韓國國防經營分析學會誌
제 29 권, 제 2 호, 2003. 12. 31.

Graphical Approaches for Planning Experiments-High Resolution Linear Graphs for Three-Level Designs

Sang-Heon Lee*

Abstract

The lack of consideration for statistical properties in Taguchi's three-level linear graphs is rectified. We propose a new set of linear graphs for the three-level orthogonal arrays according to the maximum resolution criterion. In the presence of two-factor interactions however, the serious bias of all the estimated effects as well as the estimated variance shows that these designs should not be employed. The various alternative designs are discussed.

* Korean National Defense University

1. Introduction

Most of the underlying physical mechanisms advocated by Taguchi (1980, 1987) are such that the effects of factors could be approximated by an additive model. Taguchi's strategy for approximating a response surface by a polynomial model is to choose the response and control factors continuously to avoid the need to allow for interactions. That is, Taguchi's designs fundamentally sacrifice informations about interactions to reduce the number of trials. In doing so, he recommends taking more than two-levels for continuous valued factors to study quadratic effects but ignores interactions.

A very limited number of interactions can be studied in some orthogonal arrays. The only possible standard orthogonal arrays which allow estimation of interactions are L18, L'36, L54, and which are originally due to Burman (1946) and Seiden (1954). The frequent recommendation of using an L18 by Taguchi is based on the ability to estimate interaction effect. However, the L18 orthogonal array only allows estimation of only one interaction effect between factors in the first column (two-level factor) and second column (one of the seven three-level factors). Likewise, the L'36 allows three interactions between the fourth column (one of the thirteen three-level factors) and first three columns (two-level factors), and L54 allows only one interaction between the first column (two-level factors) and second column (one of the twenty five three-level factors). Each of the other interactions is partially confounded with several main effects.

The study of interaction is the essential subject of fractional factorial plans. The presence or absence of interactions can have profound impacts in product or process design. It is necessary to construct plans which permit uncorrelated estimation of all main effects as well as orthogonal estimation of two-factor interaction effects.

Orthogonal array approach however, is performed poorly when an experimenter considers models with second-order components. For the 2n-m fractional factorial designs by orthogonal arrays if quadratic terms exist, the inability to estimate quadratic coefficients does not bias either the first-order effects or two-factor interaction effects since they bias only constant.

However unlike two-level disigns, 3^{n-m} fractional factorial designs by orthogonal arrays fail to obtain unbiased estimates for all coefficients. Most of the three-level fractional factorials, 3^{n-m} designs, by orthogonal arrays are Latin square type desins. Those are Latin square, Graeco-Latin square, and hyper-Graeco-Latin square designs. For example, an L_9 which is equivalent to the 3^{4-2} fractional factorial design is obtained from a Graeco-Latin square. As discussed in detail by Hunter(1985) however, the fractional factorials from Latin square type designs seriously bias all the estimated effects as well as the estimate of variance if two-factor interaction effects exist.

Orthogonal array approach in planning fractional factorial designs are not all misleading. The orthogonal array approach is basically different from the response surface methodology(RSM). The components of RSM assume that an underlying model include system curvature and interactions so that an experimenter design accordingly. The orthogonal approach often benefits from a main effect model to determine the best settings of factors that is most likely to produce product or process improvement, but not estimating the model. The main emphasis is that an experimenter must choose a right tool among experimental designs carefully by considering relative importance differs in different circumstances and tailor the design to fit the required experimental situations.

This paper provides improved linear graphs for the three-level designs under the framework of Taguchi's basic approach and describe better alternatives. By doing so, we have drawn not only improved Taguchi's three-level orthogonal array's linear graphs but also on those of the better alternatives. In section 2, we present the high resolution linear graphs for the three-level designs as opposed to Taguchi's in an uncritical way, following to the orthogonal array approaches for the two-level designs. In section 3, we briefly overview better alternatives.

2. High Resolution Linear Graphs for Three-Level Designs

2.1 L27(3^{13}) Orthogonal Array

A three-level fractional factorial orthogonal array with 3n rows, where n=2,3,4,..., can be construted in a similar way to the two-level fractional arrangements(Lee, 2001). The main difference between two-level and three-level orthogonal array is the two column appearance in three-level orthogonal array for the two-factor interaction between two factors.

The $a \times b$, which denotes two-factor interactions between factors $\boldsymbol{a}$ and $\boldsymbol{b}$, are assigned into two columns, $\boldsymbol{ab}$ and $\boldsymbol{ab^2}$, with two degrees of freedom each since there are two orthogonal contrasts for each set of three planes; $a+b \equiv 0,1,2 (\mathrm{mod}\ 3)$ and $a+2b \equiv 0,1,2 (\mathrm{mod}\ 3)$. The three-factor interactions $\boldsymbol{a \times b \times c}$ among three factors $\boldsymbol{a, b}$ and $\boldsymbol{c}$ are assigned to four columns, $\boldsymbol{abc}$, $\boldsymbol{ab^2c}$, $\boldsymbol{abc^2}$ and $\boldsymbol{ab^2c^2}$ with a total of eight degrees of freedom. Those four components are corresponding to sets of hyperplanes; $a+b+c \equiv 0,1,2$, $a+2b+c \equiv 0,1,2$, $a+b+2c \equiv 0,1,2$, and $a+2b+2c \equiv 0,1,2$, respectively in modulo-3 arithmetic.

After rearranging the entries 0,1 and 2 in a systematic pattern, we can match the labels of the columns notation with the column numbers of the $L_{27}(3^{13})$ orthogonal array. Thus the generators of the $L_{27}(3^{13})$ orthogonal array. Thus the generators of the $L_{27}(3^{13})$ orthogonal array are $\boldsymbol{d=ab}$, $\boldsymbol{e=ab^2}$ $\boldsymbol{f}$=**ac**, **g**=$\boldsymbol{ac^2}$, $\boldsymbol{h=bc}$, $\boldsymbol{i=abc}$, $\boldsymbol{j=ab^2c^2}$, $\boldsymbol{k=bc^2}$, $\boldsymbol{l=ab^2c}$ and $\boldsymbol{m=abc^2}$. The confounding patterns are as follows:

$$l_a = a + bd^2 + be^2 + cf^2 + cg^2 + de^2 + fg^2 + hi^2 + hj^2 + ij^2 + kl^2 + km^2 + lm^2$$

$$l_b = b + ad^2 + ae + ch^2 + ck + de + fi^2 + fl^2 + gj^2 + gm^2 + hk^2 + il^2 + jm^2$$

$$l_c = c + af^2 + ag + bh^2 + bk + di^2 + dm + ej^2 + el + fg + hk + im + jl$$

$$l_d = d + ab + ae^2 + be + ci^2 + cm^2 + fj^2 + fk^2 + gh^2 + gl^2 + hl^2 + im^2 + jk^2$$

$$l_e = e + ab^2 + ad + bd + cj^2 + cl^2 + fh^2 + fm^2 + gi^2 + gk^2 + hm^2 + ik^2 + jl^2$$

$$l_f = f + ac + ag^2 + bi^2 + bl + cg + dj^2 + dk + eh^2 + em + hm + il + jk$$

$$l_g = g + ac^2 + af + bj^2 + bm + cf + dh^2 + dl + ei^2 + ek + hk + ik + jm$$

$$l_h = h + ai^2 + aj + bc + bk^2 + ck + dg + dl^2 + ef + em^2 + fm + gk + ij$$

$$l_i = i + ah + aj^2 + bf + bl^2 + cd + cm + dm^2 + eg + ek^2 + fl + gk + hj$$

$$l_j = j + ah^2 + ai + bg + bm^2 + ce + cl + df + dk^2 + el^2 + fk + gm + hi$$

$$l_k = k + am + al^2 + bc^2 + bh + ch + df^2 + dj + eg^2 + ei + fj + gi + lm$$

$$l_l = l + ak + am^2 + bf^2 + bi + ce^2 + cj + dg^2 + ah + ej + fi + gh + lm$$

$$l_m = m + am^2 + al + bg^2 + bj + cd^2 + ci + di + ef^2 + eh + fh + gj + kl$$

Table 1 lists all the possible highest resolution that can be obtained for the various requirements set.

For an experiment with more than four main factors, neither the main effects nor the two-factor interactions can be estimated without confoundings. For the case 3, for example, the resulting confounding patterns for five factors *A*, *B*, *C*, *D* and *E* are as follows:

Table 1: Possible Highest Resolution by L_{27}

Case Number	Requirements Set		Possible Highest Resolution
	Number of Main Factors	Maximum Number of 2-Factor Interactions	
1	3	3	V
2	4	3	Ⅳ
3	5	4	Ⅲ
4	6,7	3	Ⅲ
5	8,9	2	Ⅲ
6	10,11	1	Ⅲ
7	12,13	0	Ⅲ

$$l_a = a + bd^2 + be^2 + de^2$$

$$l_b = b + ad^2 + ae + de$$

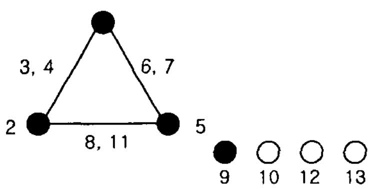
$$l_c = c$$

$$l_d = d + ab + ae^2 + be$$

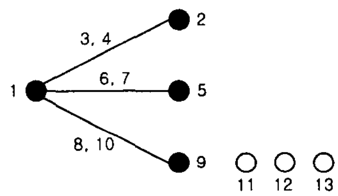
$$l_e = e + ab^2 + ad + bd.$$

Four factors, A, B, D and E are confounded with three components of two-factor interactions, while factor C is not confounded with any. component of a two-factor interaction. Hence the resulting design is of resolution Ⅲ. For the case 2, four main factors with up to three two-factor interactions, linear graphs in Figure 1 can be used in constructing a design of resolution Ⅳ. Note that a solid node indicates a resolution Ⅳ column and a hollow node indicates a resolution Ⅲ column.

For example, consider an experiment with four three-level factors A, B, C, D and three two-factor interactions AB, AC, AD. We can obtain a resolution Ⅳ design by choosing a linear graph in Figure 1(b). By assigning factors A, B, C and D to nodes 1, 2, 5 and 9 respectively, the generator of the design is $D=ABC$ which generate a design of resolution Ⅳ. For case 1, three main factors with up to three two-factor interactions, we have to choose linear graph in Figure 1 (a) and use only triangular part to obtain a resolution V design.



(a)



(b)

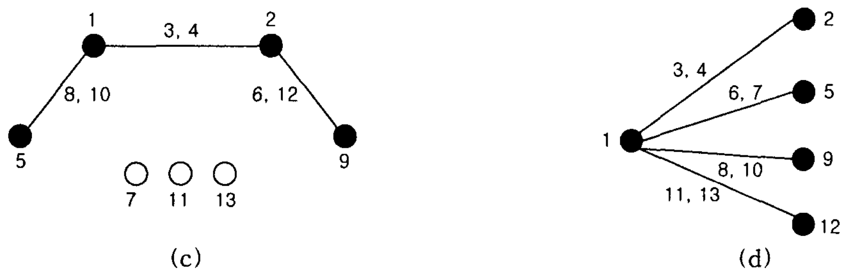


Figure 1. High Resolution Linear Graphs for L_{27}

Note that the same design by L_{27} for studying the main effects of 13 three-level factors could be obtained by constructing a 3^{13-10} design. Addeman's(1962) orthogonal main effect plan for the 3^{13} experiment in 27 trials is differ only slightly from the L_{27} orthogonal array. An almost orthogonal array $L'_{27}(3^{22})$ is another class of array with 27 runs used by Taguchi (1987). This array is constructed with L_{27} by adding nine columns (column 14-22) which are mutually orthogonal and are also orthogonal to columns 1, 11, 12 and 13, so that two separate sources of class could be analyzed by using columns 1-13 and columns 1, 11, 12, and 13-22.

2.2 L81(340) Orthogonal Array

The four-factor interaction $\boldsymbol{a\times b\times c\times d}$ has a total of 16 degrees of freedom and 8 components, $\boldsymbol{abcd}$, $\boldsymbol{ab^2cd}$, $\boldsymbol{abc^2d}$, $\boldsymbol{abcd^2}$, $\boldsymbol{ab^2c^2d}$, $\boldsymbol{ab^2cd^2}$, $\boldsymbol{abc^2d^2}$, and $\boldsymbol{ab^2c^2d^2}$, corresponding to sets of hyperplanes; $a+b+c+d \equiv 0,1,2$, $a+2b+c+d \equiv 0,1,2$, $a+b+2c+d \equiv 0,1,2$, $a+b+c+2d \equiv 0,1,2$, $a+2b+2c+d \equiv 0,1,2$, $a+2b+c+2d \equiv 0,1,2$, $a+b+2c+2d \equiv 0,1,2$, $a+2b+2c+2d \equiv 0,1,2$, respectively in modulo-3 arithmetic.

Thus the generators of the $L_{81}(3^{40})$ orthogonal array are $\boldsymbol{e=ab}$, f=ab2, $\boldsymbol{g=ac}$, $\boldsymbol{h=ac^2}$, $\boldsymbol{i=bc}$, $\boldsymbol{j=abc}$, $\boldsymbol{k=ab^2c^2}$, $\boldsymbol{l=bc^2}$, $\boldsymbol{m=ab^2c}$, $\boldsymbol{n=abc^2}$, $\boldsymbol{o=ad}$, $\boldsymbol{p=ad^2}$, $\boldsymbol{q=bd}$, $\boldsymbol{r=abd}$, $\boldsymbol{s=ab^2d^2}$, $\boldsymbol{t=bd^2}$, $\boldsymbol{u=ab^2d}$, $\boldsymbol{v=abd^2}$, $\boldsymbol{w=cd}$, $\boldsymbol{x=acd}$, $\boldsymbol{y=ac^2d^2}$, $\boldsymbol{z=bcd}$, $\boldsymbol{A=abcd}$, $\boldsymbol{B=ab^2c^2d^2}$, $\boldsymbol{C=bc^2d^2}$, $\boldsymbol{D=ab^2cd}$, $\boldsymbol{E=abc^2d^2}$, $\boldsymbol{F=cd^2}$, $\boldsymbol{G=ac^2d}$, $\boldsymbol{H=acd^2}$, $\boldsymbol{I=bc^2d}$, $\boldsymbol{J=abc^2d}$, $\boldsymbol{K=ab^2cd^2}$, $\boldsymbol{L=bcd^2}$, $\boldsymbol{M=ab^2c^2d}$ and $\boldsymbol{N=abcd^2}$. For an experiment with more than eight factors, neither the main effects nor the two-factor interactions can be estimated without confoundings.

The possible high resolution that can be obtained for the various requirements set is listed in Table 2. A set of thirteen high resolution linear graphs with three to eight factors and up to seven two-factor interactions for an L81 orthogonal array is listed in figure 2. We suggest the following procedures to be effective for using *three-level high resolution linear graphs*.

- Make a complete list for requirements set which are to be estimated.
- Select the smallest orthogonal array that could accommodate a plan
- Select the high resolution linear graph corresponding to the chosen orthogonal array, which can accommodate the required form of two-factor interactions.
- Assign to the solid-nodes of the high resolution linear graph those factors that appear most often in the list of the requirements set and whose interactions are appeared in the requirements set.
- Assign to the free nodes (solid-nodes if available) or free lines of the high resolution linear graph those factors whose interactions are not assigned yet.
- Finally, assign the remaining factors to the free nodes (solid-nodes if available) and lines.

Suppose we want to plan an experimental design that has 7 three-level factors and 7 two-factor interactions among them. The requirements set in this case study is $\{A,B,C,D,E,F,G,AB,AC,AD,AE,BC,BD,CD\}$. The total number of degrees of freedom is $7+7\times2 = 21$. It turns out that L_{27} is insufficient. So L_{81} is the appropriate orthogonal array.

Now note that the desired interaction form is consisted of all interactions among four factors A, B, C and D and one additional interaction AE, which interacts to one of the four interacting factors. In the L_{81} high resolution linear graphs, we have two choices to select a linear graph which could be implemented in a design of resolution Ⅳ with satisfying the required conditions of design. Those are two high resolution linear graphs in Figures 2 (b) and (c).

If we choose a former linear graph (Figure 2 (b)), first assign interacting four factors A,B,C,D to the solid-nodes 1, 2, 5 and 14 in rectangular-shape part. The interaction effects AB, AC, AD, BC, BD and CD are then represented columns (3,4), (6,7), (15,16), (8,11), (17,20) and (23,32), respectively. Now we are ready to assign factor E because the interaction AE is not assigned yet. We assigned the factor E to the solid- node representing column 26. The interaction, AE, is then represented by columns 27 and 28. The remaining factors F and G can be assigned arbitrarily to the remaining nodes but we assign two of the three remaining solid-nodes in order to obtain a design of resolution Ⅳ. Those nodes are 9 and 18. The generators of this design are $E=ABC$, $F=ACD$ and $G=BCD$.

Note that, likewise in the two-level orthogonal arrays, our collection of high resolution linear graphs in 81-runs design are exhaustive to accommodate all design cases listed in Table 2.

Table 2: Possible Highest Resolution by L_{81}

Case Number	Requirements Set		possible highest resolution
	Number of Main Factors	Maximum Number of 2-Factor Interactions	
1	4	6	V
2	5	7	Ⅳ
3	6	7	Ⅳ
4	7,8	7	Ⅳ
5	9,10	15	Ⅲ
6	11,12	14	Ⅲ
7	13,14	13	Ⅲ
8	15,16	12	Ⅲ
9	17,18	11	Ⅲ
10	19,20	10	Ⅲ
11	21,22	9	Ⅲ
12	23,24	8	Ⅲ
13	25,26	7	Ⅲ
14	27,28	6	Ⅲ
15	29,30	5	Ⅲ
16	31,32	4	Ⅲ
17	33,34	3	Ⅲ
18	35,36	2	Ⅲ
19	37,38	1	Ⅲ
20	39,40	0	Ⅲ

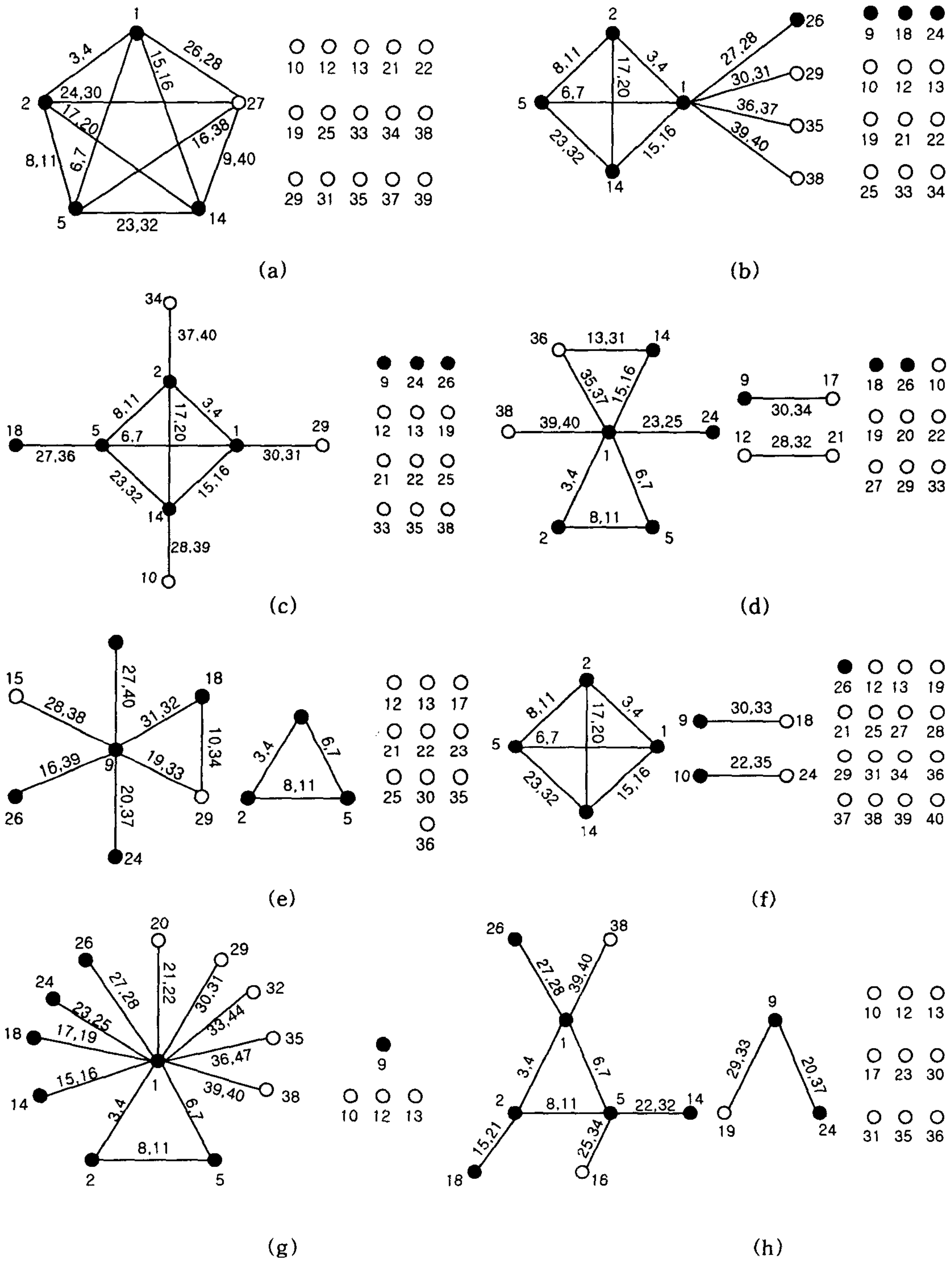


Figure 2. High Resolution Linear Graphs for L_{81}

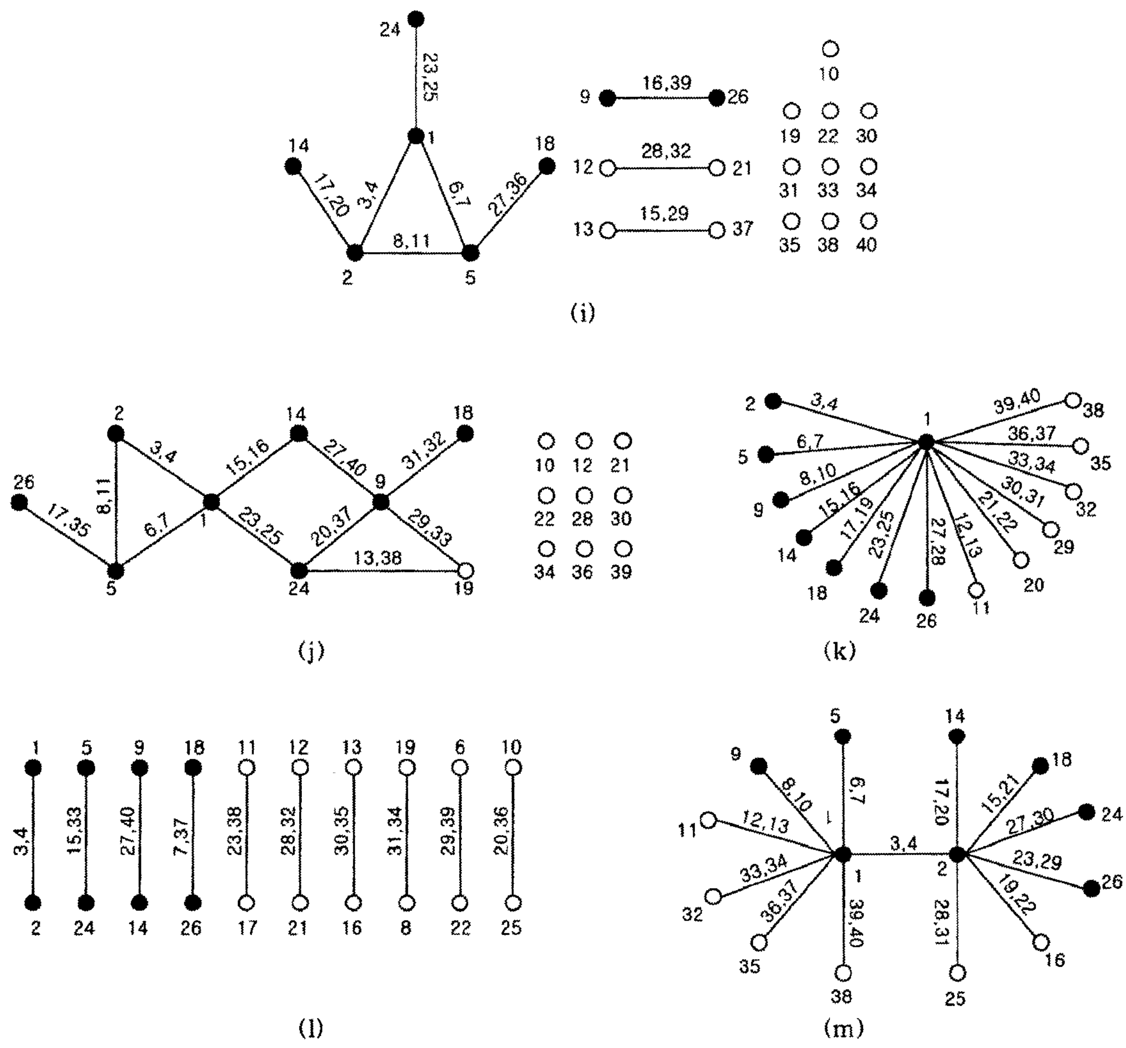


Figure 2. (Continued)

3. Conclusion

Taguchi's linear graphs for the three-level designs have only some limited partial informations for the interaction relationships among columns in the orthogonal arrays. Under the Taguchi's (1987, Volume Ⅱ) framework for constructing three- level designs, we improved his linear graphs by adding highest possible resolution. The orthogonal approach however only concentrates on identifying the factors that reflect most variability. The source of variability is not of primary interest whether it is due to average effects, interactions or curvature. In addition to knowing which factors affect the most variability, it is necessary to learn in which way they contribute.

The response surface methodology is very effective to be able to identify which interactions occur so that the underlying causes can be better understood. In the context of the robust parameter designs, the following features are most recommended for the second-order designs compared to Box and Drapper(1975)

- Ensure the fitting of a full second degree equations in all the factors including quadratic and interactions terms.
- Provide an economical experimentation. A minimum number of experimental runs is required.
- Ensure easy calculation of coefficients and analysis.
- Allow easy generative designs.

In addition, the particular design characteristics are required depending on the experimental circumstances. In general, central composite designs and Box- Behnken designs can be used extensively for the continuous-valued factors. Small composite and hybrid designs are recommended for experimental situations in which cost is an important concern.

References

[1] Box, G.E.P and Behnken, D.W., "Some New Three Level Designs for the Stu of Quantitative Variables", *Technometrics*, 2, pp.455-475, 1960.

[2] Box, M.J. and Draper, N.R., "On Minimum-Point Second-Order Designs", *Technometrics*, 16, pp.613-616, 1974.

[3] Draper, N.R., "Small Composite Designs", *Technometrics*, 27, pp.173-180, 1985.

[4] Draper, N.R., and Smith, H., *Applied Regression Analysis* 2nd ed. New York: Wiley, 1981.

[5] Hartley, H.O., "Smallest Composite Designs for Quadratic Response Surface", *Biometrics*, 15, pp.611-6124, 1959.

[6] Hoke, A.T., "Economical Second-Order Designs Based on Irregular Fractions of the 3n Factorial", *Technometrics*, 16, pp.375-384, 1974.

[7] Hunter, J.S., "Statistical design Applied to Product Design", *Journal of Quality Technology*, 17, pp.210-221, 1985.

[8] John, P.W.M., *Statistical Design and Analysis of Experiments*, The Macmillan Company, New York, 1971.

[9] Koshal, R.S., "application of the Mehod of Maximum Likelihood to the Improvement of Curves Fitted by the Method of Moments", *Journal of the Royal Statistical Society*, A96, pp.303-313, 1933.

[10] Lee, S.H., "High Resolution Linear Graphs : Graphical Aids for Designing Off-line Process Control", *Journal of Military Operation Society of Korea,* Vol.27, No 1, 2001.

[11] Lin, D. and Tu., W. "Dual Response Surface Optimization", *Journal of Quality Technology* 27, pp.34-39. 1995.

[12] Roquemore, K.G., "Hybrid designs for Quadratic Response Surfaces", *Technometrics*, 18, pp.419-243, 1976.

[13] Taguchi, G., Systems of Experimental Design. 2 vols., Translated and published in English by UNIPUB, New York, 1987.

[14] Taguchi, G and Wu, Y., *Introduction to Off-line Quality Control,* Japan, Central Japan Quality Contol Association, 1980.

[15] Westlake, W.J., "Composite Designs Based On Irregular Fractions of Factorials", *Biometrics*, 21, pp.324-336, 1965.