퍼지 및 지능시스템학회 논문지 2003, Vol. 13, No. 2, pp. 231-236

Notes on the compatibility between defuzzification and t-norm based fuzzy arithmetic operations

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요 약

Recently, Oussalah [Fuzzy Sets and Systems 128(2002) 247-260] investigated some theoretical results about some invariance properties concerning the relationships between the defuzzification outcomes and the arithmetic of fuzzy numbers. But, in this note we introduce some explicit calculations of the resulting fuzzy set or possibility distribution when the matter is the determination of the defuzzified value pertaining to the result of some manipulation of fuzzy quantities under t-norm based fuzzy arithmetic operations.

Key words : T-Extension principle, Fuzzy numbers, Defuzzification

1. Introduction

Solving a multivariable non-linear optimazation problem under operations of fuzzy numbers using Zadeh's extension principle [39] may be complex and highly time consuming. Many authors [2-9] investigated both parameterized formulations and approximate formulas to overcome the deficiency.

It is well known that reasoning in terms of fuzzy arithmetic may lose some basic properties that trivially hold for real arithmetic. For instance, the multiplication and the division of convex fuzzy sets may lead to a non-convex set (see, for instance, [6, 34]).

Further, the shape of the resulting fuzzy set is not preserved for the multiplication (and division) operation. Consequently, the problem of finding some invariance properties both in theoretical and computation points of view has been investigated by many researchers since the introduction of fuzzy arithmetic. Mesiar [32, 33], Hong [24], Kolesárová [27] investigated an invariance property in terms of shape preservation under addition of fuzzy intervals expressed in terms of L-R representation.

Recently, Oussalah [35] investigated the calculus of some arithmetic operations when reasoning in terms of the defuzzified values. Particularly, he proved that the defuzzification method based on level sets averaging method ALC induces some invariance properties. He also carried out an attempt to generalize the result to the case of T-extension principle, particularly for those t-norms whose value lie between the min and the Lukasiewicz operations.

Some results on t-norm based fuzzy arithmetic operations and shape preserving arithmetic operations of L-R fuzzy intervals can be found in [1, 13-22, 24]. Using these results in this note, we will introduce some explicit calculations of ALC of the resulting fuzzy set or possibility theory when the matter is the determination of the defuzzified value pertaining to the result of some manipulation of fuzzy quantities under t-norm based fuzzy arithmetic operations.

접수일자 : 2002년 10월 9일
완료일자 : 2003년 2월 24일
본 연구는 2003학년도 대구가톨릭대학교 일반연구비에 의해 지원받았습니다.

2. Preliminaries

2.1 Fuzzy arithmetic operations

A fuzzy number is a convex subset of the real line R with a normalized membership function. A triangular fuzzy number $\tilde{a}$ denoted by (a, α, β) is defined as

$$\tilde{a}(t) = \begin{cases} 1 - \frac{|a-t|}{\alpha} & \text{if } a-\alpha \le t \le a, \\ 1 - \frac{|a-t|}{\beta} & \text{if } a \le t \le a+\beta, \\ 0 & \textit{otherwise}, \end{cases}$$

where $a \in R$ is the center and $\alpha > 0$ is the left spread, $\beta > 0$ is the right spread of $\tilde{a}$.

If $\alpha = \beta$, then the triangular fuzzy number is called a symmetric triangular fuzzy number and denoted by (a, α).

Following Dubois and Prade [5], a fuzzy interval A is a so-called $L-R$ fuzzy interval, $A = (a, b, \alpha, \beta)_{LR}$, if the corresponding membership function satisfies, for all $x \in R$,

$$\tilde{A}(x)=\begin{cases}1 & a\le x\le b,\\ L\left(\frac{a-x}{\alpha}\right) & a-\alpha\le x\le a,\ \alpha>0,\\ R\left(\frac{x-b}{\beta}\right) & b\le x\le b+\beta,\ \beta>0,\\ 0 & otherwise,\end{cases}$$

where $[a,b]$ is the peak A, $\alpha>0$ and $\beta>0$ are the left and right spreads, respectively, and L and R are nonincreasing continuous functions from $[0,1]$ to $[0,1]$ such that $L((0,1))=R((0,1))=(0,1)$. Recall that L and R are called the left and right shape functions, respectively. If $R(x)=L(x)=1-x$, we denote $(a,b,\alpha,\beta)_{LR}$ by (a,b,α,β). If $L=R$ and $\alpha=\beta$, then the symmetric $L-L$ fuzzy number is denoted $(a,\alpha)_L$.

A binary operation T on the unit interval is said to be triangular norm (t-norm for short) iff T is associative, commutative, non-decreasing and $T(x,1)=x$ for each $x\in[0,1]$. Moreover, every t-norm satisfies the inequality,

$T_W(a,b)\le T(a,b)\le \min(a,b)=T_M(a,b)$,

where

$$T_W(a,b)=\begin{cases}a & \text{if } b=1,\\ b & \text{if } a=1,\\ 0 & otherwise.\end{cases}$$

The usual arithmetical operations of reals can be extended to the arithmetical operations on fuzzy numbers by means of Zadeh's extension principle [39] based on a triangular norm T. Let $\tilde{A}$, $\tilde{B}$ be fuzzy numbers of the real line R. The fuzzy number arithmetic operations are summarized as follows:

Fuzzy number addition $\oplus$:

$(\tilde{A}\oplus\tilde{B})(z)=\sup_{x+y=z}T(\tilde{A}(x),\tilde{B}(y))$.

Fuzzy number multiplication $\otimes$:

$(\tilde{A}\otimes\tilde{B})(z)=\sup_{x\cdot y=z}T(\tilde{A}(x),\tilde{B}(y))$.

The addition rule for $L-R$ fuzzy numbers is well known in the case of T_M-based addition and then the resulting sum is again on $L-R$ fuzzy numbers, i.e., the shape is preserved. It is also known that T_W-based addition preserves the shape of $L-R$ fuzzy numbers [27, 33]. In practical computation, it is natural to preserve the shape of fuzzy numbers during the multiplication. Unfortunately, the characteristics of multiplication have not yet been clarified well enough. Of course, we know that T_M-based multiplication does not preserve the shape of $L-R$ fuzzy numbers

The following theorem is due to Kolesárová [27, 28].

Theorem 1. (a) Let T be an arbitrary t-norm weaker than or equal to the Lukasiewicz t-norm T_L; $T(x,y)\le T_L(x,y)=\max(0,x+y-1)$, $x,y\in[0,1]$. Then the addition $\oplus$ based on T coincides on linear fuzzy intervals with the addition $\oplus$ based on the weakest t-norm T_W; i.e.,

$$(a_1,b_1,\alpha_1,\beta_1)\oplus(a_2,b_2,\alpha_2,\beta_2)$$
$$=(a_1+a_2,b_1+b_2,\max(\alpha_1,\alpha_2),\max(\beta_1,\beta_2)).$$

(b) Let T be a continuous Archimedean t-norm with convex additive generator f. Then the addition $\oplus$ based on T preserves the linearity of fuzzy intervals if and only if the t-norm T is a member of Yager's family of nilpotent t-norms with parameter $p\in[1,\infty)$, $T=T_p^Y$, and $f(x)=(1-x)^p$. Then $T_1^Y=T_L$ and for $p\in(0,\infty)$,

$$(a_1,b_1,\alpha_1,\beta_1)\oplus(a_2,b_2,\alpha_2,\beta_2)$$
$$=(a_1+a_2,b_1+b_2,(\alpha_1^q+\alpha_2^q)^{\frac{1}{q}},(\beta_1^q+\beta_2^q)^{\frac{1}{q}}).$$

where $(\frac{1}{q}+\frac{1}{q})=1$, i.e. $q=\frac{p}{p-1}$.

The above Theorem 1 gives some sufficient conditions for a t-norm T to serve as a basis for a linearity-preserving addition. Mesiar [33] conjectured that the only t-norm-based additions preserving the linearity of fuzzy intervals are those described in Theorem 1.

Hong [24] proved Mesiar's open problem as follows.

Theorem 2. Let a continuous t-norm T be not weaker than or equal to T_L. Then the addition $\oplus$ based on T preserves the linearity of fuzzy intervals if and only if the t-norm T is either T_M or a member of Yager's family of nilpotent t-norms with parameter $p\in(1,\infty)$, $T=T_p^Y$, and $f(x)=(1-x)^p$.

But it is known that, for given shapes L and R, T_W induces a shape-preserving multiplication of $L-R$ fuzzy numbers [23].

Hong and Do [23] proved the following result.

Lemma 1. Let $T=T_W$ and let $\tilde{A}_i=(a_i,\alpha_i,\beta_i)_{LR}$, $i=1,2,\cdots,p$ and let $a_i>0$, $x_i>0$, $i=1,2,\cdots,p$. Then, the possibilistic linear function with fuzzy parameter A_i and fuzzy variables $\tilde{X}_i$, $i=1,2,\cdots,p$, is given by

$$\tilde{Y}=(\tilde{A}_1\otimes\tilde{X}_1)\oplus(\tilde{A}_2\otimes\tilde{X}_2)\oplus\cdots\oplus(\tilde{A}_n\otimes\tilde{X}_n)$$
$$=\left(\sum_{i=1}^{p}a_ix_i,\ \max_{1\le i\le p}(a_i\gamma_i,x_i\alpha_i),\ \max_{1\le i\le p}(a_i\delta_i,x_i\beta_i)\right)_{LR}.$$

Recall that a t-norm T is Archimedean iff T is continuous and $T(x,x)<x$ for all $x\in(0,1)$. A well-known theorem (see [36]) asserts that for each Archimedean t-norm there exists a continuous, decreasing function $f:[0,1]\to[0,\infty]$ with $f(1)=0$ such that

$$T(x_1,\cdots,x_n)=f^{[-1]}(f(x_1)+\cdots+f(x_n))$$

for all $x_i \in (0,1)$, $a \le i \le n$.

Here $f^{[-1]}:[0,\infty] \to [0,1]$ is defined by

$$f^{[-1]}(y)=\begin{cases} f^{-1}(y) & \text{for } y \in (o, f(0)], \\ 0 & \text{if } y > f(0). \end{cases}$$

The function f is called the additive generator of T.

If T is a t-norm and $\widetilde{a_1}, \widetilde{a_2}, \cdots, \widetilde{a_n}$ are fuzzy sets of the real line then their T-sum $\widetilde{A_n} = \widetilde{a_1} \oplus \widetilde{a_2} \oplus \cdots \oplus \widetilde{a_n}$ is defined by (see [36])

$$\widetilde{A_n}(z) = \sup_{x_1+x_2+\cdots+x_n=z} T(\widetilde{a_1}(x_1), \widetilde{a_2}(x_2), \cdots, \widetilde{a_n}(x_n)), \quad z \in \Re.$$

Since f is continuous and decreasing, $f^{[-1]}$ is also continuous and non-increasing, we have

$$\widetilde{A_n}(z) = \sup_{x_1+\cdots+x_n=z} f^{[-1]}\left(\sum_{i=1}^{n}(f(\widetilde{a_i}(x_i))\right)$$
$$= f^{[-1]}\left(\inf_{x_1+\cdots+x_n=z}\left(\sum_{i=1}^{n}(f(\widetilde{a_i}(x_i))\right)\right).$$

The exact output of a T-sum of some $L-R$ fuzzy numbers in an analytical form was found only for some special cases. For Archimedean t-norm T with additive generator f, the most general published result is due to Hong and Hwang [18], generalizing the earlier results of Fullér and Keresztfalvi [13].

Theorem 3. Let T be an Archimedean t-norm with additive generator f and let $\widetilde{a_i} = (a_i, \alpha, \beta)_{LR}$, $i=1,\cdots,n$, be $L-R$ fuzzy numbers. If L and R are concave functions, and f is a convex function, then the membership function of the T-sum $\widetilde{A_n} = \widetilde{a_1} \oplus \cdots \oplus \widetilde{a_n}$ is

$$\widetilde{A_n}(z)=\begin{cases} f^{[-1]}\left(n \cdot f\left(L\left(\frac{A_n - z}{n \cdot \alpha}\right)\right)\right) & \text{if } A_n - n\alpha \le z \le A_n, \\ f^{[-1]}\left(n \cdot f\left(R\left(\frac{z - A_n}{n \cdot \beta}\right)\right)\right) & \text{if } A_n \le z \le A_n + n\beta, \\ 0 & \textit{otherwise}, \end{cases}$$

where $A_n = a_1 + \cdots + a_n$.

2.2 Some defuzzification methods

Zhao and Govidn [40] inverstigated some defuzzification methods on fuzzy intervals. More recently, an interesting axiomatic comparison of different defuzzification methods is pointed out by Van Leekwijck and Kerre in [37].

The most common used methods for the defuzzification are the center of area (COA) and the mean of the maxima (MOM). They correspond, respectively, to the center of gravity of the air supported by the distribution and the mean of the core of the distribution.

Yager and Filev [10, 38] have formulated a general defuzzification method called basic defuzzification distribution (BADD).

Yager and Filev have proposed another approach called semi-linear defuzzification (SLIDE) [38].

Dubois and Prade [8] have discussed another defuzzification procedure by averaging the α-cut sets. This method is justified by the insufficient reason principle. One denotes this method as averaging level cuts (ALC). More formally, let $[\pi^1_{\alpha_i}, \pi^2_{\alpha_i}]$ be the boundaries of the α_i-cut set (with $i=1-N$ and $0 < \alpha_1 < \cdots < \alpha_N = 1$) pertaining to the distribution π, then the defuzzified value is given by

$$(x^*)_{ALC} = \int_0^1 \left(\frac{\inf \pi_\alpha + \sup \pi_\alpha}{2}\right) d\alpha,$$

which, in discrete case, comes down to the mean value of the different α_i-sets, or, more specifically, the arithmetic mean of the middles of the different intervals pertaining delimiting the different α_i-sets.

$$(x^*)_{ALC} = M(\pi) = \frac{1}{N}\sum_{i=1}^{N}\frac{\pi^1_{\alpha_i} + \pi^2_{\alpha_i}}{2}.$$

This method has also been used for ranking fuzzy numbers by a method based on area compensation introduced by Fortemps and Roubens [12].

Yager and Filev [11, 38] have also proposed a defuzzification method based on level set method. Their proposal yields an optimistic value and a pessimistic one.

It is easy to see that for symmetrical distribution, the defuzzification result remains unchanged when using COA, MOM or ALC method.

3. Explicit calculations of defuzzified values of the resulting fuzzy sets

Oussalah [35], recently, investgated some relationships between fuzzy arithmetic and defuzzified values by using t-norm based arithmetic operations based on level sets averaging method ALC. In this section, we will introduce some explicit calculations of defuzzified values of the fuzzy numbers based on level sets averaging method ALC under t-norm based fuzzy arithmetic operations using known results about T-sum and shape preserving operation of $L-R$ fuzzy numbers. The cases for $L-R$ fuzzy intervals is similar.

Lemma 2. Let $\widetilde{A} = (a, \alpha, \beta)_{LR}$ with strictly decreasing L and R functions, then we have

$$(\widetilde{A})_{ALC}=a+\frac{1}{2}\left(\beta\int_0^1 R^{-1}(t)dt-\alpha\int_0^1 L^{-1}(t)dt\right).$$

Proof.
Since $\{\widetilde{A}\geq t\}=[a-\alpha L^{-1}(t),\ a+\beta R^{-1}(t)]$,

$$(\widetilde{A})_{ALC}=\int_0^1\frac{[(a-\alpha L^{-1}(t))+(a+\beta R^{-1}(t))]dt}{2}$$
$$=a+\frac{1}{2}\left(\beta\int_0^1 R^{-1}(t)dt-\alpha\int_0^1 L^{-1}(t)dt\right).$$

Proposition 1. Let T be an Archimedean t-norm with additive generator f and let $\widetilde{a}_i=(a_i,\alpha,\beta)_{LR}$, $i=1,\cdots,n$, be $L-R$ fuzzy numbers. If L and R are strictly decreasing concave functions, and f is a convex function, then

$$(\widetilde{a}_1\oplus\cdots\oplus\widetilde{a}_n)_{ALC}$$
$$=(a_1+\cdots+a_n)+\frac{1}{2}\int_0^1 n\Big[\beta(f\circ R)^{-1}\Big(\frac{f(t)}{n}\Big)-\alpha((f\circ L)^{-1}\Big(\frac{f(t)}{n}\Big)\Big]dt.$$

Proof. From Theorem 1, we have for $0<t<1$,
$\{\widetilde{A}_n\geq t\}=\Big[A_n-n\alpha(f\circ L)^{-1}\Big(\frac{f(t)}{n}\Big),$
$A_n+n\beta(f\circ R)^{-1}\Big(\frac{f(t)}{n}\Big)\Big]$, and hence the result follows immediately from Lemma 2.

Example 1.
Let T be a member of Yager's family of t-norms i.e., $T(u,v)=1-\min\{1,\sqrt{(1-u)+(1-v)}\}$ with generator $f(x)=(1-x)$, and $R(x)=1-x^2$, $L(x)=1-\sqrt{x}$. Let $\widetilde{a}_1=\widetilde{a}_2=(3,1,2)_{LR}$.
Then $(f\circ L)^{-1}(t)=t^2$ and $(f\circ R)^{-1}(t)=\sqrt{x}$ and hence, by Proposition 1,

$$(\widetilde{a}_1+\widetilde{a}_2)_{ALC}=6+\left[2\int_0^1\sqrt{\frac{1-t}{2}}\,dt-\int_0^1\Big(\frac{1-t}{2}\Big)^2dt\right]$$
$$=6+\Big(\frac{2\sqrt{2}}{3}-\frac{1}{12}\Big)=6+\frac{8\sqrt{2}-1}{12}$$
$$\approx 6.86\,.$$

Proposition 2. Let T be an arbitrary t-norm weaker than or equal to Lukasiewicz t-norm and $\widetilde{a}_i=(a_i,\alpha_i,\beta_i)$, $i=1,2,\cdots,n$. Then we have

$$(\widetilde{a}_1\oplus\cdots\oplus\widetilde{a}_n)_{ALC}$$
$$=(a_1+\cdots+a_n)+\frac{1}{4}\Big(\max_{1\leq i\leq n}\beta_i-\max_{1\leq i\leq n}\alpha_i\Big)$$

Proof. It comes immediately from Theorem 1 and Lemma 2.
Similarly we have the following result.

Proposition 3. Let T be a member of Yager's family of t-norms with parameter $p\in[1,\infty)$ and let $\widetilde{a}_i=(a_i,\alpha_i,\beta_i)$, $i=1,\cdots,n$. Then we have

$$(\widetilde{a}_1\oplus\cdots\oplus\widetilde{a}_n)_{ALC}$$
$$=(a_1+\cdots+a_n)+\frac{1}{4}\left(\Big(\sum_{i=1}^n\beta_i^q\Big)^{\frac{1}{q}}-\Big(\sum_{i=1}^n\alpha_i^q\Big)^{\frac{1}{q}}\right)$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Note. Proposition 2 has remarkably nice and simple representation for the defuzzified value of sum of triangular fuzzy numbers whereas Proposition 5, 6 and 7 of Oussalah [35] do not.

Finally we have an exact calculation of the defuzzified valued of the possibilistic linear function with fuzzy parameter and fuzzy variables using Lemma 1 and 2.

Proposition 4. Let $T=T_W$ and let
$\widetilde{A}_i=(a_i,\alpha_i,\beta_i)_{LR}$, $X_i=(x_i,\gamma_i,\delta_i)_{LR}$, $i=1,2,\cdots,p$
and let $a_i>0$, $x_i>0$, $i=1,\cdots,p$. Let
$\widetilde{Y}=(\widetilde{A}_1\otimes\widetilde{X}_1)\oplus(\widetilde{A}_2\otimes\widetilde{X}_2)\oplus\cdots\oplus(\widetilde{A}_n\otimes\widetilde{X}_n)$,
then we have

$$(\widetilde{Y})_{ALC}=\sum_{i=1}^p a_ix_i+\frac{1}{2}\Big[\max_{1\leq i\leq p}(a_i\delta_i,x_i\beta_i)\int_0^1 R^{-1}(t)dt-\max_{1\leq i\leq p}(a_i\gamma_i,x_i\alpha_i)\int_0^1 L^{-1}(t)dt\Big].$$

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