behavior of the system for finite n. These asymptotic analyses are especially useful whenever the convergence to the asymptotic results is so fast that even for moderate n they yield results close to the true values. This tutorial paper illustrates this principle by applying it to capacity calculations of multiple-antenna systems.**

Index Terms: **Multiple antennas, MIMO channels, channel capacity, space-time codes, asymptotic analysis, random matrices, fading channels.**

I. INTRODUCTION AND SYSTEM MODEL

Asymptotic theorems are among the most widely used tools in applications of probability theory: examples of these are the law of large numbers and the central limit theorem. Other examples come from random-matrix theory, which has recently been widely recognized as an important tool for the study of large systems like CDMA [19]–[20], OFDM [6] and [21], and multiple-antenna systems [4]. Asymptotic theorems are especially useful when applied to the analysis of systems whose behavior depends on a set of n random parameters, and such that the convergence is so fast that for moderate, or even small, values of n the asymptotic results come close to exact values (a simple example of this is given by early computer programs for the generation of Gaussian random numbers, obtained by summing a small number of uniform random variables and invoking the central limit theorem). In this tutorial paper we show how this concept finds application to the evaluation of capacities of multiple-antenna systems: in fact, the results obtained approximate very closely those referring to a nonasymptotic regime.

Here we consider a radio communication system with t trasmit and r receive antennas (Fig. 1). Assuming two-dimensional elementary constellations throughout, the input-output relation between the observed vector $y \in \mathbb{C}^r$ and the input vector $x \in \mathbb{C}^t$ is given by

$$y = Hx + z, \qquad (1)$$

where the t components of x are the signals transmitted by each antenna, the r components of y are the signals received, $H \in \mathbb{C}^{r \times t}$ is a complex matrix whose entries h_{ij} describe the gains of each transmission path to a receive from a transmit antenna. In the following, unless otherwise stated, we assume that
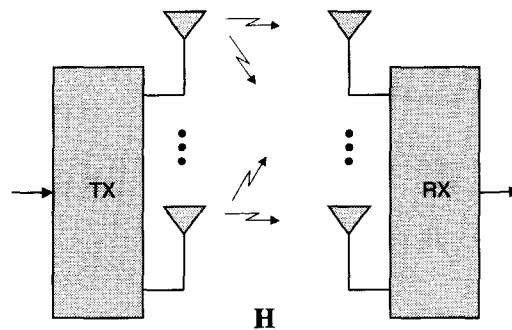

Fig. 1.   Transmission and reception with t transmitting and r receiving antennas. The channel gains are described by the $r \times t$ matrix H.

the entries of H are iid zero-mean circularly-symmetric complex Gaussian distributed as $\mathcal{N}_c(0,1)$ (i.e., their real and imaginary parts are independent and have variance $1/2$). The vector z is circularly-symmetric complex Gaussian distributed and represents the additive channel noise. We assume that $\mathbb{E}[zz^\dagger] = I_r$,[1] that is, the noises affecting the different receivers are independent, and the signal power $\mathbb{E}[x^\dagger x]$ coincides with the signal-to-noise ratio. The signal power is constrained by $\mathbb{E}[x^\dagger x] \le P$, so that P/t is the maximum average power transmitted by each antenna.

It is of interest to evaluate the capacity of the transmission system described in (1). Two scenarios will be considered. First, we assume that H is a random matrix, and each transmission of one vector x corresponds to an *independent* realization of H. Next, we assume that H remains constant during the transmission of an entire code word. The two channel models will be referred to as *ergodic* and *nonergodic*, respectively (see [2] for details).

II. ERGODIC-CHANNEL CAPACITY

We assume here that the entries of H are iid and distributed as $\mathcal{N}_c(0,1)$ and every transmission of a vector x corresponds to an independent value of H. This choice models fast Rayleigh fading with enough separation within antennas such that the fades for each TX/RX antenna pair are independent. We also assume that the channel state information (that is, the realization of H) is known at the receiver, while only the distribution of H is known at the transmitter (the latter assumption is necessary for capacity computations, since the transmitter must choose an optimum code for that specific channel).

[1]Here and in the following $(\cdot)^\dagger$ denotes the Hermitian conjugate of a matrix/vector.

The capacity, which is achieved by a transmitted signal $\mathbf{x} \sim \mathcal{N}_c(0, (P/t)\mathbf{I}_t)$, is given by [18]

$$C = \mathbb{E}\left[\log \det \left(\mathbf{I}_r + \frac{P}{t}\mathbf{H}\mathbf{H}^\dagger\right)\right]. \qquad (2)$$

Here and hereafter $\mathbf{I}_a$ denotes the $a \times a$ identity matrix; moreover, log denotes base-2 logarithm, so that C is expressed in bits per dimension pair, or, if one dimension pair per second is transmitted in a 1 Hz bandwidth, in bit/s/Hz. Exact calculation of the expectation in (2) yields [18]

$$C = \int_0^\infty \log\left(1 + \frac{P}{t}\lambda\right) \sum_{k=0}^{m-1} \frac{k!}{(k+n-m)!}$$
$$\cdot \left[L_k^{n-m}(\lambda)\right]^2 \lambda^{n-m} e^{-\lambda} d\lambda, \qquad (3)$$

where $m \triangleq \min\{t, r\}$, $n \triangleq \max\{t, r\}$, and $L_i^j(\cdot)$ are Laguerre polynomials. An alternative expression, avoiding the summation in the integrand, can be obtained by using the Christoffel–Darboux identity for Laguerre polynomials:

$$\sum_{k=0}^{m-1} \frac{k!}{(k+n-m)!} L_k^{n-m}(\lambda_1) L_k^{n-m}(\lambda_2)$$
$$= \frac{m!}{(n-1)!} \frac{L_{m-1}^{n-m}(\lambda_1)L_m^{n-m}(\lambda_2) - L_{m-1}^{n-m}(\lambda_2)L_m^{n-m}(\lambda_1)}{\lambda_1 - \lambda_2}, (4)$$

which, for $\lambda_1 = \lambda_2$, yields

$$\sum_{k=0}^{m-1} \frac{k!}{(k+n-m)!} \left[L_k^{n-m}(\lambda)\right]^2$$
$$= \frac{m!}{(n-1)!} \left[\frac{d}{d\lambda}\left[L_{m-1}^{n-m}(\lambda)\right] L_m^{n-m}(\lambda)\right.$$
$$\left. - L_{m-1}^{n-m}(\lambda)\frac{d}{d\lambda}\left[L_m^{n-m}(\lambda)\right]\right]$$
$$= \frac{m!}{(n-1)!} \left[L_{m-1}^{n-m}(\lambda)L_{m-1}^{n-m+1}(\lambda)\right.$$
$$\left. - L_m^{n-m}(\lambda)L_{m-2}^{n-m+1}(\lambda)\right]. \qquad (5)$$

As expression (3) is quite involved, and can hardly give immediate insight into the behavior of capacity, asymptotic analyses can succor us in the task of examining the latter in depth.

It is convenient to observe that we have

$$C = \sum_{i=1}^{\min(t,r)} \log\left(1 + \frac{P}{t}\lambda_i\right), \qquad (6)$$

where λ_i denote the eigenvalues of $\mathbf{H}\mathbf{H}^\dagger$ (there are no more than $\min(t, r)$ nonzero eigenvalues).
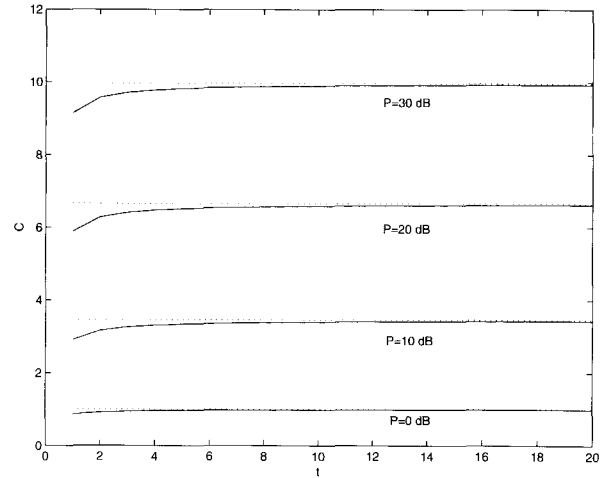

Fig. 2. Capacity (in bit/s/Hz) of the ergodic Rayleigh MIMO channel with $r = 1$ (continuous line). The asymptotic approximation $C \approx \log(1 + P)$ is shown by the dotted line.
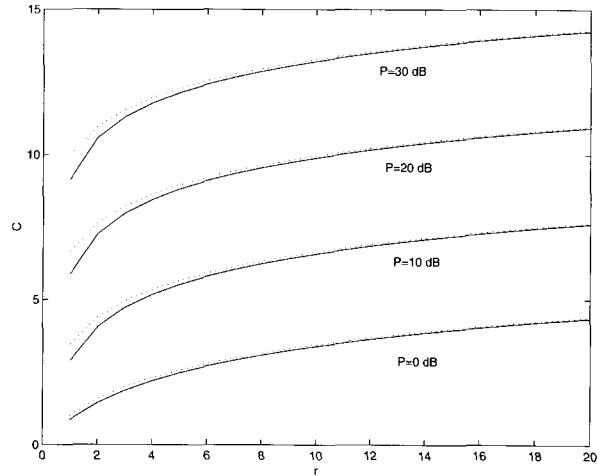

Fig. 3. Capacity (in bit/s/Hz) of the ergodic Rayleigh MIMO channel with $t = 1$ (continuous line). The asymptotic approximation $C \approx \log(1 + Pr)$ is also shown (dotted line).

A. Two Special Cases: $r = 1$ and $t = 1$

The simplest special case of capacity calculation occurs when $\min(t, r) = 1$, that is, either $r = 1$ or $t = 1$, so that from (6) we have $C = \log(1 + (P/t)\lambda)$, with λ the only nonzero eigenvalue of $\mathbf{H}\mathbf{H}^\dagger$. When $r = 1$ this eigenvalue is $\lambda = \sum_{i=1}^t |h_{1i}|^2$. Invoking the law of large numbers, as $t \to \infty$ we have $(1/t)\lambda \to 1$, so that $C \to \log(1 + P)$. The accuracy of the approximation $C \approx \log(1 + P)$ is shown in Fig. 2.

Consider now the case $t = 1$. We have $\lambda = \sum_{i=1}^r |h_{i1}|^2$, which as $r \to \infty$ tends to r. Thus, we obtain the asymptotic approximation $C \approx \log(1 + rP)$, shown in Fig. 3 along with the exact value obtained numerically from (3). It is seen that the asymptotic expression of C comes very close to the true capacity even for values of r as small as 1.

We can observe that in both cases of $r = 1$ and $t = 1$ the asymptotic expression of C is valid independently of the assumption of Gaussian entries for $\mathbf{H}$: actually, it suffices to have independent, identically distributed (iid) unit-variance entries.

B. General Case

More generally, we can evaluate an asymptotic approximation to (2) by using a result from random-matrix theory [16] (see Appendix A for details):

Theorem 1: Let $\mathbf{H}_n$ be a sequence of $m \times n$ random matrices with iid entries with zero mean and unit variance, such that $m/n \to \alpha$ as $n \to \infty$. Let $F_n(x)$ denote the empirical eigenvalue distribution of $\frac{1}{n}\mathbf{H}_n\mathbf{H}_n^\dagger$, i.e.,

$$F_n(x) \triangleq \sum_{k=1}^{m} \frac{1}{m} u(x - \lambda_k(\mathbf{H}_n\mathbf{H}_n^\dagger/n)), \qquad (7)$$

where $u(\cdot)$ denotes the unit step function, and $\lambda_k(\mathbf{W})$ denotes the kth eigenvalue of $\mathbf{W}$. Then, $F_n(x)$ converges weakly, as $n \to \infty$, to a deterministic distribution $F_\infty(x)$ whose corresponding pdf is

$$f_\infty(x) = (1 - \alpha^{-1})_+ \delta(x) + \frac{\sqrt{(x - a)_+(b - x)_+}}{2\pi\alpha x}, \qquad (8)$$

where $(\cdot)_+ \triangleq \max(0, \cdot)$ and $a, b \triangleq (1 \mp \sqrt{\alpha})^2$.

Application of Theorem 1 yields the following asymptotic result:

$$\frac{C}{t} \to \int_0^\infty \log(1 + \alpha^{-1} Px) f_\infty(x)dx = \mathcal{C}(\alpha, P), \qquad (9)$$

where

$$\mathcal{C}(\alpha, P) \triangleq \int_a^b \log(1 + \alpha^{-1} Px) \frac{\sqrt{(x - a)(b - x)}}{2\pi\alpha x} dx. \qquad (10)$$

(Notice that a scaling factor r has been taken into account due to the different definitions of the matrices $\mathbf{H}$ and $\mathbf{H}_n$ in the Theorem.)

This integral can be computed in a rather straighforward way as shown in Appendix B, yielding the result

$$\lim \frac{C}{m} \to \max(1, \alpha) \cdot \mathcal{C}(\alpha, \mathcal{P})$$

$$= \frac{\max(1, \alpha)}{\alpha \ln 2}$$

$$\cdot \Big(\ln(w_+ P) - \alpha\, w_- + (1 - \alpha)\ln(1 - w_-) \Big), \qquad (11)$$

where again $m \triangleq \min\{t, r\}$,

$$w_\pm \triangleq (w \pm \sqrt{w^2 - 4/\alpha})/2, \qquad (12)$$

and

$$w \triangleq 1 + \frac{1}{\alpha} + \frac{1}{P}. \qquad (13)$$

Evaluation of this integral was also performed in [14] and [15] in the context of CDMA analysis. In [20] the calculation is done indirectly, without actually computing the integral. In [14] the integral is computed directly (using a method that differs from
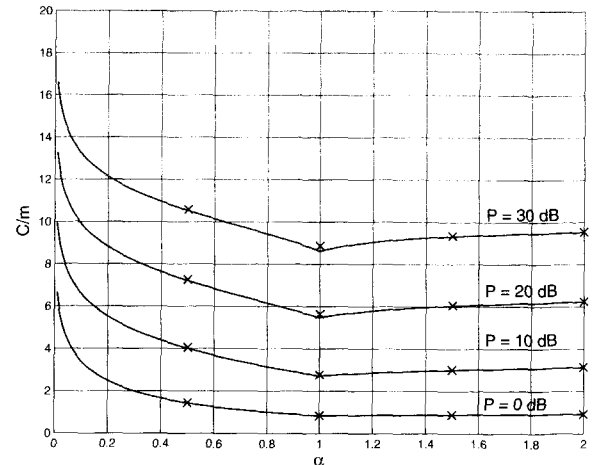

Fig. 4. Asymptotic ergodic capacity per antenna (C/m) with independent Rayleigh fading as $t, r \to \infty$ and $t/r \to \alpha$ (solid curves). The ergodic capacity per antenna for $r = 2$ is also shown for comparison $(\times)$.
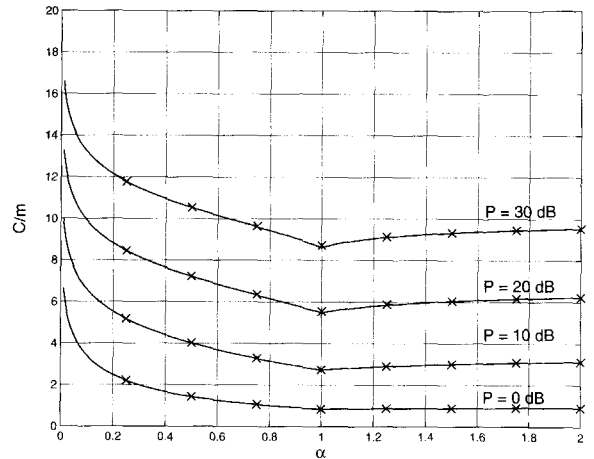

Fig. 5. Asymptotic ergodic capacity per antenna (C/m) with independent Rayleigh fading as $t, r \to \infty$ and $t/r \to \alpha$ (solid curves). The ergodic capacity per antenna for $r = 4$ is also shown for comparison $(\times)$.

that of Appendix B). Yet another technique to derive (11) is described in [13] and [15]. This asymptotic result is plotted in Figs. 4 and 5. The figures show also the values of C/m corresponding to $r = 2$ and 4, respectively and show how good the asymptotic approximation is even for such small values of r. By setting $\alpha = t/r$, (11) yields C as a function of t and r, and provides values very close to the true capacity even for small r, t. We stress that, for the validity of (11), it is not necessary to assume that the entries of $\mathbf{H}$ are Gaussian, as needed for the preceding nonasymptotic result (3): a sufficient condition is that $\mathbf{H}$ have iid entries with zero mean and unit variance [16].

C. A Reciprocity Formula

For the asymptotic ergodic capacity, from equality

$$\frac{1}{t}\mathbb{E} \log \det \left(\mathbf{I}_r + \frac{P}{t}\mathbf{H}\mathbf{H}^\dagger \right) = \frac{r}{t}\frac{1}{r}\mathbb{E} \log \det \left(\mathbf{I}_t + \frac{r}{t}\frac{P}{r}\mathbf{H}^\dagger\mathbf{H} \right),$$

the following reciprocity relation can be derived:

$$C(\alpha, P) = \alpha^{-1} C(\alpha^{-1}, \alpha^{-1} P). \qquad (14)$$

III. NONERGODIC CHANNEL CAPACITY

When $\mathbf{H}$ is chosen randomly at the beginning of the transmission, and held fixed for all channel uses, average capacity has no meaning [2]. In this case the quantity to be evaluated is, rather than capacity, outage probability, that is, the probability that the transmission rate R cannot be sustained by the channel. The capacity (or, more precisely, the mutual information exchanged in the multiple-antenna channel) is the random variable

$$C(\mathbf{H}) = \log \det \left(\mathbf{I}_r + \frac{P}{t} \mathbf{H} \mathbf{H}^{\dagger} \right), \qquad (15)$$

and the outage probability is defined as

$$P_{\text{out}}(R) = \mathbb{P}(C(\mathbf{H}) < \mathbf{R}). \qquad (16)$$

The evaluation of (16) can be done by Monte Carlo simulation. However, one can profitably use an asymptotic result which states that, as t and r grow to infinity, the capacity $C(\mathbf{H})$ tends to a Gaussian random variable. This result has been recently obtained in [11] for the case of iid zero-mean circularly-symmetric complex Gaussian distributed entries. A related result applying to matrices with iid real random entries is reported in [10] and numerical evidence is provided by several other authors [7] and [22].

Moreover, it was observed in [11], [17], and [22] that $C(\mathbf{H})$ follows very closely a normal distribution even for small values of r and t: thus, the computation of the mean and variance of $C(\mathbf{H})$ allows one to obtain a good approximation of its probability density function. The expected value of $C(\mathbf{H})$ is the ergodic capacity discussed in the previous section. Its variance was evaluated independently in the form of an integral for finite t and r by Smith and Shafi [17] and by Wang and Giannakis [22] (the latter reference actually derives the moment generating function of $C(\mathbf{H})$, and hence all of its moments). The case of asymptotically large t and r is dealt with by [15] in a more general case of correlated Gaussian entries of $\mathbf{H}$ where the authors resort to the *replica method*.

In practice, the asymptotic mean and variance of $C(\mathbf{H})$, denoted μ_C and σ_C^2, respectively, yield a close approximation to the statistics of $C(\mathbf{H})$ even for small r and t. Thus, the outage probability can be obtained in the form

$$P_{\text{out}}(R) \approx Q \left(\frac{\mu_C - R}{\sigma_C} \right). \qquad (17)$$

A. Asymptotic Variance Calculation

The integral expressions [17] and [22] do not lead easily to an asymptotic expression of the variance, whose calculation would
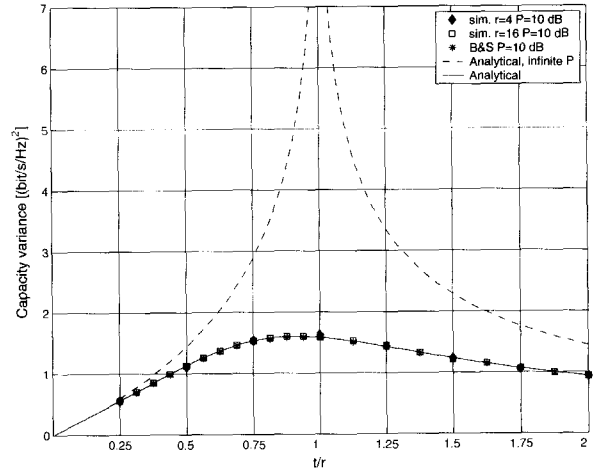

Fig. 6.   Variance of the nonergodic capacity with $P = 10$ dB. Continuous line: Asymptotic variance from [15]. (∗): Numerical integration of Bai-Silverstein [1] expression for the variance. (♦) and (□): Monte Carlo simulation with $r = 4$ and 16.

require determining the limit joint distribution of pairs of eigenvalues of $\mathbf{H} \mathbf{H}^{\dagger}$. Approximations to σ_C^2 are derived in [11], while exact results are obtained in [1] and [15]. Sengupta and Mitra [15] use the *replica method* to derive results that can be specialized to our problem in the form

$$\sigma_C^2 = -\log e \log \left(1 - \frac{q^2 \rho^2}{\beta} \right), \qquad (18)$$

with $w \triangleq \sqrt{1/P}$, $\beta \triangleq \alpha^{-1}$, and q, ρ are given by

$$q \triangleq \frac{\beta - 1 - w^2 + \sqrt{(\beta - 1 - w^2)^2 + 4\beta w^2}}{2w}, \qquad (19)$$

$$\rho \triangleq \frac{1 - \beta - w^2 + \sqrt{(1 - \beta - w^2)^2 + 4w^2}}{2w}. \qquad (20)$$

An alternative expression in the form of an integral for the asymptotic capacity variance was obtained by Bai and Silverstein in [1]. Using their (1.17) we have the following expression with $a, b \triangleq 1 + \alpha \mp 2\sqrt{\alpha}$. By the change of variables
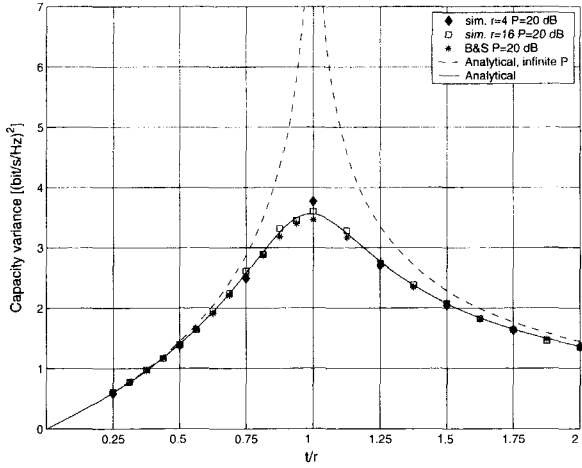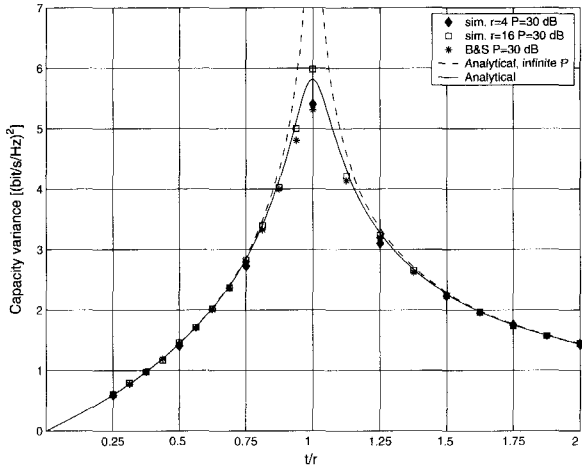
$$\begin{cases} x = 1 + \alpha + 2\sqrt{\alpha} \cos \varphi \\ y = 1 + \alpha + 2\sqrt{\alpha} \cos \theta \end{cases}, \qquad (22)$$

we obtain (23).

This integral can be evaluated numerically. However, its calculation is made hard by the fact that the integrand exhibits a logarithmic discontinuity at the line $\varphi = \theta$. Numerical results are shown in Figs. 6–8.

The values of the variance for $P \to \infty$ are obtained by observing that, as $w \to 0$, equations (19)–(20) yield

$$\sigma_{C'}^2 = \left(\frac{\log e}{2\pi} \right)^2 \int_a^b \int_a^b \frac{1}{(x + \alpha/P)(y + \alpha/P)}$$

$$\times \ln \left\{ 1 + \frac{4xy\sqrt{(x-a)(b-x)}\sqrt{(y-a)(b-y)}}{(\alpha - 1)^2 (x-y)^2 + (y\sqrt{(x-a)(b-x)} - x\sqrt{(y-a)(b-y)})^2} \right\} dx \, dy \qquad (21)$$

Fig. 7. Same as Fig. 6 but $P = 20$ dB.

Fig. 8. Same as Fig. 6 but $P = 30$ dB.

Fig. 9. Outage probability for $r = t = 4$ and a nonergodic Rayleigh channel vs. R, the transmission rate in bit/s/Hz. The continuous line shows the results obtained by Monte Carlo simulation, while the dashed line shows the normal approximation.

$$1 - \frac{q^2\rho^2}{\beta} = \begin{cases} 1 - \beta + \dfrac{2\beta}{1-\beta}w^2 + O(w^4) & \text{for } \beta < 1 \\ 1 - \dfrac{1}{\beta} + \dfrac{2}{\beta(\beta-1)}w^2 + O(w^4) & \text{for } \beta > 1, \end{cases} \tag{26}$$

which allows us to compute the limit

$$\lim_{P \to \infty} \sigma_C^2 = \log e \cdot \log(1 - \min(\alpha, \alpha^{-1})). \tag{27}$$

B. Numerical Results

Fig. 9, which plots P_{out} versus R for $r = t = 4$ and two values of SNR, shows the quality of the Gaussian approximation for a Rayleigh channel.

Based on these results, we can evaluate the outage probabilities as in Figs. 10 and 11. These figures show the rate R that can be supported by the channel for a given SNR and a given outage probability, that is, from (17):

$$R = \mu_C - \sigma_C Q^{-1}(P_{\text{out}}).$$

Notice how as r and t increase the outage probabilities curves come closer to each other: in fact, as r and t grow to infinity the channel tends to an ergodic channel. This fact has already been observed in [5] (for $r = t \to \infty$) where they say that the mutual information $C(\mathbf{H})$ tends to be insensitive to the realization of $\mathbf{H}$.

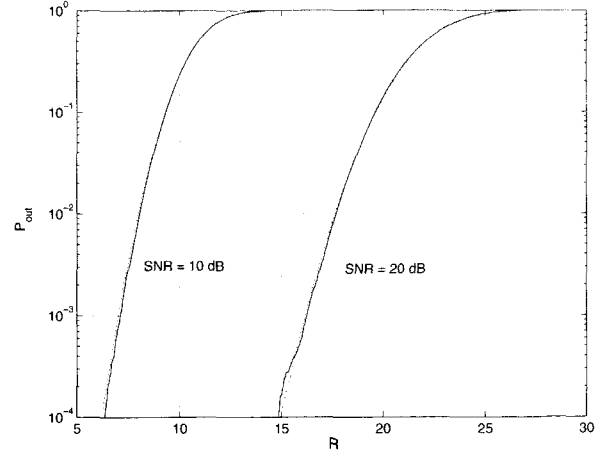Fig. 12 shows the outage capacity (at $P_{\text{out}} = 0.01$) of an independent Rayleigh fading MIMO channel.

$$q = \begin{cases} \dfrac{\beta}{1-\beta}w - \dfrac{\beta}{(1-\beta)^3}w^3 + O(w^5) & \text{for } \beta < 1 \\ \dfrac{\beta-1}{w} + \dfrac{1}{\beta-1}w - \dfrac{\beta}{(\beta-1)^3}w^3 + O(w^5) & \text{for } \beta > 1, \end{cases} \tag{24}$$

and

$$\rho = \begin{cases} \dfrac{1-\beta}{w} + \dfrac{\beta}{1-\beta}w - \dfrac{\beta}{(1-\beta)^3}w^3 + O(w^5) & \text{for } \beta < 1 \\ \dfrac{1}{\beta-1}w - \dfrac{\beta}{(\beta-1)^3}w^3 + O(w^5) & \text{for } \beta > 1, \end{cases} \tag{25}$$

so that

$$\sigma_C^2 = \alpha \left(\frac{\log e}{\pi}\right)^2 \int_0^\pi \int_0^\pi \frac{\sin\varphi \sin\theta}{(1 + 2\sqrt{\alpha}\cos\varphi + \alpha(1 + 1/P))(1 + 2\sqrt{\alpha}\cos\theta + \alpha(1 + 1/P))}$$
$$\times \ln\left\{1 + \frac{4(1 + \alpha + 2\sqrt{\alpha}\cos\varphi)\sin\varphi(1 + \alpha + 2\sqrt{\alpha}\cos\theta)\sin\theta}{(\alpha - 1)^2(\cos\varphi - \cos\theta)^2 + ((1 + \alpha)(\sin\varphi - \sin\theta) + 2\sqrt{\alpha}\sin(\varphi - \theta))^2}\right\} d\varphi d\theta \tag{23}$$
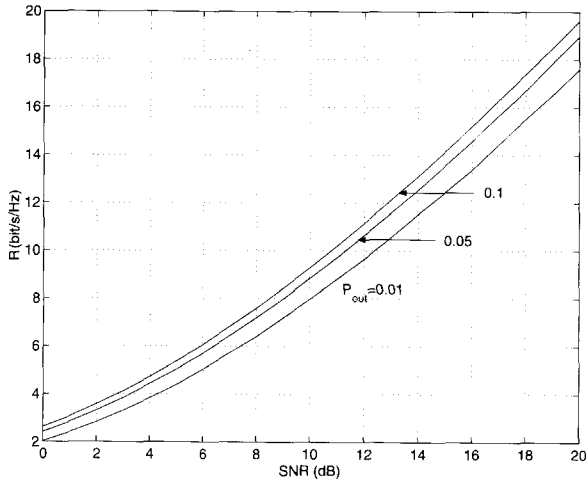
Fig. 10. Transmission rate that can be supported with $r = t = 4$ and a give outage probability by a nonergodic Rayleigh channel. The results are based on the Gaussian approximation.
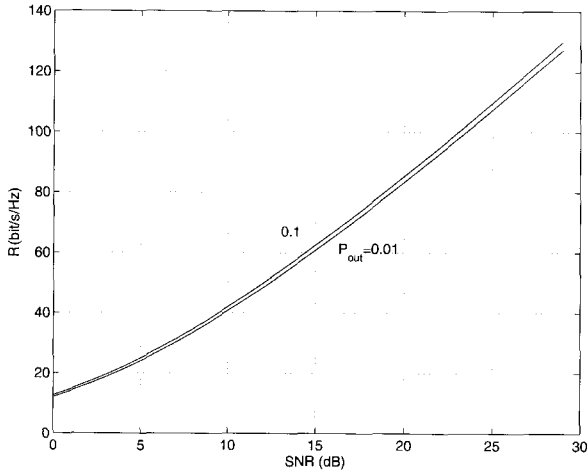

Fig. 12. Outage capacity (at $P_{out} = 0.01$) with independent Rayleigh fading and $r = t$ antennas.

It is assumed that $m_F(z) \in \mathcal{C}^+ = \{z : \mathrm{Im(z)} > 0\}$ whenever $z \in \mathcal{C}^+$. The inverse Stieltjes transform returns the corresponding pdf at its points of continuity:

$$F'(\lambda) = \frac{1}{\pi} \lim_{\omega \to 0^+} m_F(\lambda + j\omega). \qquad (29)$$

The result above derives from the following:

$$\frac{1}{\pi} \lim_{\omega \to 0^+} \mathrm{Im} \int (\mathrm{x} - \lambda - \mathrm{j}\omega)^{-1} \mathrm{dF(x)}$$

$$= \frac{1}{\pi} \lim_{\omega \to 0^+} \int \frac{\omega}{(x - \lambda)^2 + \omega^2} dF(x)$$

$$= \int \delta(x - \lambda) dF(x)$$

$$= F'(\lambda). \qquad (30)$$

The Stieltjes transform can be used to obtain the limiting eigenvalue distribution of the matrix $\mathbf{H}^\dagger \mathbf{H}/r$, where $\mathbf{H}$ is a $t \times r$ random matrix with iid entries having zero mean and unit variance and we assume that $t/r \to \alpha$ as the matrix dimensions grow to infinity.

According to [16], the Stieltjes transform $m(z)$ of this distribution satisfies the following equation:

$$
\begin{aligned}
m(z) &= \frac{1}{1 - \alpha - \alpha z m(z) - z} \\
&\Longrightarrow \alpha z \, m(z)^2 + (z + \alpha - 1)m(z) + 1 = 0 \\
&\Longrightarrow m(z) \\
&= \frac{1 - \alpha - z + \sqrt[+]{z^2 - 2(\alpha + 1)z + (\alpha - 1)^2}}{2\alpha z},
\end{aligned}
$$

$$(31)$$

where $\sqrt[+]{z}$ denotes the square root of z in $\mathbb{C}^+$.

It is straightforward to obtain the corresponding probability density function:

Fig. 11. Transmission rate that can be supported with $r = t = 16$ and a give outage probability by a nonergodic Rayleigh channel. The results are based on the Gaussian approximation.

IV. CONCLUSION

We have shown that asymptotic analyses of multiple-antenna systems can be profitably used to obtain results which often yield accurate approximations of the ergodic and nonergodic capacities even when the parameters take on small values.

ACKNOWLEDGMENTS

APPENDIX A: LIMITING DISTRIBUTION OF THE EIGENVALUES OF $\mathbf{HH}^\dagger/r$

The *Stieltjes transform* of a probability distribution $F(\lambda)$ is defined as

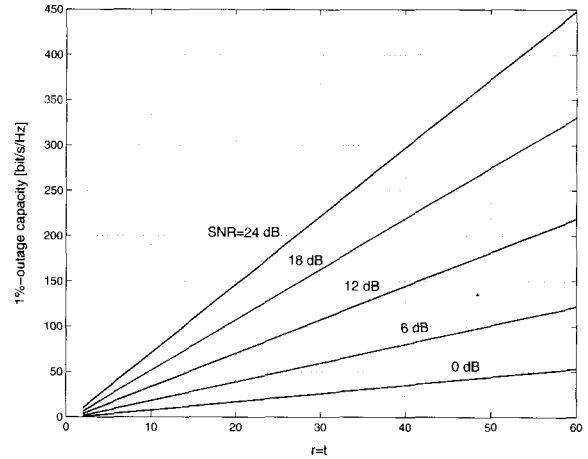$$m_F(z) \triangleq \int (\lambda - z)^{-1} dF(\lambda). \qquad (28)$$

$$f(x) = (1 - \alpha^{-1})_+ \delta(x)$$
$$+ \frac{\sqrt{(-x^2 + 2(\alpha + 1)x - (\alpha - 1)^2)_+}}{2\pi\alpha x}, \quad (32)$$

where $(x)_+ \triangleq \max(0, x)$.

Notice that the first term appears only when $\alpha > 1$ since, in that case, its Stieltjes transform $-(1 - \alpha^{-1})/z \in \mathcal{C}^+$. This agrees with the fact that only in that case the matrix $\mathbf{H}^\dagger\mathbf{H}$ has $(t - r)$ eigenvalues equal to zero, i.e., a fraction $(1 - \alpha^{-1})$ of all its eigenvalues.

APPENDIX B: COMPUTATION OF ERGODIC CAPACITY

With the change of variables $x = 1 + \alpha + 2\sqrt{\alpha}\cos\theta$ in (10) we obtain

$$\mathcal{C}(\alpha, P) = \frac{1}{\pi} \int_0^{2\pi} \log\left(1 + \alpha^{-1}P(1 + \alpha + 2\sqrt{\alpha}\cos\theta)\right)$$
$$\cdot \frac{\sin^2\theta}{1 + \alpha + 2\sqrt{\alpha}\cos\theta} d\theta, \quad (33)$$

which is a real number. Next, notice that

$$1 + \alpha^{-1}P(1 + \alpha + 2\sqrt{\alpha}\cos\theta)$$
$$= \alpha^{-1/2}Pe^{-j\theta}(e^{j\theta} + \alpha^{1/2}w_+)(e^{j\theta} + \alpha^{1/2}w_-), \quad (34)$$

where $w \triangleq 1 + \alpha^{-1} + P^{-1}$ and $w_\pm \triangleq (w \pm \sqrt{w^2 - 4/\alpha})/2$. From the inequality

$$(\alpha^{1/2}w_+ - 1)(\alpha^{1/2}w_- - 1)$$
$$= -\alpha^{-1/2}(1 - \alpha^{1/2})^2 - \alpha^{1/2}P^{-1} < 0, \quad (35)$$

we have $\alpha^{1/2}w_- < 1$ and $\alpha^{1/2}w_+ > 1$, which makes it convenient to rewrite the above expression in the form

$$1 + \alpha^{-1}P(1 + \alpha + 2\sqrt{\alpha}\cos\theta)$$
$$= w_+ P\left(1 + \frac{\exp(j\theta)}{\alpha^{1/2}w_+}\right)\left(1 + \frac{\alpha^{1/2}w_-}{\exp(j\theta)}\right) < 0. \quad (36)$$

We can now expand the logarithm of the above function in a uniformly-convergent series:

$$\ln(1 + \alpha^{-1}P(1 + \alpha + 2\sqrt{\alpha}\cos\theta))$$
$$= \ln(w_+ P)$$
$$+ \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}\left[\left(\frac{\exp(j\theta)}{\alpha^{1/2}w_+}\right)^k + \left(\frac{\alpha^{1/2}w_-}{\exp(j\theta)}\right)^k\right]. \quad (37)$$

This can be integrated term-by-term. To do this, we need to calculate

$$I_k \triangleq \frac{1}{\pi}\int_0^{2\pi} e^{jk\theta}\frac{\sin^2\theta}{1 + \alpha + 2\sqrt{\alpha}\cos\theta}d\theta, \quad (38)$$

for every integer k. After some algebra, we obtain

$$I_k = \begin{cases} 1 & k = 0 \\ -\frac{1}{2}\sqrt{\alpha} & |k| = 1 \\ (-1)^{k+1}\frac{1-\alpha}{2\alpha}\alpha^{|k|/2} & |k| > 1. \end{cases} \quad (39)$$

Thus, term-by-term integration of the series (37) yields

$$\ln 2 \cdot \mathcal{C}(\alpha, \mathcal{P})$$
$$= \ln(w_+ P) - \frac{1}{2\alpha}\left[w_+^{-1} + \alpha w_-\right]$$
$$+ \frac{1-\alpha}{2\alpha}\sum_{k=1}^{\infty}\frac{1}{k}\left[w_+^{-k} + (\alpha w_-)^k\right]$$
$$= \ln(w_+ P) - w_-$$
$$- \frac{1-\alpha}{2\alpha}\left[\ln(1 - w_+^{-1}) + \ln(1 - \alpha w_-)\right]$$
$$= \ln(w_+ P) - w_- - \frac{1-\alpha}{\alpha}\ln(1 - w_+^{-1})$$
$$= \frac{1}{\alpha}\ln(w_+ P) - w_- - \frac{1-\alpha}{\alpha}\ln((w_+ - 1)P)$$
$$= \frac{1}{\alpha}\ln(w_+ P) - w_- + \frac{1-\alpha}{\alpha}\ln(1 - w_-). \quad (40)$$

Both series in brackets converge because $w_+ > 1$ and $\alpha w_- < 1$: in fact,

$$(w_+ - 1)(\alpha w_- - 1) = 2 - w_+ - \alpha w_-$$
$$= -\frac{(w_+ - 1)^2}{w_+} < 0. \quad (41)$$

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