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Weak positive implicative hyperBCK-ideal

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Abstract

In this paper we define a weak positive implicative hyperBCK-ideal of hyperBCK-algebra. Also we investigate that every positive implicative hyperBCK-algebra is a positive implicative hyperK-algebra and then we prove that every positive implicative hyperK-algebra is a weak positive implicative hyperK-algebra.

Key words : weak positive implicative hyperBCK-algebra, weak positive implicative hyperBCK-ideal, positive implicative hyperK-algebra, weak positive implicative hyperK-algebra.

1. Introduction

The study of BCK-algebras was initiated by K.Iséki in 1966 as a generalization of the concept of set-theoretic difference and propositional calculus. Since then a great deal of literature has been produced on the theory of BCK-algebras. In particular, emphasis seems to have been put on the ideal theory of BCK-algebras.

The hyperstructure theory (called also multialgebras) is introduced in 1934 by F.Marty [7] at the 8th Congress of Scandinavian Mathematicians. Around the 40's, several authors worked on hypergroups, especially in France and in the United States, but also in Italy, Russia and Japan. Over the following decades, many important results appeared, but above all since the 70's onwards the most luxuriant flourishing of hyperstructures has been seen. Hyperstructures have many applications to several sectors of both pure and applied sciences. In [6], Y. B. Jun et al. applied the hyperstructures to BCK-algebras, and introduced the concept of a hyperBCK-algebra which is a generalization of a BCK-algebra, and investigated some related properties. They also introduced the notion of a hyperBCK-ideal and a weak hyperBCK-ideal, and gave relations between hyperBCK-ideals and weak hyperBCK-ideals. Y. B. Jun et al. [3] gave a condition for a hyperBCK-algebra to be a BCK-algebra, and introduced the notion of a strong hyperBCK-ideal, a weak hyperBCK-ideal and a reflexive hyperBCK-ideal. They showed that every strong hyperBCK-ideal is a hypersubalgebra, a weak hyperBCK-ideal and a hyperBCK-ideal. In this paper we define a weak positive implicative hyperBCK-ideal of hyperBCK-algebra. Also we investigate that every positive implicative hyper BCK-algebra is a positive implicative hyperK-algebra and then we prove that every positive implicative hyperK-algebra is a weak positive implicative hyperK-algebra.

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2. Preliminaries

In this paper, we introduce some definitions and properties by [4,5,6]. Note that the identity $x*(x*(x*y))=x*y$ holds in a BCK-algebra. A non-empty subset I of a BCK-algebra X is called an ideal of X if $0\in I$, and $x*y\in I$ and $y\in I$ imply $x\in I$ for all $x, y\in X$.

Let H be a non-empty set endowed with a hyperoperation "$\circ$" For two subsets A and B of H, denote by $A\circ B$ the set $\bigcup_{a\in A, b\in B} a\circ b$. We shall use $x\circ y$ instead of $x\circ\{y\}$, $\{x\}\circ y$, or $\{x\}\circ\{y\}$.

Definition 2.1 ([6]). By a *hyperBCK-algebra* we mean a non-empty set H endowed with a hyperoperation "$\circ$" and a constant 0 satisfying the following axioms:

(HK1) $(x\circ z)\circ(y\circ z)\ll x\circ y$
(HK2) $(x\circ y)\circ z=(x\circ z)\circ y$,
(HK3) $x\circ H\ll\{x\}$,
(HK4) $x\ll y$ and $y\ll x$ imply $x=y$.

for all $x, y, z\in H$, where $x\ll y$ is defined by $0\in x\circ y$ and for every $A, B\subseteq H$, $A\ll B$ is defined by $\forall a\in A$, $\exists b\in B$ such that $a\ll b$.

Proposition 2.2 ([6]). In a hyperBCK-algebra H, the condition (HK3) is equivalent to the condition:

$$x\circ y\ll\{x\} \text{ for all } x, y\in H.$$

Proposition 2.3 ([5]). In any hyperBCK-algebra H, the following hold :

(i) $0\circ 0=\{0\}$,
(ii) $0\ll x$,
(iii) $x\ll x$,
(iv) $A\ll A$,
(v) $A\subseteq B$ implies $A\ll B$,

(vi) $0 \circ x = \{0\}$,
(vii) $0 \circ A = \{0\}$,
(viii) $A \ll \{0\}$ implies $A = \{0\}$,
(ix) $A \circ B \ll A$,
(x) $x \in x \circ 0$,
(xi) $x \circ 0 \ll \{y\}$ implies $x \ll y$,
(xii) $y \ll z$ implies $x \circ z \ll x \circ y$,
(xiii) $x \circ y = \{0\}$ implies $(x \circ z) \circ (y \circ z) = \{0\}$ and $x \circ z \ll y \circ z$,
(xiv) $A \circ \{0\} = \{0\}$ implies $A = \{0\}$,
(xv) $x \circ 0 \ll \{x\}$, $0 \circ x \ll \{0\}$ and $0 \circ 0 \ll \{0\}$ for all $x \in H$,
(xvi) $(A \circ B) \circ C = (A \circ C) \circ B$, $A \circ B \ll A$ and $0 \circ A \ll \{0\}$ for all $x, y, z \in H$ and for all non-empty subsets A, B and C of H.

Definition 2.4 ([6]). Let$(H, \circ)$ be a hyperBCK-algebra and let S be a subset of H containing 0. If S is a hyperBCK-algebra with respect to the hyperoperation "$\circ$" on H, we say that S is a *hyper-subalgebra* of H.

By Definition 2.4, if S is a non-empty subset of a hyperBCK-algebra $(H, \circ)$ and $x \circ y \subseteq S$ for all $x, y \in S$, then $0 \in S$. Therefore S is a hyper subalgebra of H if and only if $x \circ y \subseteq S$ for all $x, y \in S$.

Proposition 2.5 ([4]). Let H be a hyperBCK-algebra. Then we have

$$(x \circ y) \circ z \ll (x \circ z) \circ (y \circ z) \text{ for all } x, y, z \in H.$$

Definition 2.6 ([4]). A hyperBCK-algebra H is said to be *weak positive implicative(resp. positive implicative)* if it satisfies the axiom

$$(x \circ z) \circ (y \circ z) \ll (x \circ y) \circ z (\text{ resp. } (x \circ z) \circ (y \circ z) = (x \circ y) \circ z)$$

for all $x, y, z \in H$.

Proposition 2.7([4]). Every positive implicative hyperBCK-algebra is a weak positive implicative hyperBCK-algebra.

Definition 2.8 ([5]). Let I be a non-empty subset of a hyperBCK-algebra $(H, 0)$. Then I is said to be a *hyperBCK-ideal* of H if

(HI1) $0 \in I, y \in I$
(HI2) $x \circ y \ll I$ and imply $x \in I$ for all $x, y \in H$.

Proposition 2.9([5]). If $\{I_\lambda \mid \lambda \in \Lambda\}$ is a family of hyperBCK-ideals of a hyperBCK-algebra $(H, 0)$, then $\bigcap\{I_\lambda \mid \lambda \in \Lambda\}$ is a hyperBCK-ideal of H.

Definition 2.10 ([4]). Let I be a non-empty subset of a hyperBCK-algebra $(H, 0)$. Then I is called *a weak hyperBCK-ideal* of H if

(HI1) $0 \in I$,
(WHI1) $x \circ y \subseteq I$ and $y \in I$ imply $x \in I$ for all $x, y \in H$.

Definition 2.11 ([4]). Let H be a hyperBCK-algebra. A non-empty subset I of H is said to be a *positive implicative hyperBCK-ideal* of H if $0 \in I$ and it satisfies:
(PI) $(x \circ y) \circ z \ll I$ and $y \circ z \subseteq I$ imply $x \circ z \subseteq I$ for all $x, y, z \in H$.

In a hyperBCK-algebra H, every positive implicative hyperBCK-ideal is a hyperBCK-ideal. Also if A is a subset of a hyperBCK-algebra H, and I is a hyperBCK-ideal of H such that $A \ll I$, then A is contained in I.

3. Weak positive implicative hyperBCK-ideal

In this chapter, we define a weak positive implicative hyperBCK-ideal and investigate some related properties.

Definition 3.1. Let H be a hyperBCK-algebra.
Then I is a *weak positive implicative hyperBCK-ideal* of H.
(i) $0 \in I$,
(WPI) $(x \circ y) \circ z \subseteq I$ and $y \circ z \subseteq I$, $x \circ z \subseteq I$, for all $x, y, z \in H$.

Example 3.2. Let $H = \{0, 1, 2\}$. Consider the following table:

o	0	1	2
0	{0}	{0}	{0}
1	{1}	{0,1}	{0,1}
2	{2}	{1,2}	{0,1,2}

Then $(H, \circ, 0)$ is a hyperBCK-algebra. Then $I_1 := \{0, 1\}$ and $I_2 := \{0, 2\}$ are weak positive implicative hyperBCK-ideal of H.

Theorem 3.3. Let $(H, \circ)$ be a hyperBCK-algebra. Then every positive impicative hyperBCK-ideal of H is a weak positive implicative hyperBCK-ideal of H.

Proof.

$$(x \circ y) \circ z \subseteq I \text{ and } y \circ z \subseteq I$$
$$\Rightarrow (x \circ y) \circ z \ll I \text{ and } y \circ z \subseteq I$$
$$\Rightarrow x \circ z \subseteq I.$$
□

The converse of above theorem may not be true. In fact, in Example 3.2 I_2 is not a positive implicative hyperBCK-ideal. Since

$(1 \circ 2) \circ 0 = \{0,1\} \ll I_2$ and $2 \circ 0 = \{2\}$, but $1 \circ 0 = \{1\} \not\subseteq I_2$.

Theorem 3.4. If $\{I_\lambda \mid \lambda \in \Lambda\}$ is a family of weak positive implicative hyperBCK-ideals of a hyperBCK-algebra $(H, \circ, 0)$, then so is $\bigcap_{\lambda \in \Lambda} I_\lambda$.

Proof. Clearly $0 \in \bigcap_{\lambda \in \Lambda} I_\lambda$. Let $x, y, z \in I_\lambda$ be such that $(x \circ y) \circ z \subseteq \bigcap I_\lambda$ and $y \circ z \subseteq \bigcap I_\lambda$. Then $(x \circ y) \circ z \subseteq I_\lambda$ and $y \circ z \subseteq \bigcap I_\lambda$, for all $\lambda \in \Lambda$. By Definition, we have $x \circ y \subseteq I_\lambda$, for all $\lambda \in \Lambda$, and hence $x \circ y \subseteq \bigcap_{\lambda \in \Lambda} I_\lambda$. □

Theorem 3.5. In a hyperBCK-algebra H, every weak positive implicative hyperBCK-ideal is a weak hyperBCK-ideal.

Proof. Let I be a weak positive implicative hyperBCK-ideal of a hyper BCK-algebra H and let $x, y \in H$ be such that $x \circ y \subseteq I$ and $y \in I$. Putting $z=0$ in (WPI), we get $(x \circ y) \circ 0 = x \circ y \subseteq I$ and $y \circ 0 = \{y\} \subseteq I$. It follows from (WPI) that $\{x\} = x \circ 0 \subseteq I$. Thus I is a weak hyperBCK-ideal of H. □

Theorem 3.6. Let H be a positive implicative hyperBCK-algebra. Then every weak hyperBCK-ideal is a weak positive implicative hyperBCK-ideal.

Proof. Let I be a positive implicative hyperBCK-algebra, and let $x, y, z \in H$ be such that $(x \circ z) \circ (y \circ z) = (x \circ y) \circ z \subseteq I$, $y \circ z \subseteq I$ which implies that $x \circ z \subseteq I$. Thus I is a weak positive implicative. □

Theorem 3.7. Let I be a weak positive implicative hyperBCK-ideal of a hyperBCK-algebra H and let $a \in H$. Then the set $I_a := \{x \in H \mid x \circ a \subseteq I\}$ is a weak hyperBCK-ideal of H.

Proof. Cleary $0 \in I_a$. Let $x, y \in H$ be such that $x \circ y \subseteq I_a$ and $y \in I_a$. Then $(x \circ y) \circ a \subseteq I$ and $y \circ a \subseteq$ I. It follows that $x \circ a \subseteq I$ or equivalently $x \in I_a$. Hence I_a is a weak hyperBCK-ideal of H. □

Theorem 3.8. Let I be a hyperBCK-ideal of a hyperBCK-algebra H. If $I_a := \{x \in H \mid x \circ a \subseteq I\}$ is a weak hyperBCK-ideal of H for all $a \in H$, then I is a weak positive implicative hyperBCK-ideal of H.

Proof. Let $x, y, z \in H$ be such that $(x \circ y) \circ z \subseteq I$ and $y \circ z \subseteq I$. Thus for each $t \in x \circ y$, we have $t \circ z \subseteq I$ or equivalently $t \in I_z$ and $y \in I_z$. Hence $x \circ y \subseteq I_z$ and $y \in I_z$. Since I_z is a weak hyperBCK-ideal of H, it follows that $x \in I_z$, i.e., $x \circ z \subseteq I$. Therefore I is a weak positive implicative hyperBCK-ideal of H. □

Theorem 3.9. Let I be a weak hyperBCK-ideal of a hyperBCK-algebra H. Then $A \circ B \subseteq I$ and $B \subseteq I$ imply that $A \subseteq I$ for every subsets A and $B(\neq \phi)$ of H.

Proof. Let $a \in A$ and $b \in B$. Then $a \circ b \subseteq A \circ B \subseteq I$. It follows from(WH1) that $a \in I$ so that $A \subseteq I$. □

Theorem 3.10. Let I be a subset of a hyperBCK-algebra H such that $x \circ x \subseteq I$ for all $x \in H$. I is a weak positive implicative hyperBCK-ideal of H for every $x, y \in H$, $(x \circ y) \circ y \subseteq I$ implies $x \circ y \subseteq I$.

Proof. Let $x, y \in H$ be such that $(x \circ y) \circ y \subseteq I$. Since $y \circ y \subseteq I$ by hypothesis, it follows from (WPI) that $x \circ y \subseteq I$. □

Definition 3.11. A hyperK-algebra H is said to be weak positive implicative(resp. positive implicative) if it satisfies the axiom

$$(x \circ z) \circ (y \circ z) < (x \circ y) \circ z \quad (\text{resp.} (x \circ z) \circ (y \circ z) = (x \circ y) \circ z)$$

for all $x, y, z \in H$.

Example 3.12. Let $H=\{0, 1, 2\}$. Consider the following table:

∘	0	1	2
0	{0}	{0}	{0}
1	{1}	{0,1}	{0,1}
2	{2}	{1,2}	{0,1,2}

$H=\{0, 1, 2\}$ is a positive implicative hyperK-algebra.

Example 3.13. Let $H=\{0, 1, 2\}$. Consider the following table:

∘	0	1	2
0	{0}	{0,1,2}	{0,1,2}
1	{1}	{0,1,2}	{0,1,2}
2	{2}	{1,2}	{0,1,2}

H={0, 1, 2} is a weak positive implicative hyperK-algebra.

The following proposition can be obtained directly from the above definition.

Proposition 3.14. Every positive implicative hyperBCK-algebra is a positive implicative hyperK-algebra.

Proposition 3.15. Every positive implicative hyperK-algebra is a weak positive implicative hyperK-algebra.

Note that the converse of Proposition 3.15 may not be true. Since $(2 \circ 1) \circ (0 \circ 1) \neq (2 \circ 0) \circ 1$.

Every hyperBCK-algebra is a hyperK-algebra. But in Example 3.13, since $(x \circ y) \circ (0 \circ y) = \{0, x, y\} \nless \{x\} = (x \circ 0)$, the converse is not true.

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