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# Tight인 퍼지 랜덤변수에 대한 강한 대수의 법칙

# Strong laws of large numbers for tight fuzzy random variables

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**요 약**

이 논문에서는 서로 독립이고 동일한 분포를 갖는 퍼지 랜덤변수의 합에 대하여 잘 알려진 강한 대수의 법칙을 tight인 퍼지 랜덤변수의 합에 대한 경우로 일반화 하였다.

**Abstract**

The present paper establishes a strong law of large numbers for sums of tight fuzzy random variables as a generalization of SLLN for sums of independent and identically distributed fuzzy random variables.

**Key words** : Fuzzy numbers, Fuzzy random variables, Strong law of large numbers, Tightness, the Hausdorff-Skorohod metric

## 1. Introduction

The theory of fuzzy sets introduced by Zadeh [16] has been extensively studied and applied in statistics and probability areas in recent years. Since Puri and Ralescu [13] introduced the concept of a fuzzy random variable, there has been increasing interests in limit theorems for fuzzy random variables. Among others, strong laws of large numbers (in brief, SLLN) for independent fuzzy random variables have been studied by several researchers. SLLN for sums of independent and identically distributed fuzzy random variables was obtained by Joo and Kim [7], Klement et al. [11], Molchanov [12]. Inoue [3] established a SLLN for sums of level-wise tight fuzzy random variables by using a different metric from that will be used in the present work. Also, Joo [4] studied a SLLN for sums of convexly tight fuzzy random variables and Kim [9] obtained SLLN for sums of strongly tight fuzzy random variables as a corollary of compact-uniformly integrable case. These results were obtained by similar arguments used in Daffer and Taylor [1] which established SLLN for sums of Banach space valued random variables. But unfortunately, independent and identically distributed fuzzy random variables are neither convexly tight nor strongly tight. Thus, the results of Joo [4] and Kim [9] cannot be a generalization of SLLN for sums of independent and identically distributed fuzzy random variables. Accordingly, it is significant to study SLLN for sums of tight fuzzy random variables as a generalization of independent and identically distributed case.

The purpose of this paper is to solve such a problem. Section 2 is devoted to describe basic facts for fuzzy numbers. The main results are given in section 3.

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## 2. Preliminaries

Let $K(R^p)$ be the family of all non-empty compact and convex subsets of $R^p$. Then $K(R^p)$ is metrizable by the Hausdorff metric $h$ defined by

$$h(A, B) = \max\{\sup_{a\in A}\inf_{b\in B}|a-b|, \sup_{b\in B}\inf_{a\in A}|a-b|\},$$

where $|\cdot|$ is the usual norm in $R^p$.

A norm of $A\in K(R^p)$ is defined by

$$\| A \| = h(A, \{0\}) = \sup_{a\in A} | a | .$$

It is well known that the metric $(K(R^p), h)$ is complete and separable (See Debreu [2]). The addition and scalar multiplication on $K(R^p)$ are defined as usual;

$$A\oplus B = \{a+b \mid a\in A, b\in B\},$$
$$\lambda A = \{\lambda a \mid a\in A\}.$$

Let $F(R^p)$ denote the space of fuzzy numbers in $R^p$, i.e., the family of all normal, fuzzy convex and upper-semicontinuous fuzzy sets $u$ in $R^p$ such that

$$supp\ u = cl\ \{x \in R^p : u(x) > 0\}$$

is compact, where $cl$ denotes the closure. For a fuzzy set $u$ in $R^p$, the $\alpha$-level set of $u$ is defined by

$$L_\alpha u = \begin{cases} \{x : u(x) \geq \alpha\}, & 0 < \alpha \leq 1, \\ supp\ u, & \alpha = 0. \end{cases}$$

Then it follows that $u \in F(R^p)$ if and only if $L_\alpha u \in K(R^p)$ for each $\alpha \in [0,1]$.

**Lemma 2.1.** For $u \in F(R^p)$, let us define $f_u : [0,1] \to (K(R^p), h)$ by $f_u(\alpha) = L_\alpha u$. Then the followings hold;

(1) $f_u$ is non-increasing; i.e., $\alpha \leq \beta$ implies $f_u(\alpha) \supset f_u(\beta)$.

(2) $f_u$ is left-continuous on $(0,1]$.

(3) $f_u$ has right limits on $[0,1)$ and is right continuous at $0$.

Conversely, if $g : [0,1] \to (K(R^p), h)$ is a function satisfying the above conditions (1) - (3), then there exists a unique $v \in F(R^p)$ such that

$$g(\alpha) = L_\alpha v \text{ for all } \alpha \in [0,1].$$

**Proof**: See Joo and Kim [6].

If we denote the right-limit of $f_u$ at $\alpha \in [0,1)$ by $L_{\alpha^+} u$, then

$$L_{\alpha^+} u = cl\ \{x \in R^p : u(x) > \alpha\}.$$

The addition and scalar multiplication on $F(R^p)$ are defined as usual;

$$(u \oplus v)(x) = \sup_{y+z=x} \min(u(y), v(z)),$$

$$(\lambda u)(x) = \begin{cases} u(x/\lambda), & \text{if } \lambda \neq 0 \\ I_{\{0\}}(x), & \text{if } \lambda = 0 \end{cases}$$

where $I_{\{0\}}$ is the indicator function of $\{0\}$.

Then it is well known that for each $\alpha \in [0,1]$,

$$L_\alpha(u \oplus v) = L_\alpha u \oplus L_\alpha v$$

and

$$L_\alpha(\lambda u) = \lambda L_\alpha u.$$

Now, the uniform metric $d_\infty$ on $F(R^p)$ is defined by

$$d_\infty(u, v) = \sup_{0 \leq \alpha \leq 1} h(L_\alpha u, L_\alpha v).$$

Also, the norm of $u \in F(R^p)$ is defined as

$$\| u \| = d_\infty(u, I_{\{0\}}) = \sup_{x \in L_0 u} | x | .$$

Then it is well-known that $(F(R^p), d_\infty)$ is complete, but is not separable. (See Klement et al. [11]). Joo and Kim [5, 6] introduced a new metric on $F(R^p)$ which makes it a separable metric space as follows:

**Definition 2.3.** Let $T$ be the class of strictly increasing continuous mappings of $[0,1]$ onto itself. For $u, v \in F(R^p)$, we define

$$d_s(u, v) = \inf\{\varepsilon > 0 : \text{there exists a } t \in T \text{ such that } \sup_{0 < \alpha \leq 1} |t(\alpha) - \alpha| \leq \varepsilon \text{ and } d_\infty(u, t(v)) \leq \varepsilon\},$$

where $t(v)$ denotes the composition of $v$ and $t$.

It follows immediately that $d_s$ is a metric on $F(R^p)$ and $d_s(u, v) \leq d_\infty(u, v)$. The metric $d_s$ will be called the Hausdorff-Skorohod metric.

## 3. Main Results

Let $(\Omega, \mathcal{F}, P)$ be a probability space. A set-valued function $X : \Omega \to (K(R^p), h)$ is called a random set if it is measurable.

A random set $X$ is called integrably bounded if $E\|X\| < \infty$. The expectation of integrably bounded random set $X$ is defined by

$$E(X) = \{E(\xi) \mid \xi \in L(\Omega, R^p) \text{ and } \xi(\omega) \in X(\omega) \text{ a.s.}\},$$

where $L(\Omega, R^p)$ denotes the class of all $R^p$ valued random variables $\xi$ such that $E|\xi| < \infty$.

A fuzzy number valued function $\tilde{X} : \Omega \to F(R^p)$ is called a fuzzy random variable if it is measurable. It is well known that $\tilde{X}$ is a fuzzy random variable if and only if for each $\alpha \in [0,1]$, $L_\alpha \tilde{X}$ is a random set (For details, see Kim [8]).

A fuzzy random variable $\tilde{X}$ is called integrably bounded if $E\|\tilde{X}\| < \infty$. The expectation of integrably bounded fuzzy random variable $X$ is a fuzzy number defined by

$$E(\tilde{X})(x) = \sup\{\alpha \in [0,1] \mid x \in E(L_\alpha \tilde{X})\}.$$

It is well-known that if $\tilde{X}$, $\tilde{Y}$ are integrably bounded, then

(1) $L_\alpha E(\tilde{X}) = E(L_\alpha \tilde{X})$ for all $\alpha \in [0,1]$.

(2) $E(\tilde{X} \oplus \tilde{Y}) = E(\tilde{X}) \oplus E(\tilde{Y})$.

(3) $E(\lambda \tilde{X}) = \lambda E(\tilde{X})$.

**Lemma 3.1.** Let $E\|\tilde{X}\| < \infty$. Then for each $\varepsilon > 0$, there exists a partition $0 = \alpha_1 < \alpha_2 < \ldots < \alpha_m = 1$ of $[0,1]$ such that

$$E[h(L_{\alpha_{k-1}^+}\tilde{X}, L_{\alpha_k}\tilde{X})] < \varepsilon \tag{3.1}$$

for all $k=1,2,\ldots,m$.

**Proof.** For $0 \le \alpha < \beta \le 1$, let

$$\phi(\alpha,\beta) = E[h(L_{\alpha^+}\tilde{X}, L_\beta \tilde{X})].$$

We define

$$\gamma_1 = \begin{cases} 1, & \text{if } \phi(0,1) < \varepsilon, \\ \inf\{\beta \mid \phi(0,\beta) \ge \varepsilon\}, & \text{otherwise,} \end{cases}$$

and for $j \ge 2$,

$$\gamma_j = \begin{cases} 1, & \text{if } \phi(\gamma_{j-1},1) < \varepsilon, \\ \inf\{\beta > \gamma_{j-1} \mid \phi(\gamma_{j-1},\beta) \ge \varepsilon\}, & \text{otherwise.} \end{cases}$$

Now we must show that $\gamma_j = 1$ for some $j$.

Suppose that $\gamma_j < 1$ for all $j$. Then by left-continuity of $L_\alpha \tilde{X}$ as a function of $\alpha$, there exists a sequence $\{\alpha_n\}$ with $\gamma_n < \alpha_n < \gamma_{n+1}$ such that

$$E[h(L_{\alpha_n}\tilde{X}, L_{\gamma_{n+1}}\tilde{X})] \ge \varepsilon/2 \tag{3.2}$$

for all $n$. Since $\{\gamma_n\}$ is monotone increasing sequence, it follows that

$$h(L_{\alpha_n}\tilde{X}, L_{\gamma_{n+1}}\tilde{X}) \to 0 \text{ pointwise.}$$

Then by the dominated convergence theorem,

$$E[h(L_{\alpha_n}\tilde{X}, L_{\gamma_{n+1}}\tilde{X})] \to 0,$$

whish is impossible by (3.2). Q.E.D.

**Remark.** If $E\|\tilde{X}\| < \infty$, by applying Lemma 2.2 of Joo and Kim [6] we can obtain the following:

For each $\varepsilon > 0$, there exists a partition $0=\alpha_1<\alpha_2<\ldots<\alpha_m=1$ of $[0,1]$ such that

$$h(L_{\alpha_{k-1}^+}E(\tilde{X}), L_{\alpha_k}E(\tilde{X})) < \varepsilon \tag{3.3}$$

for all $k=1,2,\ldots,m$. Since

$$h(L_{\alpha^+}E(\tilde{X}), L_\beta E(\tilde{X})) \le E[h(L_{\alpha^+}\tilde{X}, L_\beta\tilde{X})],$$

it follows that (3.1) is stronger than (3.3).

**Definition 3.2.** Let $\{\tilde{X}_n\}$ be a sequence of fuzzy random variables. $\{\tilde{X}_n\}$ is said to be tight if for each $\varepsilon>0$, there exists a compact subset $A$ of $F(R^p)$ relative to $d_s$-topology such that

$$P(\tilde{X}_n \notin A) < \varepsilon \text{ for all } n.$$

The purpose of this paper is to obtain SLLN for sums of tight fuzzy random variables as a generalization of independent and identically distributed case. By lemma 3.1, it is natural that the condition (3.1) should be assumed to this end.

**Theorem 3.3.** Let $\{\tilde{X}_n\}$ be a sequence of independent and tight fuzzy random variables. Suppose that the following two conditions are satisfied;

(1) $\sup_n E\|\tilde{X}_n\|^r = M < \infty$ for some $r > 1$.

(2) For each $\varepsilon > 0$, there exists a partition $0=\alpha_1<\alpha_2<\ldots<\alpha_m=1$ of $[0,1]$ such that

$$E[h(L_{\alpha_{k-1}^+}\tilde{X}_n, L_{\alpha_k}\tilde{X}_n)] < \varepsilon$$

for all $n$ and all $k=1,2,\ldots,m$.

Then

$$\lim_{n\to\infty}\frac{1}{n} d_\infty(\oplus_{i=1}^n \tilde{X}_i, \oplus_{i=1}^n E(\tilde{X}_i)) = 0 \quad a.s.$$

**Proof:** Let $\varepsilon>0$ be given. By assumption (2), we choose a partition $0=\alpha_1<\alpha_2<\ldots<\alpha_m=1$ of $[0,1]$ such that

$$E[h(L_{\alpha_{k-1}^+}\tilde{X}_n, L_{\alpha_k}\tilde{X}_n)] < \varepsilon \tag{3.4}$$

for all $n$ and all $k=1,2,\ldots,m$.

Now, if $\alpha_{k-1} < \alpha \le \alpha_k$, then

$$\begin{aligned}
& h(\oplus_{i=1}^n L_\alpha \tilde{X}_i, \oplus_{i=1}^n E(L_\alpha \tilde{X}_i)) \\
&\le h(\oplus_{i=1}^n L_\alpha \tilde{X}_i, \oplus_{i=1}^n L_{\alpha_k}\tilde{X}_i) \\
&\quad + h(\oplus_{i=1}^n L_{\alpha_k}\tilde{X}_i, \oplus_{i=1}^n E(L_{\alpha_k}\tilde{X}_i)) \\
&\quad + h(\oplus_{i=1}^n E(L_\alpha \tilde{X}_i), \oplus_{i=1}^n E(L_{\alpha_k}\tilde{X}_i)) \\
&\le \sum_{i=1}^n h(L_{\alpha_{k-1}^+}\tilde{X}_i, L_{\alpha_k}\tilde{X}_i) \\
&\quad + h(\oplus_{i=1}^n L_{\alpha_k}\tilde{X}_i, \oplus_{i=1}^n E(L_{\alpha_k}\tilde{X}_i)) \\
&\quad + \sum_{i=1}^n h(E(L_{\alpha_{k-1}^+}\tilde{X}_i), E(L_{\alpha_k}\tilde{X}_i)).
\end{aligned}$$

Thus,

$$\begin{aligned}
& \frac{1}{n}\sup_{\alpha_{k-1}<\alpha\le\alpha_k} h(\oplus_{i=1}^n L_\alpha \tilde{X}_i, \oplus_{i=1}^n E(L_\alpha \tilde{X}_i)) \\
&\le \frac{1}{n}\sum_{i=1}^n h(L_{\alpha_{k-1}^+}\tilde{X}_i, L_{\alpha_k}\tilde{X}_i) \\
&\quad + \frac{1}{n} h(\oplus_{i=1}^n L_{\alpha_k}\tilde{X}_i, \oplus_{i=1}^n E(L_{\alpha_k}\tilde{X}_i)) \\
&\quad + \frac{1}{n}\sum_{i=1}^n h(E(L_{\alpha_{k-1}^+}\tilde{X}_i), E(L_{\alpha_k}\tilde{X}_i)) \\
&= (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III})
\end{aligned}$$

For (I). first we note that

$$E[h(L_{\alpha_{k-1}^+}\tilde{X}_n, L_{\alpha_k}\tilde{X}_n)^r] \le 2^r E\|\tilde{X}_n\|^r \le 2^r M$$

Thus, by SLLN for real-valued random variables, we have

$$\frac{1}{n}\sum_{i=1}^n [h(L_{\alpha_{k-1}^+}\tilde{X}_i, L_{\alpha_k}\tilde{X}_i) - Eh(L_{\alpha_{k-1}^+}\tilde{X}_i, L_{\alpha_k}\tilde{X}_i)] \to 0 \quad a.s.$$

Therefore, by (3.4),

$$(\mathrm{I}) = \frac{1}{n}\sum_{i=1}^n [h(L_{\alpha_{k-1}^+}\tilde{X}_i, L_{\alpha_k}\tilde{X}_i) - Eh(L_{\alpha_{k-1}^+}\tilde{X}_i, L_{\alpha_k}\tilde{X}_i)]$$

$$+\frac{1}{n}\sum_{i=1}^{n}E[h(L_{\alpha_{k-1}^{+}}\tilde{X}_i, L_{\alpha_k}\tilde{X}_i)]$$
$$< \varepsilon \ a.s.$$

It is obvious that by SLLN for tight random sets( Taylor and Inoue [14], Uemura [15]),

$$(\mathrm{II}) \rightarrow 0 \ a.s.$$

Now for (III), we have by (3.4).

$$(\mathrm{III}) \leq \frac{1}{n}\sum_{i=1}^{n}Eh(L_{\alpha_{k-1}^{+}}\tilde{X}_i, L_{\alpha_k}\tilde{X}_i)$$
$$< \varepsilon.$$

Hence,

$$\frac{1}{n}\sup_{\alpha_{k-1}<\alpha\leq\alpha_k}h(\oplus_{i=1}^{n}L_\alpha\tilde{X}_i, \oplus_{i=1}^{n}E(L_\alpha\tilde{X}_i))$$
$$< 2\varepsilon \ a.s.$$

and so,

$$\frac{1}{n}d_\infty(\oplus_{i=1}^{n}\tilde{X}_i, \oplus_{i=1}^{n}E(\tilde{X}_i)) < 2\varepsilon \ a.s.$$

Since $\varepsilon$ is arbitrary, this completes the proof.
Q.E.D.

**Corollary 3.4.** Let $\{\tilde{X}_n\}$ be a sequence of independent and tight fuzzy random variables. Suppose that the following two conditions are satisfied;

(1) There exists a compact subset $A$ of $F(R^p)$ relative to the metric $d_s$ such that $P(\tilde{X}_n \in A) = 1$ for all $n$.

(2) For each $\varepsilon > 0$, there exists a partition $0 = \alpha_1 < \alpha_2 < \ldots < \alpha_m = 1$ of $[0,1]$ such that

$$E[h(L_{\alpha_{k-1}^{+}}\tilde{X}_n, L_{\alpha_k}\tilde{X}_n)] < \varepsilon$$

for all $n$ and all $k = 1, 2, \ldots, m$.

Then

$$\lim_{n\to\infty}\frac{1}{n}d_\infty(\oplus_{i=1}^{n}\tilde{X}_i, \oplus_{i=1}^{n}E(\tilde{X}_i)) = 0 \quad a.s.$$

**Corollary 3.5.** Let $\{\tilde{X}_n\}$ be a sequence of independent and identically distributed fuzzy random variables. If $E\|\tilde{X}_1\|^r < \infty$ fore some $r > 1$, then

$$\lim_{n\to\infty}d_\infty(\frac{1}{n}\oplus_{i=1}^{n}\tilde{X}_i, E(\tilde{X}_1)) = 0 \quad a.s.$$

**Example.** Let $A$ be a relatively compact and convex subset of $F(R^p)$ w.r.t. the metric $d_s$ and let $\{\tilde{X}_n\}$ be independent fuzzy random variables taking values in $A$. Then, by Theorem 3.11 of Kim [10], for each $\varepsilon > 0$, there exists a partition $0 = \alpha_1 < \alpha_2 < \ldots < \alpha_m = 1$ of $[0,1]$ such that

$$h(L_{\alpha_{k-1}^{+}}u, L_{\alpha_k}u)] < \varepsilon$$

for all $u \in A$ and all $k = 1, 2, \ldots, m$. Then

$$E[h(L_{\alpha_{k-1}^{+}}\tilde{X}_n, L_{\alpha_k}\tilde{X}_n)] < \varepsilon$$

for all $n$ and all $k = 1, 2, \ldots, m$. Thus, by Corollary 3.4,

$$\lim_{n\to\infty}\frac{1}{n}d_\infty(\oplus_{i=1}^{n}\tilde{X}_i, \oplus_{i=1}^{n}E(\tilde{X}_i)) = 0 \ a.s.$$

## 4. Conclusion

The result of this paper is a generalization of SLLN for sums of independent and identically distributed fuzzy random variables as indicated in Corollary 3.5. But it remains an open problem whether the SLLN holds if we exclude the condition (2) of Theorem 3.3.

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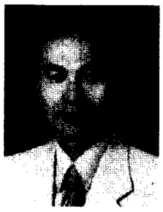

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