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Application of the Boundary Element Method to FInite Deflection of Elastic Bending Plates

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Abstract : The present study deals with an approximate integral equation approach to finite deflection of elastic plates with arbitrary plane form. An integral formulation leads to a system of boundary integral equations involving values of deflection, slope, bending moment and transverse shear force along the edge. The basic principles of the development of boundary element technique are reviewed. A computer program for solving for stresses and deflections in a isotropic, homogeneous, linear and elastic bending plate is developed. The fundamental solution of deflection and moment is employed in this program. The deflections and moments are assumed constant within the quadrilateral element. Numerical solutions for sample problems, obtained by the direct boundary element method, are presented and results are compared with known solutions.

Key words : elastic plate, boundary integral equation, fundamental solution, boundary element method

1. Introduction

The boundary element method has become an important analysis tool for obtaining approximate solutions of engineering problems. It is a relatively flexible and diverse method that has a number of distinctive features which make it superior to most other methods. One of the main attractions of boundary element methods is the ease with which they can be applied to problems involving geometrically complicated shapes. This study deals with the boundary integral theory of plate bending and some of its applications using boundary elements. A large number of research works [3, 5, 6, 8, 10, 13] has been reported in the area of boundary element of plate-flexure problems such as plate vibrations, large displacement analysis, stress concentration problems, plates on elastic foundations, sandwich plate bending application etc. The first application of boundary element method to plate bending problems appears to be due to Jawon and Maiti [7]. They used an indirect formulation in which the corners were rounded off to avoid numerical instabilities. Maiti and Chakraborty [9] analyzed simply supported polygonal plates subjected to uniform loading including the effect of corners. Alterio and Sikarskie [1] proposed the use of solutions other than Green's function in an unbounded domain. These formulations make use of source distribution densities and not the natural variables of plate flexure problems. One of the early proposals for treating plate bending problems by direct boundary element method is due to Stern [12]. He considered a direct treatment for the general case of finite plates with arbitrary boundary conditions. Paris and Leon [11] proposed an alternative procedure in which the biharmonic equation was uncoupled into two harmonic equations, the problem thus being reduced to the integral treatment of Posson's equation. However, their method involves domain discretization. Recently, Ameen [2] has solved the plate problem using the direct method. The elements are quadrilateral of the boundary which are determined at corner point. This study is primarily concerned with the direct method in plate problems. The method hinges here on the Rayleigh-Green identity, which is the analogue of Green's second identity and Betti's theorem. The governing equations are derived starting from the equations of theory of elasticity. The reciprocal theorem due to Betti is used to arrive at the integral equation formulation of the problem. The complete set of fundamental solutions corresponding to a concentrated load and a concentrated couple in rectangular Cartesian coordinates which is not very convenient for general polygonal plates. Numerical solutions for

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sample problems, obtained by the direct boundary element method, are presented.

2. Governing Differential Equations

The middle plane of the undeformed plate is the x_1, x_2 is the deflection of the middle plane in the x_3 direction as indicated in Fig. 1. The applied forces are per unit area inside the plate and per unit of length along Γ. The thickness of the plate ish h. The governing differential equation of a plate with elastic modulus E and Poisson's ratio μ can be expressed in terms of the lateral displacemenet of the plate middle plane as

$$\nabla^4 w(x_1, x_2) = \frac{1}{D} p(x_1, x_2) \quad \text{in} \quad \Omega \tag{1}$$

where $\nabla^4 = \frac{\partial^4}{\partial x_1^4} + 2\frac{\partial^4}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4}{\partial x_2^4}$ in x_1 and x_2,

$D = \frac{Eh^3}{12(1-\mu^2)}$ is the flexural rigidity of the plate and $p(x_1, x_2)$ is the vertical load. Since Eq.(1) is a fourth order partial differential equation in w, there must be four boundary variables at any boundary point.

Essential boundary conditions are

$$w = \bar{w}$$

$$\theta_n = \bar{\theta}_n$$

$$\theta_t = \bar{\theta}_t \quad \text{on} \quad \Gamma_1$$

Natural boundary conditions are

$$V = \bar{V}$$

$$M_n = \bar{M}_n$$

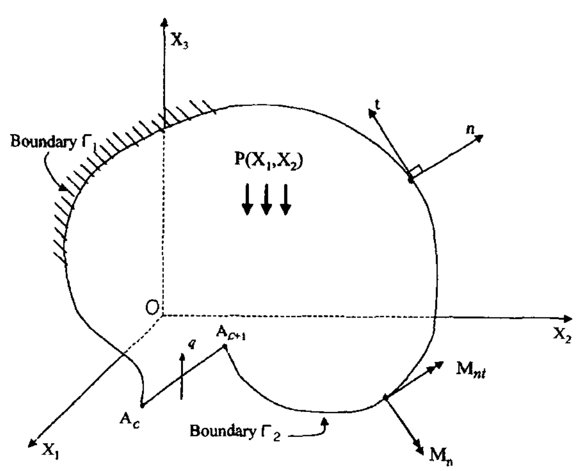

Fig. 1. Flat plate and coordinate system.

$$M_{nt} = \bar{M}_{nt} \quad \text{on} \quad \Gamma_2$$

where $\theta_n \left(= \frac{\partial w}{\partial n}\right)$ and $\theta_{nt} \left(= \frac{\partial w}{\partial t}\right)$ are the normal and tangential components of slope to the boundary. At a point (x_1, x_2) of the boundary Γ with the outward unit normal $n = [n_1, n_2]$ and the tangent $t = [t_1, t_2]$ four quantities are defined.

3. Boundary Integral Formulations

This formulation is based on Green's theorem. Consider a thin plate whose interior is represented by Ω and the boundary by Γ subjected to two distinct sets of loadings. Let w, θ_i, M_{ij}, V and p indicate the transverse deflection of the plate, the slopes, the moments, the shears and the surface loading on the plate respectively corresponding to the given set of loading. Let w^*, θ_i^*, M_i^*, V^* and p^* represent the set of corresponding quantities due to the second set of loading.

Then, we consider the bilinear form which is symmetric with respect to w and w^* defined as

$$B(w, w^*) = \int_\Omega (\nabla^2 \nabla w^2 w^* + (1-\mu)L(w, w^*))d\Omega \tag{2}$$

$L(w, w^*)$ denotes a differential operator which is given by

$$L(w, w^*) = 2\frac{\partial^2 w}{\partial x_1 \partial x_2}\frac{\partial^2 w^*}{\partial x_1 \partial x_2} - \frac{\partial^2 w}{\partial^2 x_1}\frac{\partial^2 w^*}{\partial^2 x_2} - \frac{\partial^2 W}{\partial^2 x_2}\frac{\partial^2 W^*}{\partial^2 x_1} \tag{3}$$

It is well known that Green's second identity for the biharmonic equation such as Eq. (1) can be expressed as

$$\int_\Omega \nabla^2 w \nabla^2 w^* d\Omega = \int_\Omega w^* \nabla^4 d\Omega + \int_\Gamma \left(\nabla^2 w \frac{\partial w^*}{\partial n} - w^* \frac{\partial \nabla^2 w}{\partial n}\right) d\Gamma \tag{4}$$

where $\frac{\partial w^*}{\partial n}$ denotes the normal derivative on theboundary. Integrating by parts Eq. (2) repeatedly with respect to w^* and using Green's identity, we can obtain [4, 12]

$$B(w, w^*) = \int_\Omega w^* \nabla^4 w d\Omega + \int_\Gamma [w^* \frac{\partial M_{nt}(w)}{\partial t} - \frac{\partial w^*}{\partial n} M_n(w) - w^* \frac{\partial(\nabla^2 w)}{\partial n}]d\Gamma + \int_\Gamma \frac{\partial}{\partial t}[w^* M_{nt}(w)]d\Gamma \tag{5}$$

The boundary torsional moment M_{nt} is combined with the boundary shear q to obtain the so called Kirchhoff's shear as

$$V = q + \frac{\partial M_{nt}}{\partial t} \tag{6}$$

where $\frac{\partial}{\partial t}$ denotes the tangential derivative on the boundary and q is the boundary shear force per unit length. In Eq. (5) M_n and M_{nt} are the differential operators defined on the plate boundary as follows:

$$M_n = -\nabla^2 + (1-\mu)(n_2^2\frac{\partial^2}{\partial x_1^2} + n_1^2\frac{\partial^2}{\partial x_2^2}$$
$$-2n_1 n_2\frac{\partial^2}{\partial x_1 \partial x_2}) \tag{7}$$

$$M_{nt} = -(1-\mu)(n_1 n_2\left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}\right)$$
$$+(n_1^2 - n_2^2)\frac{\partial^2}{\partial x_1 \partial x_2}) \tag{8}$$

It notes that when the boundary is smooth the last integral in Eq. (6) vanishes and this equation results in the so called Rayleigh-Green identity. If the boundary has cn corners, corner force A_c and the coresponding deflection w_c, we must add the following term

$$\int_\Gamma \frac{\partial}{\partial t}[w^* M_{nt}]d\Gamma = \sum_{c=1}^{cn}[w_c^* A_c] \tag{9}$$

to the right hand side of Eq. (5) and calculate the discontinuity jump. Using the symmetric property of the bilinear form $B(w, w*)$ with respect to the arguments, we can derive

$$\int_\Omega (w\nabla^4 w^* - w^*\nabla^4 w)d\Omega$$

$$= \int_\Gamma\left(w^* V_n - \frac{\partial V^*}{\partial n}M_n + M_n^*\frac{\partial V}{\partial n} - V_n^* w\right)d\Gamma \tag{10}$$

As the deflection function w^* we shall use the fundamental solution to Eq.(1), which is governed by

$$\nabla^4 w^*(\zeta, \eta) = \Delta(\zeta, \eta) \tag{11}$$

for an infinite plate made of the same material as that of the plate to be analyzed. Here we denote by ζ and η arbitrary points in the infinite domain and by $\Delta(\zeta, \eta)$ the Dirac delta function. Substitution of Eq. (1) and (11) into Eq. (10) and use of the property of the Dirac delta function lead to

$$w(\eta) + \int_\Gamma (M_n^*\theta_n + V^* w)d\Gamma + \sum_{c=1}^{cn} A_c^* w_c$$

$$= \int_\Omega p w^* d\Omega + \int_\Gamma (M_n\theta_n^* + Vw^*)d\Gamma + \sum_{c=1}^{cn} A_c w_c^* \tag{12}$$

Notice that we have now two unknowns per node, i.e., displacement w or effective shear force V and rotation θ or moment M. Hence we need another equation to solve the problem. This equation is given by finding the derivative of Eq. (1) with respect to the normal, which gives

$$\theta_i(\eta) + \int_\Gamma (\overline{M_n^*}\,\theta_n + \overline{V_i^*}w)d\Gamma + \sum_{c=1}^{cn}\overline{A_c^*}\,w_c = \int_\Omega p\overline{w_i^*}\,d\Omega$$

$$+\int_\Gamma (M_n\overline{\theta_n^*} + V\overline{w_i^*})d\Gamma + \sum_{c=1}^{cn} A_c\overline{w_c^*} \tag{13}$$

where the following notations are used:

$$\frac{\partial w}{\partial n} = \theta_i,\ \frac{\partial M^*}{\partial n} = \overline{M_n^*},\ \frac{\partial V^*}{\partial n} = \overline{V_i^*}$$

$$\frac{\partial A_c^*}{\partial n} = \overline{A_c^*},\ \frac{\partial w^*}{\partial n} = \overline{w_c^*}$$

4. Fundamental solutions

It is well known that the fundamental solution w^* satisfying Eq.(11) is given by

$$w^*(\zeta, \eta) = \frac{1}{8\pi D}r^2 \ln r \tag{14}$$

In the above, w^* represents the transverse displacement at the field point η due to a unit load applied at the source point ζ and r represents the distance between the field point η and the source point ζ. The corresponding slope θ_n^* along a direction n can be obtained as $\theta_n^* = w_{,n}^*$. Thus

$$\theta_n^* = \frac{1}{8(\pi)D}(1 + 2\ln r)r\frac{\partial r}{\partial r} \tag{15}$$

The boundary shear is obtained as

$$q^* = -\frac{1}{2\pi r}\frac{\partial r}{\partial n} \tag{16}$$

Hence, the Kirchhoff shear is

$$V^* = q^* + \frac{\partial M_{nt}^*}{\partial t}$$

$$= -\frac{1}{4\pi r}\left(2 + (1-\mu)\left(1 - 2\frac{\partial r}{\partial t}\frac{\partial r}{\partial t}\right)\right)\frac{\partial r}{\partial n} \tag{17}$$

The fundamental solution due to a unit moment is given by [4]

$$\overline{w_i^*} = \frac{1}{4\pi D} r \ln r \tag{18}$$

The above solution bar $\overline{w_i^*}$ represents the deflection at a field point η due to a unit moment applied at the source point ζ with the moment vector directed along x_i. The slope at a boundary point whose outward unit normal vector n is obtained as

$$\overline{\theta_n^*} = \frac{\partial \overline{w_i^*}}{\partial n} = \frac{1}{4\pi D}(n_i \ln r + r_{,i} r_{,n}) \tag{19}$$

The bending moment bar $\overline{M_n^*}$ at the field point can be obtained as

$$\overline{M_n^*} = -\frac{1}{4\pi r}[(1-\mu)(2r_{,n}n_i - 2r_{,i}r_{,n}r_{,n}) + (1+\mu)r_{,i} \tag{20}$$

The boundary shear is

$$\overline{V_i^*} = \frac{1}{4\pi r^2}[n_i(2(1-\mu)r_{,t}r_{,t} - 3 + \mu) + (6-2\mu)r_{,i}r_{,n}$$
$$+4(1-\mu)r_{,n}r_{,t}t_{,i} - 8(1-\mu)r_{,i}r_{,n}r_{,t}r_{,t}] \tag{21}$$

In the case of a uniformly distributed load, the domain integral present in Eq. (2) is

$$\int_\Omega p w^* d\Omega = \frac{p}{128\pi r}\int_\Gamma (4\ln r - 1) r^3 r_{,n} d\Gamma \tag{22}$$

$$\int_\Omega p \overline{w_i^*} d\Omega = \frac{p}{32\pi D}\int_\Gamma r^2 n_k(\ln r(\Delta_{ik} + 2r_{,k}r_{,i})$$
$$-\left(\frac{3}{4}\Delta_{ik} + \frac{1}{2}r_{,k}r_{,i}\right))d\Gamma \tag{23}$$

5. Boundary Discretization

Now, since the source point and the field point never coincide, all the above integrals are nonsingular and hence can easily be evaluated numerically. The boundary of the domain is discretised into a number of boundary elements. The transverse displacement, the normal boundary slope, the shear, the bending moment and the boundary geometry are all interpolated by using chosen interpolation polynomials. Supposing that for the discretization of the plate, n_e elements on Γ are considered and applying the Eq. (12) and (13) on the whole boundary, we get as

$$cw(\eta) + \sum_{e=1}^{n_e}\int_\Gamma (M_n^*\theta_n + V^* w - M_n\theta_n^* - Vw^*)d\Gamma + \sum_{c=1}^{cn} A_c^* w_c$$

$$= \sum_{e=1}^{n_e}\int_\Omega p w^* d\Omega + \sum_{c=1}^{cn} A_c w_c^* \tag{24}$$

and

$$cw(\eta) + \sum_{e=1}^{n_e}\int_\Gamma (\overline{M_n^*}\theta_n + \overline{V_i^*}w - M_n\overline{\theta_n^*} - V\overline{w_i^*})d\Gamma$$

$$+\sum_{c=1}^{cn}\overline{A_c^*}w_c = \sum_{c=1}^{n_e}\int_\Omega p\overline{w_i^*}d\Omega + \sum_{c=1}^{cn} A_c\overline{w_c^*} \tag{25}$$

The interior of the domain is discretized into panel, at the n_p nodal points of which we define the value w and that of the equivalent transverse load. By performing a static condensation for a domane with n_e nodes with simple mathematical manipulations, we may write with the Eq. (24) and (25) in matrix form as

$$[H]\{D\} = [G]\{P\} + \{F\} \tag{26}$$

where
$$[H] = \begin{pmatrix} h_{1,1} & h_{1,2} & h_{1,2n_e} \\ h_{2,1} & h_{2,2} & h_{2,2n_e} \\ .. & .. & .. \\ h_{2n_e,1} & h_{2n_e,2} & h_{2n_e,2n_e} \\ h_{2n_e+1,1} & h_{2n_e+1,2} & h_{2n_e+1,2n_e} \\ .. & .. & .. \\ h_{2n_e+cn,1} & h_{2n_e+cn,2} & h_{2n_e+cn,2n_e} \end{pmatrix},$$

$$[G] = \begin{pmatrix} g_{1,1} & g_{1,2} & .. & g_{1,2n_e+cn} \\ g_{2,1} & g_{2,2} & .. & g_{2,2n_e+cn} \\ .. & .. & .. & .. \\ g_{2n_e,1} & g_{2n_e,2} & .. & g_{2n_e,2n_e+cn} \\ g_{2n_e+1,1} & g_{2n_e+1,2} & .. & g_{2n_e+1,2n_e+cn} \\ .. & .. & .. & .. \\ g_{2n_e+cn,1} & g_{2n_e+cn,2} & .. & g_{2n_e+cn,2n_e+cn} \end{pmatrix},$$

$$\{D\} = \{w_1\theta_1 w_2\theta_2 ... w_{n_e}\theta_{n_e}\}^T,$$

$$\{P\} = \{V_1M_1V_2M_2...V_{n_e}M_{n_2}A_1A_2...A_c\}^T$$

$$\{F\} = \{f_1 f_2 f_3 .. f_{n_p}\}^T$$

$[H]$ is $2n_e + cn$ by $2n_e$ rectangular matrix and $[G]$ is $2n_e + cn$ by $2n_e + cn$ square matrices whose coefficients stem from the curvilinear integrals of Eq. (12) and (13). $\{F\}$ is a column vector whose n_p components are the concentrated loads equivalent to the transverse load per unit area, at the n_p nodal points inside the domain Ω

After reordering the Eq. (26), we obtain the final form as

$$[A]\{X\} = \{B\} \tag{27}$$

where $[A]$ is a fully populated rectangular matrix and $\{X\}$ is formed by the unknown displacemenets and slopes. The contributions of the prescribed values are included in vector $\{B\}$. Eq. (27) may now be solved to yield all remaining unknown displacemenets and slopes on the boundary.

6. Numerical Implementation

To investigate the validity of the proposed equations and calculation procedure, two distinct problems pertaining to rectangular plates are considered.

The whole plate edge is either clamped or simply supported. In each case Poisson's ratio μ is taken to be 0.3 and all our results pertain to a rectangular plate loaded in a state of uniformly distributed load of 10 psi. Due to the symmetry of the problem only one quarter of the plate is considered in the analysis. The boundary of the quarter plate is discretized into 12 elements and we compute the moments and slopes on all boundaries of plate. Since we will verify numerical solutions of this computational model against analytical solutions, only moment with the simply supported boundary condition was considered in the x–y coordinate system. The Fourier series solution M_x is obtained as

$$M_x(x, y) = \frac{16p}{\pi^4}\sum_m^\infty \sum_n^\infty \frac{(m/a)^2 + \mu(n/b)^2}{mn[(m/a)^2 + (n/b)^2]^2} \sin\frac{m\pi x}{a} \sin\frac{m\pi y}{b}$$

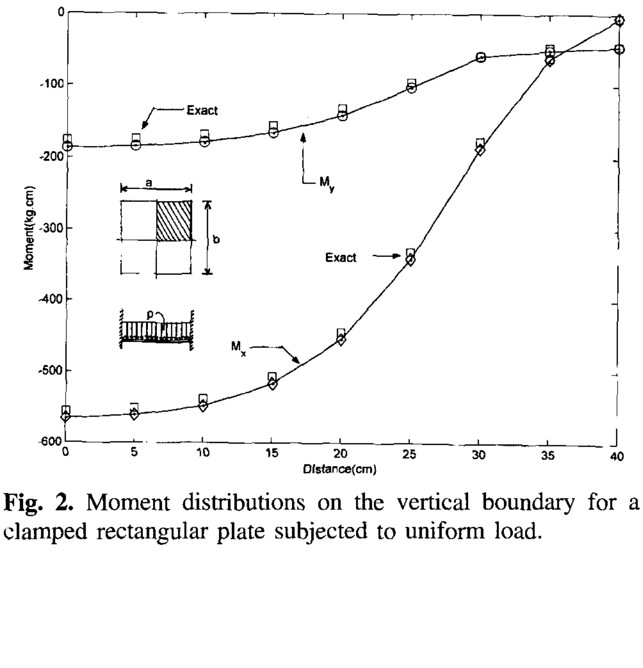

Fig. 2. Moment distributions on the vertical boundary for a clamped rectangular plate subjected to uniform load.

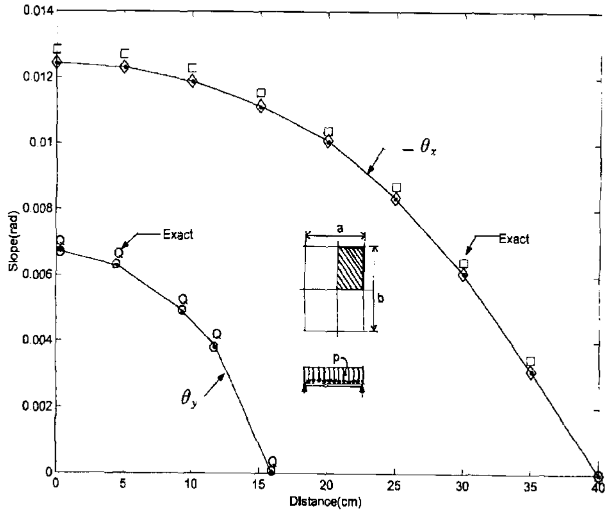

Fig. 3. Slope distributions on the side boundary for a simply supported rectangular plate subjected to uniform load.

Comparing the results with the analytical solution, maximum moment at the center of vertical boundary is found to be within 2.3 percentage of the analytical solution. Maximum slope, which occur at the center of vertical boundary, gives 2.6 percentage error for this dicretization. The numerical results from the boundary element method show good agreements with the exact solutions as shown in in Fig. 2 and 3. The boundary element method produces results that are much more accurate for the solution of plates.

7. Conclusions

The boundary element method can be applied to solve finite deflection of elastic bending plate accurately. Integral equation formulations based on the von Karman theory has been described. This formulation gives a coupled system of field equations which is reduced to an uncoupled system for the latter. For the problems dealing with the large domains and loaded in a relatively small area, the boundary element method offers a very convenient and accurate method for analysis. The convenience is in the input data preparation. The method is applicable to various problems in the area of foundation engineering where large mass of soil medium are to be considered.

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