Trends in Researches for Fourth Order Elliptic Equations with Dirichlet Boundary Condition*

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Abstract

The nonlinear fourth order elliptic equations with jumping nonlinearity was modeled by McKenna. We investigate the trends for the researches of the existence of solutions of a fourth order semilinear elliptic boundary value problem with Dirichlet boundary condition, $\Delta^2 u + c\Delta u = b_1[(u+1)^+ - 1] + b_2 u^+$ in Ω, where Ω is a bounded open set in R^N with smooth boundary $\partial\Omega$.

0. Historical Review

The nonlinear fourth order elliptic equations with jumping nonlinearity was modeled by McKenna. In this article we investigate the trends for the researches of the existence of solutions of a fourth order semilinear elliptic equation

$$\Delta^2 u + c\Delta u = b_1[(u+1)^+ - 1] + b_2 u^+ + f \quad \text{in } \Omega \tag{1}$$

$$u = 0, \quad \Delta u = 0 \quad \text{on } \partial\Omega$$

where Δ^2 denote the biharmonic operator, Δ is the laplacian on R^N, $u^+ = \max\{u, 0\}$, Ω is a smooth open bounded set of R^N. Here $\lambda_1 < c < \lambda_2$ where $\{\lambda_k\}_{k\geq 1}$ denote the

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sequence of the eigenvalue of $-\Delta$ in $H_0^1(\Omega)$ and b_1, b_2 are not eigenvalue of $\Delta^2+c\Delta$.

The existence of solutions of the fourth order elliptic boundary value problem

$$\Delta^2 u + c\Delta u = bg(x, u) \quad \text{in} \quad \Omega \tag{2}$$
$$u=0, \quad \Delta u = 0 \quad \text{on} \ \partial\Omega$$

have been extensively studied by many authors. Choi and Jung [1] proved by the Variation of Linking Theorem that problem (2) has at least two solutions under some condition on g. Micheletti and Pistoia [7] proved that problem (2) has at least two solutions by the classical mountain pass theorem when $\lambda_1 < c$ and $b < \lambda_1(\lambda_1 - c)$. Choi and Jung [2] studied the problem, with Dirichlet boundary condition,

$$\Delta^2 u + c\Delta u = b_2 u^+ + f \quad \text{in} \quad \Omega \tag{3}$$

They proved by variational reduction method that (3) has at least two solutions when $\lambda_1 < c < \lambda_2$ and $b < \lambda_1(\lambda_1 - c)$ and $f = s > 0$, or $c < \lambda_1$, $\lambda_k(\lambda_k - c) < b < \lambda_{k+1}(\lambda_{k+1} - c)$ $(k = 1, 2, \cdots)$ and $f = s < 0$.

The nonlinear biharmonic equation with jumping nonlinearity problem

$$\Delta^2 u + c\Delta u = b_1[(u+1)^+ - 1] \quad \text{in} \quad \Omega \tag{4}$$
$$u=0, \quad \Delta u = 0 \quad \text{on} \quad \partial\Omega$$

was studied by some authors. Lazer and McKenna [3] has been pointed out that this kind of nonlinearity furnishes a good model to study travelling waves in a suspension bridge. In [4] the authors proved the existence of $2k-1$ solutions of (4) when $\Omega \subset R$ is an interval and $b > \lambda_k(\lambda_k - c)$, by global bifurcation method. Tarantello [9] showed by degree theory that if $b \geq \lambda_1(\lambda_1 - c)$, then (4) has a solution u such that $u(x) < 0$ in Ω, when $c < \lambda_1$. Micheletti and Pistoia [7] showed that there exist two solutions when $b \geq \lambda_1(\lambda_1 - c)$ and three solutions when b is close to $\lambda_k(\lambda_k - c)$ for a more general nonlinearity source term.

It is natural to consider equation (1) that the nonlinear term has both bu^+ and $b[(u+1)^+ - 1]$. The multiplicity of solutions of problem (1) is closely related to the position of c, b_1 and b_2.

In this article we investigate the existence and multiplicity of solutions for the Dirichlet boundary value problem in different six regions of (b_1, b_2) when $\lambda_1 < c < \lambda_2$ and $f=0$. We also show the existence of solutions when the source term is non-constant.

1. Sobolev Space and Linking Theorem

We introduce the Sobolev space spanned by the eigenfunctions of the operator $\Delta^2 + c\Delta$ with Dirichlet boundary condition.

Let λ_k denote the eigenvalues and e_k the corresponding eigenfunctions, suitably normalized with respect to $L^2(\Omega)$ inner product, of the eigenvalue problem $\Delta u + \lambda u = 0$ in Ω, with Dirichlet boundary condition, where each eigenvalue λ_k is repeated as often as its multiplicity.

We recall that $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$, $\lambda_i \to +\infty$ and that $e_1 > 0$ for all $x \in \Omega$.

The eigenvalue problem with Dirichlet boundary condition, $\Delta^2 u + c\Delta u = \lambda u$ in Ω. has infinitely many eigenvalues $\Lambda_k(c) = \lambda_k(\lambda_k - c)$, $k = 1, 2, \cdots$ and corresponding eigenfunctions e_k. Set

$$H_k = \text{span}\{e_1, \cdots, e_k\}, \qquad H_k^{\perp} = \{w \in H \mid (w, v)_H = 0, \ \forall v \in H_k\}.$$

Let $H = H_2(\Omega) \cap H_0^1(\Omega)$ be the Hilbert space equipped with the inner product $(u, v)_H = \int \Delta u \Delta v + \int \nabla u \nabla v$. When $f=0$ the functional $G: H \to R$ corresponding to (1) is given by

$$G(u) = \frac{1}{2}\int [(\Delta u)^2 - c|\nabla u|^2 - b_2(u^+)^2] - \frac{b_1}{2}\int \{[(u+1)^+]^2 - 2u - 1\}. \tag{5}$$

We rewrite $G(u)$ as follows

$$G(u) = \frac{1}{2}\int [(\Delta u)^2 - c|\nabla u|^2 - b_2(u^+)^2] - \frac{b_1}{2}\int u^2 + \frac{b_1}{2}\int [(u+1)^-]^2. \tag{6}$$

We note that the functional G is $C^1(H, R)$ (cf. [3, 11]) and its critical points are weak solutions of problem (1).

Definition 1.1. We say G satisfies the (PS) condition if any sequence $u_k \in H$ for which $G(u_k)$ is bounded and $G'(u_k) \to 0$ as $k \to \infty$ possesses a convergent subsequence.

The (PS) condition is a convenient way to build some "compactness" into the functional G. Indeed observe that (PS) implies that $K_c \equiv \{u \in H \mid G(u) = c \text{ and } G'(u) = 0\}$, i.e. the set of critical points having critical value c, is compact for any $c \in R$. For any $c \in R$, $b_1 + b_2 \neq \Lambda_j(c)$ $(j = 1, 2, \cdots)$ and b_1, $b_2 \neq 0$ the funcional G satisfies the (PS) condition.

To introduce a Variational Linking Theorem, we define the following sets.

Definition 1.2. Let X be an Hilbert space, $Y \subset X$, $\rho > 0$, and $e \in X/Y$ $e \neq 0$. Set

$$B_\rho(Y) = \{x \in Y \mid |x|_X \leq \rho\},$$

$$S_\rho(Y) = \{x \in Y \mid |x|_X = \rho\},$$

$$\Delta_\rho(e, Y) = \{\sigma e + v \mid \sigma \geq 0,\ v \in Y,\ |\sigma e + v|_X \leq \rho\}$$

$$\Sigma_\rho(e, Y) = \{\sigma e + v \mid \sigma \geq 0,\ v \in Y,\ |\sigma e + v|_X = \rho\} \cup \{v \mid v \in Y,\ |v|_X \leq \rho\}$$

We recall a theorem of existence of two critical levels for a functional which is a variation of linking theorem.

Theorem 1.3 (a Variation of Linking). Let X be an Hilbert space, which is topological direct sum of the subspaces X_1 and X_2. Let $f \in C^1(X, R)$. Moreover assume

(a) dim $X_1 < +\infty$,

(b) there exist $\rho > 0$, $R > 0$ and $e \in X_1$, $e \neq 0$ such that $\rho < R$ and

$$\sup_{S_\rho(X_1)} f < \inf_{\Sigma_R(e,\, X_2)} f,$$

(c) $-\infty < a = \inf_{\Delta_R(e,\, X_2)} f$,

(d) $(\text{PS})_c$ condition holds for any $c \in [a, b]$ where $b = \sup_{S_\rho(X_1)} f$.

Then there exist at least two critical levels c_1 and c_2 for the functional f such that $a \le c_1 \le \sup_{S_\rho(X_1)} f < \inf_{\Sigma_R(e,\, X_2)} f \le c_2 \le b$.

2. The Investigation for Nontrivial Solution

In this section we suppose that $f = 0$. We investigate the existence and multiplicity of solutions for problem (1) in six regions of $(b_1,\ b_2)$ when $\lambda_1 < c < \lambda_2$.

(i) Uniqueness

The following three theorem is the uniqueness results for problem (1).

Theorem 2.1. Assume that $b_1 + b_2 < \Lambda_1(c)$, $\Lambda_1(c) < b_1 < 0$. Then problem (1) has only the trivial solution.

Theorem 2.2. Assume that $b_1 + b_2 < \Lambda_1(c)$, $b_1 > 0$. Then problem (1) has only the trivial solution.

Let $L = \Delta^2 + c\Delta$ and $k = \dfrac{\Lambda_1 + \Lambda_2}{2}$. We rewrite (1) as

$$Lu = b_1[(u+1)^+ - 1] + b_2 u^+$$

or equivalently

$$Lu - ku = b_1[(u+1)^+ - 1] + b_2 u^+ - ku.$$

Since $\lambda_1 < c < \lambda_2$ the eigenvalue of operator L satisfies $\Lambda_1(c) < 0 < \Lambda_2(c) \le \Lambda_3(c)$, we have the following uniqueness result.

Theorem 2.3. Let $|\Lambda_1| < |\Lambda_2|$. Suppose that $b_1 > |k|$, $b_2 > |k|$, and $b_1 + b_2 < \Lambda_2$. Then equation (7) has only the trivial solution.

(ii) At least two solutions

We prove the existence of two solutions for problem (1).

Theorem 2.4. Assume that $b_1 + b_2 < \Lambda_1(c)$ and $b_2 > 0$. Then problem (1) has at least two solutions.

To prove Theorem 2.4 we need the following two lemmas.

Lemma 2.5. Let $b_1 + b_2 < \Lambda_1(c)$ and $b_2 > 0$. Then we have $\lim_{r \to +\infty} G(-re_1) = -\infty$.

Remark 2.6. We have $\lim_{|u|_H \to 0} \int \frac{[(u+1)^-]^2}{|u|_H^2} = 0$.

From the above equation we have $\int [(u+1)^-]^2 \le |u|_H^2 \cdot o(|u|_H)$.

Consider the values of G in the set $\Gamma_\rho(H) = \{u_1 + u_2 \in \text{span}\{e_1\} \oplus H_1^\perp \mid \int u_1^2 + \int (\Delta u_2)^2 - c\int |\nabla u_2|^2 \le \rho^2\}$. The set $\Gamma_\rho(H)$ is homeomorphic to a ball in H, whose boundary is the set $\gamma_\rho(H) = \{u_1 + u_2 \in \text{span}\{e_1\} \oplus H_1^\perp \mid \int u_1^2 + \int (\Delta u_2)^2 - c\int |\nabla u_2|^2 = \rho^2\}$.

Lemma 2.7. Let $b_1 + b_2 < \Lambda_1(c)$ and $b_2 > 0$. Then there exists a small $\rho > 0$ such that $\inf_{u \in \gamma_\rho(H)} G(u) > 0$.

Proof of Theorem 2.4. Since $\lambda_1 < c < \lambda_2$, $b_1 + b_2 < \Lambda_1(c)$ and $b_2 > 0$, by Lemma 2.7 there is a small $\rho > 0$ such that $\inf_{u \in \gamma_\rho(H)} G(u) > 0$. By definition of G we have $G(0) = 0$ with $0 \in \Gamma_\rho(H)$. Set $A = \{-re_1, 0\}$, $B = \gamma_\rho(H)$. Then A links B. By Lemma 2.5 there is sufficiently large $r > 0$ such that $-re_1 \notin \Gamma_\rho(H)$ and $G(-re_1) < 0$. Thus

$$\sup_A G(u) < \inf_B G(u).$$

By the Mountain Pass Theorem G possesses a critical value

$$c_1 \geq \inf_{u \in \gamma_\rho(H)} g(u) > 0 \quad \text{and} \quad 0 = \min_{u \in \Gamma_\rho(H)} G(u).$$

So G has two critical values. Hence problem (1) has at least two solutions, one of which is nontrivial.

(iii) At least three solutions

Lemma 2.8. Suppose $b_2 > 0$ and $b_1 > \Lambda_2(c)$. Then there exists a small $\rho > 0$ such that $\sup_{|u| = \rho,\ u \in H_2} G(u) < 0$.

Lemma 2.9. Let $\lambda_1 < c < \lambda_2$ and

$$T = \max\{ \int (u^+)^2 \mid u \in H_1^\perp,\ \int (\Delta u)^2 - c|\nabla u|^2 = 1\}.$$

Then we have $T < \dfrac{1}{\Lambda_2(c)}$.

Lemma 2.10. Suppose $b_2 > 0$ and $b_1 + b_2 < \frac{1}{T}$. Then there exist a large $R > 0$ such that $\inf\{G(u) \mid u = \sigma e_2 + v,\ \sigma \geq 0,\ v \in H_2^\perp,\ \int (\Delta u)^2 - c\int |\nabla u|^2 = R^2\} > 0$.

Theorem 2.11. Let $\lambda_1 < c < \lambda_2$. Suppose $b_1 + b_2 < \frac{1}{T}$ and $b_1 > \Lambda_2(c)$, $b_2 > 0$. Then problem (1) has at least three solutions, two of which are nontrivial.

Proof. Since $b_1 + b_2 < \frac{1}{T}$ and $b_1 > \Lambda_2(c)$, $b_2 > 0$, by Lemma 2.8 and 2.10 there exist $R > \rho > 0$ such that

$$\sup_{|u| = \rho,\ u \in H_2} G(u) < 0 < \inf_{v \in \Sigma_R(e_2,\ H_2^\perp)} G(v),$$

where $\Sigma_R(e_2,\ H_2^\perp)$ is the boundary of the set

$$\{G(u) \mid u = \sigma e_2 + v,\ \sigma \geq 0,\ v \in H_2^\perp,\ \int (\Delta u)^2 - c\int |\nabla u|^2 \leq R^2\}.$$

By the Variational Linking Theorem $G(u)$ has at least two nonzero critical values c_1, c_2:

$$c_1 \leq \sup_{|u|=\rho,\ u\in H_2} G(u) < 0 < \inf_{v\in \Sigma_R(e_2,\ H_2^\perp)} G(v) \leq c_2.$$

Therefore (1) has at least two nontrivial solutions. This implies that (1) has at least three solutions.

3. Nonconstant Source Terms

We let $Lu = \Delta^2 u + c\Delta u$. We investigate relations between multiplicity of solutions and source terms $f(x)$ of the fourth order nonlinear elliptic boundary value problem, under the condition: $\lambda_1 < c < \lambda_2$, $b < \Lambda_1(c)$.

$$Lu - bu^+ = f \text{ in } H, \tag{7}$$

where we assume that $f = c_1\phi_1 + c_2\phi_2$ (c_1, $c_2 \in R$).

Let $V = \text{span}\{\phi_1, \phi_2\}$ and we set

$$R_1 = \{d_1\phi_1 + d_2\phi_2 \mid d_1 \geq 0,\ |d_2| \leq \frac{-b+\Lambda_2}{-b+\Lambda_1} kd_1\},$$

$$R_2 = \{d_1\phi_1 + d_2\phi_2 \mid d_1 \geq 0,\ \frac{-b+\Lambda_2}{-b+\Lambda_1} kd_1 \leq d_2 \leq \frac{\Lambda_2}{|\Lambda_1|} kd_1\},$$

$$R_3 = \{d_1\phi_1 + d_2\phi_2 \mid d_1 \geq 0,\ |d_2| \leq \frac{\Lambda_2}{|\Lambda_1|} kd_1\}.$$

Then we have the result([cf. 2]).

Theorem 3.1. Suppose $-bu^+$ satisfies $\frac{\Lambda_2}{|\Lambda_1|} > \frac{-b+\Lambda_2}{-b+\Lambda_1}$. Let $f = c_1\phi_1 + c_2\phi_2 \in V$. Then we have:

(a) If $f \in \text{Int}\, R_1$, then (7) has exactly two solutions, one of which is positive and the other is negative.

(b) If $f \in \text{Int} R_2 \cup \text{Int} R_4$, then (7) has a negative solution and at least one sign changing solution.

(c) If $f \in \partial R_3$, then (7) has a negative solution.

(d) If $f \in R_3^c$, then (7) has no solution.

References

1. Q.H. Choi and T. Jung, "An application of a variational reduction method to a non-linear wave equation," *J. Differential Equations* 117(1995), 390-410.
2. Q.H. Choi and T. Jung, "Multiplicity of solutions and source terms in a fourth order nonlinear elliptic equation," *Acta Mathematica Scientia* 19(1999), 361-374.
3. A.C. Lazer and P.J. McKenna, "Large amplitude periodic oscillations in suspension bridge: some new connections with nonlinear analysis," *SIAM Review* 32(1990), 537-578.
4. A.C. Lazer and P.J. McKenna, "Global Bifurcation and a Theorem of Tarantello," *J. Math. Anal. Appl.* 181(1994), 648-655.
5. A. Marino and C. Saccon, "Some variational theorems of mixed type and elliptic problem with jumping nonlinearities," *Annali S.N.S.* XXV(1997), 631-665.
6. P.J. McKenna and W. Walter, "Travelling waves in a suspension bridge," *SIAM J. Appl. Math.* 50(1990), 703-715.
7. A.M. Micheletti and A. Pistoia, "Nontrivial solutions for some fourth order semilinear elliptic problems," *Nonlinear Analysis* 34(1998), 509-523.
8. P.H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, CBMS. Regional Conf. Ser. Math. Amer. Math. Soc. 65, Providence, Rhode Island, 1986.
9. G. Tarantello, "A note on a semilinear elliptic problem," *Differential Integral Equations* 5(1992), 561-565.