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Recursive Thinking and Solving Methods

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Recursive thinking is iterative, self-referential, and building on itself continuously. Moreover, it is becoming a more prominent feature of the mathematical scope because of the availability of computers and languages like Logo, Excel, and Pascal that support recursion. This study investigates the way to create students' recursive thinking in mathematics classroom and to use various methods to solve problems using a spreadsheet, the Excel program where technology could be accessible.

Keywords: recursive thinking, problem solving, self-referencing.
ZDM classification: D53, U73
MSC2000 classification: 97D50, 97U70

INTRODUCTION

Problems and solving strategies of discrete mathematics have rapidly developed in mathematics education by combining with the increased computational power of technology. Discrete topics are algorithmic thinking, critical reasoning, counting, discrete probability, matrices, graph theory, and recursion (Dossey 1991). Standards (NCTM 1989, p. 176) indicates the purposes of discrete mathematics as follows.

In grades 9–12, the mathematics curriculum should include topics from discrete mathematics so that students can do the followings:

- Represent problem situations using discrete structures such as finite graphs, matrices, sequences, and recurrence relations;
- Represent and analyze finite graphs using matrices;
- Develop and analyze algorithms;
- Solve enumeration and finite probability problem.

Of discrete topics, recursion is a powerful mathematical progress for generating sequences whenever we have enough information at step $n-1$ to give us knowledge of step *n* (Pirie & Kieren 1989; Quesada 1999). Flusser (1929) mentioned that solving recursive relations is an important part of discrete mathematics and it should be taught in the middle grades.

Mingus & Grassl (1998) define the terms of algorithmic thinking and recursive thinking:

> Algorithmic thinking is a method of thinking and guiding thought processes that uses step-by-step procedures, requires inputs and produces outputs.
>
> Recursive thinking is a never ending, step-by-step method of thinking in which each step is dependent on the step or steps that come immediately before it.

Miller (1990) indicated that the use of concrete materials and pictorial representations allows middle school and high school students to investigate informally the progress of recursion. Also, teaching a few techniques of recurrence can lead students into areas of surprise and delight where they can ask, and maybe answer, some significant questions. Kieren & Pirie (1991) list the important characteristics of recursive thinking as the followings:

- The process is self-referencing.
- Each level of the process builds on the previous level and is then incorporated as a condition called on later to serve as a foundation (p. 79).

Many educators have recognized technology's potential and effects on the future of mathematics education (Kaput 1992; NCTM 1989, 1991, 2000; Papert 1988, 1993). Willoughby (2000) mentioned that all people should have the ability to use technology intelligently so that people might decide which technology is most appropriate to a given situation. However, the most troublesome factor advisedly affecting the impact of computer-related technologies in K-12 schools lies in the fact that most students have had little experience in logical thinking applications where many problems need complex calculations. Regarding this situation, Hart, Maltas & Rich (1991) suggested that one effective way to teach and use recursion is to use computer through programming languages. As a result, recursive thinking is becoming a prominent feature of the mathematical scope because of the availability of computers and languages like Logo, Excel, and Pascal that support recursion.

For many problems, the use of recursion makes it possible to solve complex problems using computer programs that are surprisingly concise, easily understood, and algorithmically efficient. For example, Dence (1993) indicated that to determine the value pi (π), many approaches were known long before to the mathematicians of antiquity, specially to Archimedes (287-232 BC). Shilgalis (1989) describes Archimedes' technique of bracketing pi between sequence values from inscribed and circumscribed regular polygons was described by. Also, the study of central polygonal numbers is an extension of students' previous experience with polygonal numbers and also serves as a good follow-up to the study of recursion. In this paper, other types in which recurrence relations occur and many kinds of solving method are investigated.

RECURRENCE RELATIONS AND SOLVING METHOD

The individual items in the list are called terms of the sequence, an infinite ordered list. An equation relating a general term to term that precede it is called recurrence relation. The equation

$$n! = \begin{cases} 1 & \text{for} \quad n = 0 \\ n(n-1)! & \text{for} \quad n \geq 1 \end{cases}$$

is an example of recurrence relation. In order to determine the values of the terms in a recursive sequence, the values of a specific set of terms in the sequence must be indicated, usually the beginning terms. The assignment of values for these terms gives a set of initial conditions for the sequence. In the case of factorial of n, a single initial condition is $0! = 1$.

In this paper, the type of recurrence relations and various kinds of solving method focused on the sequence of 'Mathematics I' curriculum are illustrated. In order to determine the values of terms in a sequence defined by an initial term, other condition, and in a recursively defined sequence, there are shown two ways such as solving algebraically as well as appropriate process using Excel program.

Example 1. Arithmetic Progress

An arithmetic progress is a sequence in which the difference of any two consecutive terms is a constant. The constant differenced between two consecutive terms is called the common difference.

For all $n, A_{n+1} - A_n = d$.

The nth term of an arithmetic sequence with common difference d is given by the formula $A_n - A_1 + (n-1) \times d$.

Problem: Write the 10th term of the sequence and formula whose first term is 2 and whose common difference is 4.

a. Solving algebraically

$A_1 = 2$ and $d = 4$

$A_n = A_1 + (n-1) \times d = 2 + (n-1) \times 4 = 4n - 2$

Therefore, we get the 10th term of sequence, $4 \times 10 - 2 = 38$.

b. Solving with Excel program

Enter 2 in cell B2 and the formula (B2+4) in cell B3.
Replicate this formula down column *B*.

column / row	A	B
1	*n*	*s(n)*
2	1	2
3	2	6
4	3	10
5	4	14
6	5	18
7	6	22
8	7	26
9	8	30
10	9	34
11	10	38

We get the value of the 10th term of sequence (cell of B11), 38.

Example 2. Geometric Sequence

A geometric sequence is a sequence in which the ratio of any two consecutive terms is a constant. The *n*th term of a geometric sequence with common ratio *r* is given by the formula $A_n = A_1 \times r^{n-1}$.

Problem: Write a formula for the *n*th term of a geometric sequence whose first term is 1 and whose common ratio is 2. Then write the 10th term of this sequence.

a. Solving algebraically

Finding the *n*th term when $A_1 = 1$ and $r = 2$; $A_n - A_1 \times r^{n-1} = 1 \times 2^{n-1}$.
Now evaluate $1 \times 2^{n-1}$ for $n = 10$. The term is 512.

b. Solving with Excel program

Enter 1 in cell B2 and the formula (B2×2) in cell B3.
Replicate this formula down column B.

N	s(n)
1	1
2	2
3	4
4	8
5	16
6	32
7	64
8	128
9	256
10	512

We get the value of the 10th term of sequence (cell of B11), 512.

Example 3. First order linear difference equations

$S_n = a \times S_{n-1} + b$ where a and b are constants and $a \neq 0$.
The first few terms of the sequence defined by this equation are

$$R_0 = R_0,$$
$$R_1 = a \times R_0 + b,$$
$$R_2 = a \times R_1 + b = a(a \times R_0 + b) + b = a^2 \times R_0 + a \times b + b,$$
$$R_3 = a \times R_2 + b = a(a^2 \times b + a \times b + b) + b = a^3 \times R_0 + a^2 \times b + a \times b + b,$$
$$R_4 = a \times R_3 + b = a(a^3 \times R_0 + s^2 \times b + a \times b + b) + b = a^4 \times R_0 + a^3 \times b + a^2 \times b + a \times b + b,$$

It appears that $R_n = a^n \times R_0 + a^{n-1} \times b + a^{n-2} \times b + \cdots + a^2 \times b + a \times b + b$
$= a^n \times R_0 + (a^{n-1} + a^{n-2} + b + \cdots + a^2 + a + 1) \times b$

Finally we get the following form:

$$R_n = a^n \times R_0 + \{(a^n - 1)/(a - b)\} \times b$$
$$= a^n \times R_0 + a\{b/(a - b)\} - \{b/a - 1)\}$$
$$= a^n \times (R_0 + c) - c,$$

where $c = \{b/a - 1)\}$.

Problem 1: The Towers of Hanoi
Given three pegs and several disks initially stacked from largest to smallest on the left

peg. How many moves are needed to transfer n disks foam one peg to another? A "move" is shifting a single disk from one peg to another, under the restriction that no disk may be placed on a disk of smaller diameter. "How many" means the least number of moves generated by any algorithm for making the transfer. For example, how many moves are needed to transfer 10 disks foam one peg to another?

The rules are simple:

1. Our goal is to move the entire tower to the middle peg.
2. We can only move one disk at a time.
3. We can never place a larger disk on a smaller one.

Let H_n denote the number of moves needed to transfer n disks.

$$H_1 = 1.$$

If $n = 2$, $H_2 = 1 + 1 + 1 = H_1 + 1 + H_1$

A similar analysis shows that $H_3 = H_2 + 1 + H_2$.

The analysis suggests the for n disks on a post we can transfer the top $n-1$ disks to a second post, which requires H_{n-1} moves, move the bottom disk to the third post, and using H_{n-1} moves, transfer the stack of $n-1$ disks onto the third post.

Thus $H_n = H_{n-1} + 1 + H_{n-1} = 1 + 2H_{n-1}$

a. Solving algebraically

$$\begin{aligned}
&H_1 = 1\\
&H_2 = 2(1) + 1 = 2 + 1\\
&H_3 = 2(2+1) + 1 = 2^2 + 2 + 1\\
&H_4 = 2(2^2 + 2 + 1) + 1 = 2^3 + 2^2 + 2 + 1\\
&H_5 = 2(2^3 + 2^2 + 2 + 1) + 1 = 2^4 + 2^3 + 2^2 + 2 + 1\\
&\qquad\vdots
\end{aligned}$$

From these calculations we can guess an explicit formula for H_n:

$$H_n = 2^{n-1} + 2^{n-2} + \cdots + 2^2 + 2 + 1.$$

By using an algebraic identity, $1 = x + x^2 + \cdots x^n = \{(x^{n+1} - 1)/(x-1)\}$,

we get this formula:

$$H_n = \{(2^n - 1)/(2-1)\} = 2^n - 1.$$

When the number of disks is 10, we get the number of the minimum movement, $2^{10} - 1$, 1023.

b. Solving with Excel program

Enter 1 in cell B2 and the formula (B2 + B2 + 1) in cell B3.

Replicate this formula down column B.

n	$s(n)$
1	1
2	3
3	7
4	15
5	31
6	63
7	127
8	255
9	511
10	1023

We get the value of the 10th term of sequence (cell B11), 1023.

Problem 2: Annuity bank interest

When Linda deposits a series of equal annual payments in an account, she is creating an annuity. If she deposits \$100 each year in an account that offers 7 percent yearly interest, what is the value of the annuity after 10 years?

The following chart shows how much is in the annuity at the beginning of each year for three years:

Year	1	2	3
Calculation	\$100	{100 + (1.07)\$100}	{100 + (1.07)\$207}
Amount	\$100	\$207	\$321.49

a. Solving algebraically

If A(n) let the amount in the annuity at the beginning of the nth year, the recursive definition of A,

$$A_n = 100 \qquad \text{if} \quad n = 1$$
$$A_n = 100 + (1.07)A_{n-1} \qquad \text{if} \quad n > 1.$$

Using the formula, $A_n = A_1 + B_1\{r^{n-1} - 1\}/(r-1\}$, we get the nth term sequence,

$$A_n = 100 + 107\{1.07^{n-1} - 1\}/(1.07 - 1\}.$$

When the term is 10, we get the value of the 10th term of sequence,

$$100 + 107\{1.07^{10-1} - 1\}/(1.07 - 1\} = 1381.6.$$

b. Solving with Excel program

Enter 100 in cell B2 and the formula (100+107×B2) in cell B3. Replicate this formula down column B.

n	*s(n)*
1	100
2	207
3	321.49
4	443.9943
5	575.073901
6	715.329074
7	865.402109
8	1025.98026
9	1197.79887
10	1381.6448

We get the value of the 10th term of sequence (cell B11), 1381.6448.

Example 4. Second-order homogeneous linear difference equations

$$S_n = a \times S_{n-1} + b \times S_{n-2} \quad \text{for} \quad n \geq 2$$

having initial values S_0 and S_1. Let r_1 and r_2 denote the roots of the equation

$$x^2 = ax + b$$

Then:

(1) If $r_1 \neq r_2$, there exist constants c_1 and c_2 such that

$$S_n = c_1 \times r \times 1^n + c_2 \times r \times 2^n \quad \text{for} \quad n = 0, 1, 2, \ldots$$

(2) If $r_1 = r_2 = r$, there exist constants c_1 and c_2 such that

$$S_n = (c_1 + n \times c_2) r^n \quad \text{for} \quad n = 0, 1, 2, \ldots$$

Problem: Fibonacci sequence

Find a formula expressing the *n*th Fibonacci number F_n as a function of *n*.

a. Solving algebraically

The recurrence relation satisfied by the Fibonacci numbers is

$$F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 3,$$

a second-order homogeneous linear difference equation with constant coefficients. Its auxiliary equation is

$$x^2 = x + 1.$$

We can find two distinct roots,

$$r_1 = \{(1+\sqrt{5})/2\}^n \text{ and } r_2 = \{(1-\sqrt{5})/2\}^n$$

There are constants c_1 and c_2 such that

$$F_n = c_1\{(1+\sqrt{5})/2\}^n + c_2\{(1-\sqrt{5})/2\}^n.$$

To find the value of c_1 and c_2, we solve the equation:

$$1 = c_1\{(1+\sqrt{5})/2\}^1 + c_2\{(1-\sqrt{5})/2\}^1 \text{ and}$$
$$1 = c_1\{(1+\sqrt{5})/2\}^2 + c_2\{(1-\sqrt{5})/2\}^2.$$

Finally, we get $c_1 = (1/\sqrt{5})$ and $c_2 = (-1/\sqrt{5})$.
Substituting these values into the formula for F_n, above gives

$$F_n = (1/\sqrt{5}) \cdot \{(1+\sqrt{5})/2\}^n - (1/\sqrt{5}) \cdot \{(1-\sqrt{5})/2\}^n.$$

b. *Solving with Excel program*

Enter 1 in both cell B2 and B3, and the formula (B2 + B3) in cell B4. Replicate this formula down column B.

n	*s(n)*
0	0
1	1
2	1
3	2
4	3
5	5
6	8
7	13
8	21
9	34
10	55

We get the value of the 10th term of sequence (cell B11), 55.

CONCLUSION

It is essential that students see mathematics as a natural, enjoyable, and realistic part of their lives. Discrete mathematics is active, exciting, and efficient mathematics in school mathematics. Moreover, the radical change in accessible technology has changed people's perspectives toward the role of mathematics. The use and integration of computer-related technology might create discrete models more powerfully and use them to answer interesting questions.

Curriculum in Korea includes sequence in 'Mathematics I' textbook and requires students to find the *n*th term equation and the sum of the terms from 1 to *n* term by algebraic way rather than using calculator or computer-related technology for complex calculations. However, finding the nth term equation of a given sequence by hand has significant limitation to solve it and to generalize the sequence. Rather, students might use technology in calculation and generalization regarding sequences and get positive motivation by achieving successfully.

In this paper, various types of recurrence relations such as arithmetic sequence, geometric sequence, first-order linear difference equation, and second-order linear difference equation focused on the sequence of 'Mathematics I' curriculum are illustrated. In addition, several solving methods such as algebraic way and using Excel program are introduced.

The concept of a recursive function underlies these models, and this concept is accessible even to young students. In addition to use of calculator, a spreadsheet or a simple computer program makes the investigation even more dramatic and fun. Therefore, students might be motivated naturally to learn mathematics necessary to continue the investigation. Because students can be relived of the aspect of horrendous computational tasks by technology, recurrence relations will provide a richer curriculum to help students get logical thinking and better-prepared students.

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