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INTUITIONISTIC FUZZY FUNCTIONS

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Abstract

In this paper, we generally introduce some of the terminology of Yalvac [10] and Azad [4] in intuitionistic fuzzy topological spaces. In addition to the fundamental concepts of intuitionistic fuzzy sets, we emphasize the usefulness of the concepts of intuitionistic fuzzy points-intuitionistic fuzzy elementhood. Mainly, this paper is devoted to the study of intuitionistic fuzzy topological spaces with specific attention to the weaker forms of fuzzy continuity.

Key words : Intuitionistic fuzzy sets, intuitionistic fuzzy points, intuitionistic fuzzy topological spaces, intuitionistic fuzzy semicontinuity, intuitionistic fuzzy $\beta-$ continuity, intuitionistic fuzzy almost continuity, intuitionistic fuzzy weakly continuity.

1. Introduction

As the study of fuzzy topology can be regarded as a special case of intuitionistic fuzzy topology, several authors (e. g. cf. [3,5,6,7,8]) continued investigations in intuitionistic fuzzy topological spaces. Basic results and weaker forms of continuity in fuzzy topology have been considered by many workers (e.g. cf. [4,9,10]). In [1,2,5] , Atanassov and Çoker introduced the fundamental concepts of intuitionistic fuzzy sets and intuitionistic fuzzy topological spaces. In this paper, some results are given concerning fuzzy points and fuzzy sets in intuitionistic fuzzy topological spaces. The definition of fuzzy Urysohn space which is defined by Yalvac [10] is extended to intuitionistic fuzzy sets. Furthermore some results are obtained in the functions of semicontinuous, $\beta-$ continuous, almost continuous and weakly continuous of the intuitionistic fuzzy topological spaces defined by Guracy et. al. [7] and those are defined here.

2. Intuitionistic fuzzy sets

First we shall present the fundamental definitions obtained by K. Atanassov and D. Çoker.

Definition 2.1 [2]. Let X be a nonempty fixed set. An intuitionistic fuzzy set (IFS, for short) U is an object having the form $U=\{\langle x,\mu_U(x),\nu_U(x)\rangle : x\in X\}$ where the functions $\mu_U : X\to I$ and $\nu_U : X\to I$ denote the degree of membership (namely $\mu_U(x)$) and the degree of nonmembership (namely $\nu_U(x)$) of each element $x\in X$ to the set U , respectively, and $0\le \mu_U(x)+\nu_U(x)\le 1$ for each $x\in X$.

Obviously, every fuzzy set U on a nonempty set X is an IFS having the form

$U=\{\langle x,\mu_U(x),1-\mu_U(x)\rangle : x\in X\}$.

Definition 2.2 [2]. Let X be a nonempty set and the IFS's U and V be in the form

$U=\{\langle x,\mu_U(x),\nu_U(x)\rangle : x\in X\}$, $V=\{\langle x,\mu_V(x),\nu_V(x)\rangle : x\in X\}$ and let $\{U_j : j\in J\}$ be an arbitrary family of IFS's in X , then:

(i) $U\le V$ *iff* $\forall x\in X[\mu_U(x)\le \mu_V(x)$ U *and* $\nu_U(x)\ge \nu_V(x)]$;

(ii) $\overline{U}=\{\langle x,\nu_U(x),\mu_U(x)\rangle : x\in X\}$;

(iii) $\wedge\ U_j=\{\langle x,\wedge\ \mu_{U_j}(x),\vee\ \nu_{U_j}(x)\rangle : x\in X\}$;

(iv) $\vee\ U_j=\{\langle x,\vee\ \mu_{U_j}(x),\wedge\ \nu_{U_j}(x)\rangle : x\in X\}$;

(v) $1=\{\langle x,1,0\rangle : x\in X\}$ and $0=\{\langle x,0,1\rangle : x\in X\}$;

(vi) $\overline{\overline{U}}=U$, $\bar{0}=1$ and $\bar{1}=0$.

Definition 2.3 [5]. Let X and Y be two nonempty sets and $f:X\to Y$ be a function.

(i) If $V=\{\langle y,\mu_V(y),\nu_V(y)\rangle : y\in Y\}$ is an IFS in Y, then the preimage of V under f is denoted and defined by

$f^{-1}(V)=\{\langle x,f^{-1}(\mu_V)(x),f^{-1}(\nu_V)(x)\rangle : x\in X\}$;

(ii) If $U=\{\langle x,\lambda_U(x),\upsilon_U(x)\rangle : x\in X\}$ is an IFS in X, then the image of U under f is denoted and defined by

$f(U)=\{\langle y,f(\lambda_U)(y),f\_(\upsilon_U)(y)\rangle : y\in Y\}$ where

$$f(\lambda_U)(y)=\begin{cases}\sup\limits_{x\in f^{-1}(y)}\lambda_U(x) & \text{if } f^{-1}(y)\ne 0\\ 0, & \text{otherwise}\end{cases},$$

$$\text{and } f\_(\upsilon_U)=1-f(1-\upsilon_U)=\begin{cases}\inf\limits_{x\in f^{-1}(y)}\nu_U(x) & \text{if } f^{-1}(y)\ne 0\\ 1, & \text{otherwise}\end{cases}.$$

Corollary 2.4 [5]. Let $U,U_j(j\in J)$ IFS's in X, V, $V_j(j\in J)$ IFS's in Y and $f:X\to Y$ be a function. Then

(i) $U_1\le U_2 \Rightarrow f(U_1)\le f(U_2)$;

(ii) $V_1\le V_2 \Rightarrow f^{-1}(V_1)\le f^{-1}(V_2)$;

(iii) $U\le f^{-1}(f(U))$ (If f is one-to-one, then $U=f^{-1}(f(U))$);

(iv) $f(f^{-1}(V))\le V$ (If f is onto, then $f(f^{-1}(V))=V$;

(v) $f^{-1}(\vee V_j)=\vee f^{-1}(V_j)$ *and* $f^{-1}(\wedge V_j)=\wedge f^{-1}(V_j)$;
(vi) $f(\vee U_j)=\vee f(U_j)$ *and* $f(\wedge U_j)\leq\wedge f(U_j)$;(if f is injective then $f(\wedge U_j)=\wedge f(U_j)$);
(vii) $f^{-1}(\underset{\sim}{1})=\underset{\sim}{1}$ and $f^{-1}(\underset{\sim}{0})=\underset{\sim}{0}$;
$(viii)$ $f(\underset{\sim}{1})=\underset{\sim}{1}$, if f is onto.
(ix) $f(\underset{\sim}{0})=\underset{\sim}{0}$;
(x) $f^{-1}(\overline{V})=\overline{f^{-1}(V)}$.

Definition 2.5 [6]. Let X be a nonempty set and $c\in X$ a fixed element in X. If $a\in(0,1]$ and $b\in[0,1)$ are two fixed real numbers such that $a+b\leq 1$, then the IFS $c(a,b)=\langle x,c_a,1-c_{1-b}\rangle$ is called an intuitionistic fuzzy point (IFP, for short) in X, where a denotes the degree of membership of $c(a,b)$,b the degree of nonmembership of $c(a,b)$ and $c\in X$ the support of $c(a,b)$.

Definition 2.6 [6]. Let $c(a,b)$ be an IFP in X and $U=\langle x,\mu_U,\nu_U\rangle$ an IFS in X. Suppose further that $a,b\in(0,1)$. $c(a,b)$ is said to be properly contained in U ($c(a,b)\in U$, *for short*) iff $a<\mu_U(c)$ and $b>\nu_U(c)$.

Definition 2.7 [6]. (i) An IFP $c(a,b)$ in X is said to be quasi-coincident with the IFS $U=\langle x,\mu_U,\nu_U\rangle$, denoted by $c(a,b)qU$, iff $a>\nu_U(c)$ or $b<\mu_U(c)$.
(ii) Let $U=\langle x,\mu_U,\nu_U\rangle$ and $V=\langle x,\mu_V,\nu_V\rangle$ be two IFS's in X. Then, U and V are said to be quasi-coincident, denoted by UqV, iff there exists an element $x\in X$ such that $\mu_U(x)>\nu_V(x)$ *or* $\nu_U(x)<\mu_V(x)$. Also, we denote the negation of UqV by $U\bar{q}V$.

Proposition 2.8 [8]. Let $f:X\to Y$ be a function and $c(a,b)$ is an IFP in X.
(i) If for IFS V in Y we have $f(c(a,b))qV$, then $c(a,b)qf^{-1}(V)$.
(ii) If for IFS U in X we have $c(a,b)qU$, then $f(c(a,b))qf(U)$.

Definition 2.9 [5]. An intuitionistic fuzzy topology (IFT, for short) on a nonempty set X is a family Ψ of IFS's in X satisfying the following axioms:
(i) $\underset{\sim}{0},\underset{\sim}{1}\in\Psi$;
(ii) $U_1\wedge U_2\in\Psi$ for any $U_1,U_2\in\Psi$;
$(iii)\vee$ $U_j\in\Psi$ for any $\{U_j:j\in J\}\subseteq\Psi$.

In this case the pair (X,Ψ) is called an intuitionistic fuzzy topological space (IFTS, for short) and each IFS in Ψ is known as an intuitionistic fuzzy open set (IFOS, for short) in X.
The complement $\overline{U}$ of IFOS U in IFTS (X,Ψ) is called an intuitionistic fuzzy closed set (IFCS, for short).

Definition 2.10 [5]. Let U be an IFS in an IFTS (X,Ψ). U is called an intuitionistic fuzzy semiopen (semiclosed) (resp. preopen (preclosed), $\beta-$ open ($\beta-$ closed), regular open (regular closed)) set, (IFSOS (IFSCS) (resp. IFPOS (IFPCS), IF β OS (IF β CS), IFROS (IFRCS),for short), if
$U\leq cl\,int(U)$ ($U\geq int\,cl(U)$)
(resp. $U\leq int\,cl(U)$ ($U\geq cl\,int(U)$) ,
$U\leq cl\,int\,cl(U)$ ($U\geq int\,cl\,int(U)$),
$U=int\,cl(U)$ ($U=cl\,int(U)$)).

Definition 2.11 . Let (X,Ψ) be an IFTS and $U=\langle x,\mu_U(x),\nu_U(x)\rangle$ be an IFS in X. Then the fuzzy interior [5](β -interior) and fuzzy closure [5] (β -closure) of U are denoted and defined by:
$clU(\beta clU)=\wedge\{K:K$ is an IFCS (IF β CS) in X and $U\leq K\}$ and $intU(\beta intU)=\vee\{G:G$ is an IFOS (IF β OS) in X and $G\leq U\}$.

Proposition 2.12 [5]. For any IFS U in IFTS (X,Ψ) we have
(i) $cl\overline{U}=\overline{intU}$, (ii) $int\overline{U}=\overline{clU}$.

Theorem 2.13 [7]. (i) Any intersection of IFSCS's is an IFSCS.
(ii) Any union of IFSOS's is an IFSOS.

Definition 2.14 [3]. Let (X,Ψ) be an IFTS on X and $c(a,b)$ an IFP in X. An IFS N is called $\varepsilon-nbd$ $(\varepsilon q-nbd)$ of $c(a,b)$ if there exists an IFOS G in X such that $c(a,b)\in G\leq N$ ($c(a,b)qG\leq N$).
The family of all $\varepsilon-nbd\,(\varepsilon q-nbd)$ of $c(a,b)$ will be denoted by N_ε $(N_\varepsilon^q)(c(a,b))$.

Definition 2.15. A function $f:(X,\Psi)\to(Y,\Phi)$ is called an intuitionistic fuzzy open (resp.preopen) (IFO (resp.IFPO, for short)) iff the image of every IFOS in X is an IFOS (resp. IFPOS) in Y.

3. Basic results

Theorem 3.1. Let $c(a,b)$ be an IFP in (X,Ψ) and U an IFS in X. Then $c(a,b)\in clU$ iff UqN for every $N\in N_\varepsilon^q(c(a,b))$.

Proof. Suppose that there exists $N\in N_\varepsilon^q(c(a,b))$ such that $U\bar{q}N$. Then there exists $G\in\Psi$ such that $c(a,b)qG\leq N$ and $G\bar{q}U$. Since $\overline{G}$ is an IFCS and by proposition 3.11 in [6] ,we have $clU\leq\overline{G}$. Also since $c(a,b)\notin\overline{G}$, we have $c(a,b)\notin clU$ which is contradiction.

Conversely, suppose $c(a,b) \notin clU$. Then there exists an IFCS V such that $c(a,b) \notin V$ and $U \leq V$. Hence by proposition 3.11 in [6], $\overline{V} \in \Psi$ such that $c(a,b)q\overline{V}$ and $U\bar{q}\overline{V}$ which is a contradiction.

Theorem 3.2. U is an IFOS in X iff for every IFP $c(a,b)qU$, $U \in N_\varepsilon^q(c(a,b))$.

Proof. Let $U = \langle x, \mu_U(x), \nu_U(x)\rangle$ be an IFOS in X and $c(a,b)qU$. Then $c(a,b)qU \leq U$. Hence $U \in N_\varepsilon^q(c(a,b))$.

Conversely, let $c(a,b) \in U$, this implies $a < \mu_U(c)$ and $b > \nu_U(c)$. Since $a,b \in (0,1)$ and $a+b \leq 1$, we have $c(a,b)qU$ and by hypothesis $U \in N_\varepsilon^q(c(b,a))$, then there exists an IFOS G such that $c(b,a)qG \leq U$ which implies $c(a,b) \in G \leq U$. Hence U is an IFOS.

Theorem 3.3. Let U,V be IFS's in (X,Ψ). Then $U \leq V$ iff $c(a,b) \in U \Rightarrow c(a,b) \in V$ for every IFP $c(a,b)$ in X.

Proof. Let $U = \langle c, \mu_U(c), \nu_U(c)\rangle$ and $V = \langle c, \mu_V(c), \nu_V(c)\rangle$ be IFS's in X such that $U \leq V$ and $c(a,b) \in U$. This implies $\mu_U(c) \leq \mu_V(c)$, $\nu_U(c) \geq \nu_V(c)$ and $a < \mu_U(c)$, $b > \nu_U(c)$. Clearly $a < \mu_U(c) \leq \mu_V(c)$ and $b > \nu_U(x) \geq \nu_V(x)$. This gives $c(a,b) \in V$.

Conversely, let $c(a,b) \in U \Rightarrow c(a,b) \in V$ but $U \not\leq V$. Then for some IFP $c(a,b)$, $V(c) \leq U(c)$. Then $\mu_V(c) < a < \mu_U(c)$ and $\nu_V(c) > b > \nu_U(c)$ then $c(a,b) \in U$ but $c(a,b) \notin V$ which is a contradiction. This completes the proof.

Corollary 3.4. Let U,V be IFS's in (X,Ψ). Then $U = V$ iff $c(a,b) \in U \Leftrightarrow$ $c(a,b) \in V$ for every IFP $c(a,b)$ in X.

Proposition 3.5. Let U,V be IFS's in (X,Ψ) and $intU \leq V \leq clU$.
(*i*) if U is an IFSOS then so is V.
(*ii*) if U is an IFSCS then so is V.

Proof. (*i*) Let U be an IFSOS and $intU \leq V \leq clU$. There exists $G \in \Psi$ such that $G \leq U \leq clG$. It follows that $G \leq intU \leq U \leq clU \leq clG$ and hence $G \leq V \leq clG$. Thus V is an IFSOS.

(*ii*) Similar to (*i*).

Theorem 3.6. Let U be an IFS in (X,Ψ). U is an IFSOS iff for every IFP

$$c(a,b) \in U \ (c(a,b)qU\ )$$

there exists an IFSOS V_c such that

$$c(a,b) \in V_c \leq U \ (c(a,b)qV_c \leq U).$$

Proof. If U is an IFSOS, then we may take $V_c = U$ for every $c(a,b) \in U$.

Conversely, from Proposition 3.8 [6], we have $U = \bigvee_{c \in U} c(a,b) \leq \bigvee_{c \in U} V_c \leq U$ and hence $U = \bigvee_{c \in U} V_c$. This shows, from Theorem 2.13, that U is an IFSOS.

The proof of the second part is similar.

Lemma 3.7. Let $f:(X,\Psi) \to (Y,\Phi)$ be a function and $c(a,b)$ be an IFP in X. If $c(a,b) \in f^{-1}(V)$, then $f(c)(a,b) \in V$, for every IFS $V = \langle f(c), \mu_V(f(c)), \nu_V(f(c))\rangle$ in Y.

Proof. Let $c(a,b)$ be an IFP in X and $c(a,b) \in f^{-1}(V)$. This implies that $a < f^{-1}(\mu_V)(c)$ and $b > f^{-1}(\nu_V)(c)$ which implies that $a < \mu_V(f(c))$ and $b > \nu_V(f(c))$. Hence $f(c)(a,b) \in V$.

Lemma 3.8. If $f:(X,\Psi) \to (Y,\Phi)$ be an IFO function and $N \in N_\varepsilon^q(c(a,b))$ for every IFP $c(a,b) \in X$, then $f(N) \in N_\varepsilon^q(f(c(a,b)))$.

Proof. (*i*) Let $N = \langle c, \mu_N(c), \nu_N(c)\rangle \in N_\varepsilon^q(c(a,b))$.
Then there exist $G \in \Psi$ such that $c(a,b)qG \leq N$. This implies that $f(c)(a,b)qf(G) \leq f(N)$ (Theorem 2.8 (*ii*)). Since f is IFO function, we have $f(G) \in \Phi$. Thus $f(N) \in N_\varepsilon^q(f(c(a,b)))$.

Theorem 3.9. If $f:(X,\Psi) \to (Y,\Phi)$ be an IFO function, then $f^{-1}(clV) \leq clf^{-1}(V)$, for every IFS V in Y.

Proof. Let $V = \langle f(c), \mu_V(f(c)), \nu_V(f(c))\rangle$ be an IFS in Y and $c(a,b) \in f^{-1}(clV)$. Then $f(c)(a,b) \in ff^{-1}(clV) \leq clV$.
Let $N = \langle c, \mu_N(c), \nu_N(c)\rangle \in N_\varepsilon^q(c(a,b))$.
From Lemma 3.8,
$f(N) = \langle f(c), f(\mu_N)(f(c)), f(\nu_N)(f(c))\rangle \in N_\varepsilon^q(f(c(a,b)))$,
$c(a,b) \in f^{-1}(clV)$
implies $f(c(a,b)) \leq clV$ and from Theorem 3.1 there exists an IFP e in Y such that $f(\mu_N)(e) > \nu_N(e)$ *or* $f_{-}(\nu_N)(e) < \mu_N(e)$.
We choose $\varepsilon > 0$ such that
$f(\mu_N)(e) - \varepsilon > \nu_V(e)$ *or* $f_{-}(\nu_N)(e) + \varepsilon < \mu_V(e)$

Since $f(\mu_N)(e) = \sup_{c \in f^{-1}(e)} \mu_N(c)$

and $f_{-}(\nu_N)(e) = \inf_{c \in f^{-1}(e)} \nu_N(c)$, there exists an IFP $c_0 \in f^{-1}(e)$ such that $f(\mu_N)(e) - \varepsilon < \mu_V(c_0)$ *and* $f_{-}(\nu_N)(e) + \varepsilon > \nu_N(c_0)$. For this c_0, $\nu_V(e) = f^{-1}(\nu_V)(c_0)$ *and* $\mu_V(e) = f^{-1}(\mu_V)(c_0)$. We have $\mu_N(c_0) > f^{-1}(\nu_V)(c_0)$ *or* $\nu_N(c_0) < f^{-1}(\mu_V)(c_0)$, which implies that $Nqf^{-1}(V)$. Since this result is true for every $N \in N_\varepsilon^q(c(a,b))$, we have $c(a,b) \in cl\, f^{-1}(V)$ (Theorem 3.1). By Theorem 3.3, hence the result.

Corollary 3.10. If $f:(X,\Psi)\to(Y,\Phi)$ be an IFO and intuitionistic fuzzy continuous (IFC, for short) function, then $f^{-1}(cl\,V)=clf^{-1}(V)$, for every IFS V in Y.

Proof. Similarly to the proof of Theorem 3.9, one can show that if f is an IFC function, then $clf^{-1}(V)\le f^{-1}(cl\,V)$. Hence the proof is clear.

4. Intuitionistic fuzzy functions

Definition 4.1 [7]. A function $f:(X,\Psi)\to(Y,\Phi)$ is called an intuitionistic fuzzy semicontinuous (IFSC, for short) if $f^{-1}(V)$ is an IFSOS in X, for every $V\in\Phi$.

Theorem 4.2. Let $f:(X,\Psi)\to(Y,\Phi)$ be a function. The following are equivalent.

(i) f is an IFSC.

(ii) For every IFP $c(a,b)$ in X and every $M\in N_\varepsilon(f(c))(a,b)$, there exists an IFSOS U in X such that $c(a,b)\in U\le f^{-1}(M)$.

(iii) For every IFP $c(a,b)$ in X and every $M\in N_\varepsilon(f(c))(a,b)$, there exists an IFSOS U in X such that $c(a,b)\in U$ and $f(U)\le M$.

(iv) For every IFP $c(a,b)$ in X and every $M\in N^q_\varepsilon(f(c))(a,b)$, there exists an IFSOS U in X such that $c(a,b)qU$ and $f(U)\le M$.

(v) $f^{-1}(V)\le cl\,int\,f^{-1}(V)$, for every $V\in\Phi$.

Proof. $(i)\Rightarrow(ii)$: Let $c(a,b)$ be an IFP in X and $M\in N_\varepsilon(f(c))(a,b)$. There exists $V\in\Phi$ such that $f(c)\in V\le M$. $f^{-1}(V)=U$ is an IFSOS in X and we have $c\in f^{-1}(V)=U\le f^{-1}(M)$.

$(ii)\Rightarrow(iii)$: Let $c(a,b)$ be an IFP in X and $M\in N_\varepsilon(f(c))(a,b)$. There exists an IFSOS U in X such that $c(a,b)\in U\le f^{-1}(M)$. So we have $c(a,b)\in U$, $f(U)\le ff^{-1}(M)\le M$.

$(iii)\Rightarrow(i)$: Let $V\in\Phi$ and let us take $c(a,b)\in f^{-1}(V)$. This shows that $f(c)\in ff^{-1}(V)\le V$. Since V is an IFOS in Y we have $V\in N_\varepsilon(f(c))(a,b)$. There exists an IFSOS U in X such that $c(a,b)\in U$ and $f(U)\le V$. This shows that $c(a,b)\in U\le f^{-1}(V)$. From Theorem 3.6, $f^{-1}(V)$ is an IFSOS. Hence f is IFSC function.

$(i)\Rightarrow(iv)$: Let $c(a,b)$ be an IFP in X and $M\in N^q_\varepsilon(f(c))(a,b)$. There exists $V\in\Phi$ such that $f(c)(a,b)qV\le M$. $f^{-1}(V)$ is an IFSOS and from Proposition 2.8, we have $c(a,b)qf^{-1}(V)$. If we take $U=f^{-1}(V)$ then $f(U)=ff^{-1}(U)\le U\le M$.

$(iv)\Rightarrow(i)$: Let $V\in\Phi$ and let us take $c(a,b)qf^{-1}(V)$. This implies $f(c)(a,b)qV$. Since V is an IFOS in Y we have $V\in N^q_\varepsilon(f(c))(a,b)$. There exists an IFSOS U in X such that $c(a,b)qU$ and $f(U)\le V$. This shows that $c(a,b)qU\le f^{-1}(V)$ which implies that $f^{-1}(V)$ is an IFSOS (Theorem 3.6).

$(i)\Rightarrow(v)$: Since $f^{-1}(V)$ is an IFSOS in X for every $V\in\Phi$, we have $f^{-1}(V)\le cl\,int\,f^{-1}(V)$.

$(v)\Rightarrow(i)$: Since any IFS U in X which satisfies the relation $U\le cl\,int\,U$ will be an IFSOS, we have that $f^{-1}(V)$ is an IFSOS for every $V\in\Phi$.

Definition 4.3 [7]. A function $f:(X,\Psi)\to(Y,\Phi)$ is called an intuitionistic fuzzy weakly continuous (IFWC, for short) if $f^{-1}(V)\le int(f^{-1}(cl\,V))$, for every $V\in\Phi$.

An IFC function is always IFWC, but the converse is need not be true.

Theorem 4.4. Let $f:(X,\Psi)\to(Y,\Phi)$ be a function. The following are equivalent:

(i) f is an IFWC.

(ii) $f^{-1}(V)\ge cl\,f^{-1}(int\,V)$, for every IFCS V in Y.

(iii) $cl\,f^{-1}(V)\le f^{-1}(cl\,V)$, for every $V\in\Phi$.

(iv) For every IFP $c(a,b)$ in X and $M\in N_\varepsilon(f(c))(a,b)$, $f^{-1}(cl\,M)\in N_\varepsilon(c)(a,b)$.

(v) For every IFP $c(a,b)$ in X and $M\in N_\varepsilon(f(c))(a,b)$, there exists $U\in\Psi$ such that $c(a,b)\in U$ and $f(U)\le cl\,M$.

Proof. $(i)\Rightarrow(ii)$: Let V be an IFCS in Y. Then $\overline{V}\in\Phi$ and $f^{-1}(\overline{V})\le intf^{-1}(cl\,\overline{V})$. This implies that $\overline{f^{-1}(V)}\le int\,f^{-1}\overline{(int\,V)}$ (Proposition 2.12), $\overline{f^{-1}(V)}\le int\,\overline{f^{-1}(int\,V)}$ $=\overline{cl\,f^{-1}(int\,V)}$. Hence $f^{-1}(V)\ge cl\,f^{-1}(int\,V)$.

$(ii)\Rightarrow(i)$: Similar to $(i)\Rightarrow(ii)$.

$(ii)\Rightarrow(iii)$: Let $V\in\Phi$. $cl\,V$ is an IFCS in Y and $V\le int\,cl\,V$. Hence
$f^{-1}(cl\,V)\ge cl\,f^{-1}(int\,cl\,V)\ge cl\,f^{-1}(V)$.

$(iii)\Rightarrow(ii)$: Let V be an IFCS in Y. This gives that $int\,V\in\Phi$ and $cl\,int\,V\le V$. Hence $cl\,f^{-1}(int\,V)\le f^{-1}(cl\,int\,V)\le f^{-1}(V)$.

$(i)\Rightarrow(iv)$,$(iv)\Rightarrow(v)$ *and* $(v)\Rightarrow(i)$: can be easily proved.

Definition 4.5. An IFTS (X,Ψ) is called regular (IFRS, for short) iff every $U\in\Psi$ is a union of IFOS's U_j's of X such that $cl\,U_j\le U$ for every j.

Theorem 4.6. Let f be a function from an IFTS (X,Ψ) into IFRS (Y,Φ). Then f is an IFWC iff f is an IFC.

Proof. Since IFC function implies IFWC, it sufficies to show that if f is an IFWC then it is IFC. Let $U \in \Phi$. Since Y is IFRS, $U = \underset{j}{\vee} U_j$, $U_j \in \Phi$ and $cl U_j \leq U$ for every j. Now f is IFWC function implies that $f^{-1}(U) = f^{-1}(\underset{j}{\vee} U_j) = \underset{j}{\vee} f^{-1}(U_j) \leq \underset{j}{\vee} int\, f^{-1}(cl U_j) \leq \underset{j}{\vee} int\, f^{-1}(U) = int\, f^{-1}(U)$, and hence $f^{-1}(U)$ is an IFOS in X. Thus f is an IFC function.

Definition 4.7. An IFTS (X, Ψ) is called Urysohn iff c_1, c_2 be IFP's in X and $c_1 \neq c_2$ implies that there exists $G = \langle x, \mu_G(x), \nu_G(x) \rangle$, $H = \langle x, \mu_H(x), \nu_H(x) \rangle \in \Psi$ with $\mu_G(c_1) = 1$, $\nu_G(c_1) = 0$, $\mu_H(c_2) = 1$, $\nu_H(c_2) = 0$ and $cl G \wedge cl H = \underset{\sim}{0}$.

Theorem 4.8. For an IFTS (X, Ψ), the following are equivalent:

(*i*) X is Urysohn.

(*ii*) For any distinct IFP's c_1 and c_2 in X, there exist IFOS's G and H such that $c_1 q G$, $c_2 q H$ and $cl G \wedge cl H = \underset{\sim}{0}$.

Proof. Obvious.

Theorem 4.9. Let $f : (X, \Psi) \to (Y, \Phi)$ be an IFWC injective function. If an IFTS (Y, Φ) is Urysohn, then (X, Ψ) is Hausdorff ([5]).

Proof. It follows directly from definitions, immeditely.

Definition 4.10. [7]. A function $f : (X, \Psi) \to (Y, \Phi)$ is called an intuitionistic fuzzy almost continuous (IFAC, for short) if $f^{-1}(V)$ is an IFOS in X, for every IFROS V in Y.

Theorem 4.11. Let $f : (X, \Psi) \to (Y, \Phi)$ be a function. The following are equivalent:

(*i*) f is an IFAC.

(*ii*) For every IFP $c(a,b)$ in X and $M \in N_\varepsilon(f(c))(a,b)$, there exists $U \in \Psi$ such that $c(a,b) \in U$ and $f(U) \leq int\, cl\, M$.

(*iii*) For every IFP $c(a,b)$ in X and $M \in N_\varepsilon^q(f(c))(a,b)$, there exists $U \in \Psi$ such that $c(a,b) q U$ and $f(U) \leq int\, cl\, M$.

(*iv*) $f^{-1}(V)$ is an IFCS in X, for every IFRCS in Y.

(*v*) $f^{-1}(V) \leq int\, f^{-1}(int\, cl\, V)$, for every IFPOS V in Y.

(*vi*) $cl\, f^{-1}(cl\, int\, H) \leq f^{-1}(H)$, for every IFPCS H in Y.

(*vii*) $f^{-1}(V) \leq int\, f^{-1}(int\, cl\, V)$, for every $V \in \Phi$.

(*viii*) $cl\, f^{-1}(cl\, int\, H) \leq f^{-1}(H)$, for every IFCS H in Y.

Proof. $(i) \Rightarrow (ii)$: Let $c(a,b)$ be an IFP in X and $M \in N_\varepsilon(f(c))(a,b)$. There exists $V \in \Phi$ such that $f(c) \in V \leq M$. Hence $c \in f^{-1}(V) \leq f^{-1}(M)$. Since $V \in \Phi$, then $V \leq int\, cl\, V$ and $f^{-1}(V) \leq f^{-1}(int\, cl\, V)$. Thus $c(a,b) \in f^{-1}(int\, cl\, V)$. Since $int\, cl\, V$ is an IFROS ([7], Theorem 2.9), then $f^{-1}(int\, cl\, V)$ is an IFOS in X and $c(a,b) \in f^{-1}(int\, cl\, V)$. Consequently, there exists $U \in \Psi$ such that $c(a,b) \in U \leq f^{-1}(int\, cl\, V) \leq f^{-1}(int\, cl\, M)$. Hence $f(U) \leq int\, cl\, M$.

$(ii) \Rightarrow (i)$: Let V be an IFROS in Y and $c(a,b) \in f^{-1}(V)$, then $f(c) \in V$. Since $V \in \Phi$ we have $V \in N_\varepsilon(f(c))(a,b)$. By (ii) there exist $U \in \Psi$ such that $c(a,b) \in U$ and $f(U) \leq int\, cl\, V = V$. Then $c(a,b) \in U \leq f^{-1}(V)$. Hence $f^{-1}(V) \in \Psi$. Therefore f is an IFAC function.

$(i) \Leftrightarrow (iii)$: it can be proved in a similar way to the proof of Theorem 4.2.

$(i) \Leftrightarrow (iv)$: Since $f^{-1}(\overline{V}) = \overline{f^{-1}(V)}$, for every IFS V in Y, the proof follows from Theorem 2.6 [7].

$(i) \Leftrightarrow (v)$: Since V is an IFPOS in Y, then $V \leq int\, cl\, V$ and hence $f^{-1}(V) \leq f^{-1}(int\, cl\, V)$. Since $int\, cl\, V$ is an IFROS in Y, $f^{-1}(int\, cl\, V) \in \Psi$. Thus $f^{-1}(V) \leq f^{-1}(int\, cl\, V) = int\, f^{-1}(int\, cl\, V)$.

$(v) \Leftrightarrow (vi)$: Let H be an IFPCS in Y, then $\overline{H}$ is an IFPOS in Y. By hypothesis $\overline{f^{-1}(H)} = f^{-1}(\overline{H}) \leq int\ f^{-1}(int\, cl\, \overline{H}) = int\ f^{-1}(int\, \overline{(int\, H)}) = int\, f^{-1}\overline{(cl\, int\, H)} = int(\overline{f^{-1}(cl\, int\, H)}) = \overline{cl\, f^{-1}(cl\, int\, H)}$. Hence $cl\, f^{-1}(cl\, int\, H) \leq f^{-1}(H)$.

$(vi) \Leftrightarrow (iv)$: Let V be an IFRCS in Y. Then $V = cl\, int\, V$ and $f^{-1}(V) = f^{-1}(cl\, int\, V)$. By (vi) we have $cl\, f^{-1}(V) = cl\, f^{-1}(cl\, int\, V) \leq f^{-1}(V)$. Thus $f^{-1}(V) = cl\, f^{-1}(V)$ shows that $f^{-1}(V)$ is an IFCS in X.

$(i) \Rightarrow (vii) \Rightarrow (viii) \Rightarrow (iv)$: Since every an IFOS is an IFPOS, the proof can be easily proved as a corollary of the proof $(i) \Rightarrow (v) \Rightarrow (vi) \Rightarrow (iv)$.

Theorem 4.12. If $f : (X, \Psi) \to (Y, \Phi)$ is an IFWC and IFPO function, then f is an IFAC.

Proof. Let $c(a,b)$ be an IFP in X and $M \in N_\varepsilon(f(c))(a,b)$. There exists $V \in \Phi$ such that $f(c) \in V$. Hence $c(a,b) \in f^{-1}(V)$. Since f is an IFWC, there exists an IFOS U in X such that $c(a,b) \in U \leq f^{-1}(cl V)$. Hence $f(U) \leq cl V$. Since f is an IFPO function and $U \in \Psi$, then $f(U)$ is an IFPOS in Y. Hence $f(U) \leq int\, cl\, f(U) \leq int\, cl\, V$. Then f is an IFAC function.

Definition 4.13 [7]. A function $f:(X,\Psi)\to(Y,\Phi)$ is called an intuitionistic fuzzy β-continuous (IF β C, for short) if $f^{-1}(V)$ is an IF β OS in X, for every $V\in\Phi$.

Theorem 4.14. For any IFS U in (X,Ψ) we have
(i) $\beta cl\overline{U}=\overline{\beta int U}$, (ii) $\beta int\overline{U}=\overline{\beta cl U}$.

Proof. (i) Let $U=\langle x,\mu_U(x),\nu_U(x)\rangle$ and suppose that the family of $IF\beta$ OS's contained in U are indexed by the family $\{\langle x,\mu_{G_j}(x),\nu_{G_j}(x)\rangle : j\in J\}$. Then we see $\beta int U=\langle x,\vee\mu_{G_j}(x),\wedge\nu_{G_j}(x)\rangle$ and hence $\overline{\beta int U}=\langle x,\wedge\nu_{G_j}(x),\vee\mu_{G_j}(x)\rangle$. Since $\overline{U}=\langle x,\nu_U(x),\mu_U(x)\rangle$ and $\mu_{G_j}(x)\le\mu_U(x)$, $\nu_{G_j}(x)\ge\nu_U(x)$ for every $j\in J$, we obtain that $\{\langle x,\nu_{G_j}(x),\mu_{G_j}(x)\rangle : j\in J\}$ is the family of IF β CS's containing $\overline{U}$, i.e. $\beta cl\overline{U}=\{\langle x,\wedge\nu_{G_j}(x),\vee\mu_{G_j}(x)\rangle$. Hence $\beta cl\overline{U}=\overline{\beta int U}$.

(ii) The proof is similar to (i).

Theorem 4.15. Let $f:(X,\Psi)\to(Y,\Phi)$ be a function. The following are equivalent.
(i) f is an IF β C.
(ii) $f^{-1}(V)$ is an IF β CS in X, for every IFCS V in Y.
(iii) $f(\beta cl U)\le cl\, f(U)$, for every IFS U in X.
(iv) $\beta cl\, f^{-1}(V)\le f^{-1}(cl V)$, for every IFS V in Y.
(v) For every IFP $c(a,b)$ in X and every $V\in N_\varepsilon(f(c))(a,b)$ there exists an IF β OS U in X such that $c(a,b)\in U$ and $f(U)\le V$.

Proof. $(i)\Rightarrow(ii)$: Let V be an IFCS in Y. Then $\overline{V}\in\Phi$ and so $f^{-1}(\overline{V})$ is an IF β OS in X. But $f^{-1}(\overline{V})=\overline{f^{-1}(V)}$. Hence $f^{-1}(V)$ is an IF β CS in X.

$(ii)\Rightarrow(iii)$: Let U be an IFS in X. Then $cl\, f(U)$ is an IFCS in Y. By (ii), $f^{-1}(cl\, f(U))$ is an IF β CS and so $f^{-1}(cl\, f(U))=\beta cl f^{-1}(cl\, f(U))$. Since $U\le f^{-1}f(U)$, we have $\beta cl U\le\beta cl f^{-1}f(U)\le\beta cl f^{-1}(cl\, f(U))=f^{-1}(cl\, f(U))$.

$(iii)\Rightarrow(iv)$: Let V be an IFS in Y. Then by (iii), we have $f(\beta cl\, f^{-1}(V))\le cl\, ff^{-1}(V)$. Hence $\beta cl\, f^{-1}(V)\le f^{-1}(cl\, ff^{-1}(V))\le f^{-1}(cl V)$.

$(iv)\Rightarrow(i)$: Let $V\in\Phi$. Then $\overline{V}$ is an IFCS. By (iv), we have $\beta cl\, f^{-1}(\overline{V})\le f^{-1}(cl\overline{V})=f^{-1}(\overline{V})$. By Theorem 4.14, we have $f^{-1}(\overline{V})\ge\beta cl\, f^{-1}(\overline{V})=\overline{\beta int\, f^{-1}(V)}$. Hence $f^{-1}(V)$ is an IF β OS in X.

$(i)\Rightarrow(v)$: Let $c(a,b)$ be an IFP in X and $V\in N_\varepsilon(f(c))(a,b)$. Then there exists $G\in\Phi$ such that $f(c)\in G\le V$. By IF β C function, $f^{-1}(G)$ is an IF β OS in X with $c(a,b)\in f^{-1}(G)\le f^{-1}(V)$ .Hence , putting $U=f^{-1}(G)$, $U\in N_\varepsilon(c)(a,b)$ such that $f(U)=ff^{-1}(G)\le G\le V$.

$(v)\Rightarrow(i)$: Let $V\in\Phi$ and $c(a,b)\in f^{-1}(V)$. Then $V\in N_\varepsilon(f(c))(a,b)$. By hypothesis, there exists an IF β OS U in X such that $c(a,b)\in U\le f^{-1}(V)$. Hence $f^{-1}(V)$ is an IF β OS.

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