

# A Goodness-Of-Fit Test for Adaptive Fourier Model in Time Series Data<sup>1)</sup>

Hoonja Lee<sup>2)</sup>

## Abstract

The classical Fourier analysis, which is the typical frequency domain approach, is used to detect periodic trends that are of the sinusoidal shape in time series data. In this article, using a sequence of periodic step functions, describes an adaptive Fourier series where the patterns may take general periodic shapes that include sinusoidal as a special case. The results, which extend both Fourier analysis and Walsh-Fourier analysis, are applied to investigate the shape of the periodic component. Through the real data, compare the goodness-of-fit of the model using two methods, the adaptive Fourier method which is proposed method in this paper and classical Fourier method.

*Keywords:* Adaptive Fourier Model, Adaptive Fourier Anova, Goodness-Of-Fit Test

## 1. Introduction

The study of periodicity in time series data is very interesting analysis because it explains the pattern of data. The classical Fourier model in frequency domain approach decomposes time series data  $x(t)$  into a sum of periodic components that have sinusoidal shapes. That is  $x(t)$  can be written in the form:

$$x(t) = \sum_{k=0}^{\infty} a_k \cos \frac{2\pi kt}{T} + b_k \sin \frac{2\pi kt}{T}, \quad (1)$$

where  $\lambda_k = 2\pi k/T$  is the frequencies for  $k=0, 1, \dots$

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1) This paper was supported by Korea Research Foundation Grant.(KRF-2001-003-D00030)

2) Assistant Professor, Department of Information Statistics, Pyongtaek University, Pyongtaek City, 450-701, Korea.

E-mail: esther@ptuniv.ac.kr

In addition, the analysis of variance (ANOVA) in  $x(t)$  as measured by  $\int_0^T |x(t)|^2 dt$  can be expressed in the following form:

$$\int_0^T |x(t)|^2 dt = \sum_{k=0}^{\infty} \int_0^T \left| a_k \cos \frac{2\pi kt}{T} + b_k \sin \frac{2\pi kt}{T} \right|^2 dt = \sum_{k=0}^{\infty} |\beta_k|^2, \quad (2)$$

where  $\beta_k$  is the Fourier coefficients. The equation (2) implies the variability in  $x(t)$  partitions into the sum of the variabilities of the sinusoidal shapes. When the time series data are sinusoidal patterns, Fourier model detect periodic trends adequately, however the Fourier model can be misleading when time series trends are not sinusoidal.

This research is based on the decomposition of  $x(t)$  in the form of (1) with the generalization that sinusoids are replaced by adaptive periodic functions that may take non-sinusoidal shapes.

The proposed approach extends both Fourier analysis and Walsh-Fourier analysis. The Walsh-Fourier analysis is based on decomposition in the form (1) with sinusoids being replaced by Walsh step function that is not periodic function. See Stoffer (1991).

The proposed analysis is not closely related wavelet analysis even though both techniques aim to represent a function as a sum of elementary functions. In wavelet analysis, the elementary functions are wavelets. Wavelets are not periodic functions, but are transient instead; and the resulting wavelet analysis is suitable for describing time series data with transient components. In addition, the concepts of frequency and of periodicity have no precise meaning in wavelet analysis. See Priestly (1996).

In contrast, the elementary functions in the proposed analysis are not transient but periodic, so the concepts of periodicity and frequency are well defined. The proposed analysis is intended for periodic time series data, and the particular purpose is for investigating the goodness-of-fit of the model and shape of the dominant periodicities.

## 2. Theoretical Backgrounds

In this section, the ANOVA of the time series data on the Fourier analysis is briefly introduced. Let  $L^2$  be the set of all continuous time, complex-valued functions,  $\{x(t), 0 \leq t \leq T\}$ , for which the Lebesgue integral  $\int_0^T |x(t)|^2 dt$  is finite. The value  $T=2\pi$  is taken to simplify notation. Then  $L^2$  forms a Hilbert space with inner product

$$\langle x, y \rangle_L = \frac{1}{2\pi} \int_0^{2\pi} x(t) \overline{y(t)} dt, \quad (3)$$

where  $\overline{y(t)}$  is the conjugate of the complex number  $y(t)$ . Let  $A_k$  be the subspace of  $L^2$  that contains all periodic functions that have the form  $ce^{ikt}$  for the complex-valued scalar  $c$ . It is known that  $\dots, A_{-1}, A_0, A_1, \dots$  are mutually orthogonal subspace of  $L^2$  and  $L^2$  is the direct sum of  $A_k$ 's,

$$L^2 = \dots \oplus A_{-1} \oplus A_0 \oplus A_1 + \dots \quad (4)$$

See Halmos (1957, p.27).

From the equation (4), the function  $x(t)$  in  $L^2$  has the Fourier series representation

$$x(t) = \sum_{k=-\infty}^{\infty} P_L(x|A_k)(t) \quad (5)$$

and squared norm  $\|X\|_L^2 = \langle X, X \rangle_L$  is

$$\|x(t)\|_L^2 = \sum_{k=-\infty}^{\infty} \|P_L(x|A_k)\|_L^2, \quad (6)$$

where  $P_L(x|A_k)$  is the projection of  $x(t)$  onto the space  $A_k$  and  $P_L(x|A_k)$  has simple form

$$P_L(x|A_k)(t) = \langle x, e^{ikt} \rangle_L e^{ikt}(t).$$

See Koopmans (1974, p.16).

In the Fourier series (5), each frequency component  $P_L(x|A_k) + P_L(x|A_{-k})$  has a sinusoidal frequency at  $k$ . The partition (6) provides the Fourier ANOVA for revealing how well the periodicities in  $x(t)$  described by the sinusoidal shapes  $P_L(x|A_k) + P_L(x|A_{-k})$ .

The proposed research is based on the generalization of the Fourier analysis. The generalization begins by replacing each space  $A_k$  in the representation in (4) with a larger space  $B_k$ . While each function in  $A_k$  has sinusoidal shapes, the functions in  $B_k$  have more general periodic shapes. The periodic shape may be a saw-toothed type or rectangular type, for examples. This generalization of Fourier analysis, called adaptive Fourier analysis, is accomplished by Foutz and Lee(2000). The elementary functions of the adaptive Fourier

analysis are  $f_{k,n,j}$ , and  $f_{-k,n,j}$ . They are defined as follows:

$$f_{k,n,j}(t) = \exp\left(i\frac{\pi}{2} \left\lfloor \frac{2kt}{\pi} + \frac{2}{\pi} - \frac{j-1}{n} + 1 \right\rfloor\right),$$

$$f_{-k,n,j}(t) = \exp\left(i\frac{\pi}{2} \left\lfloor -\frac{2kt}{\pi} - \frac{2}{\pi} + \frac{j-1}{n} \right\rfloor\right), \text{ and}$$

$$f_{0,n,j}(t) = 1, \text{ for } k=0; \text{ where } j=1, 2, \dots, n, \tag{7}$$

where  $\lfloor s \rfloor$  be the largest integer no larger than  $s$ , for example,  $\lfloor 2.4 \rfloor = 2$  and  $\lfloor -2.4 \rfloor = -3$ . The functions  $f_{k,n,j}$  and  $f_{-k,n,j}$  in (7) are the particular time-shifted version of (8).

$$\exp\left(i\frac{\pi}{2} \left\lfloor \frac{2kt}{\pi} \right\rfloor\right) = \cos\left(\frac{\pi}{2} \left\lfloor \frac{2kt}{\pi} \right\rfloor\right) + i \sin\left(\frac{\pi}{2} \left\lfloor \frac{2kt}{\pi} \right\rfloor\right) \tag{8}$$

The functions  $f_{k,n,j}$  and  $f_{-k,n,j}$  are conjugates of each other and they are periodic step functions in  $L^2$  that takes the values  $1, i, -1$  and  $-i$ . These  $f_{k,n,j}$  and  $f_{-k,n,j}$  functions are the basis of the adaptive Fourier analysis which explained in the next section.

### 3. The Adaptive Fourier Analysis of Time Series Data

In this section, the procedure on the adaptive Fourier analysis is introduced.

#### 3.1 The Space $B_{k,n}$

Construct the space  $B_{k,n}$  to be the subspace of  $L^2$  spanned by  $\{f_{k,n,1}, \dots, f_{k,n,n}\}$  in (7). Then note that the functions in  $B_{k,n}$  are all periodic with period  $2\pi/k$ . Since the subspaces  $B_{k,n}$  for  $k = \dots, -1, 0, 1, \dots$ , are not mutually orthogonal with respect to the usual inner product  $\langle x, y \rangle_L$  in (3), we need to construct the new inner product space,  $\langle x, y \rangle_G$ , which  $B_{k,n}$  are mutually orthogonal.

### 3.2 The Inner Product Space $\langle x, y \rangle_G$

Foutz and Lee(2000) showed that the subspace  $\{B_{-m,n}, \dots, B_{m,n}\}$  are mutually orthogonal with respect to a new inner product  $\langle x, y \rangle_G$ ,

$$\langle x, y \rangle_G = \int_0^1 \langle x_u, y_u \rangle_L du, \quad (9)$$

where  $x_u(t) = \sum_{k=-m}^m x_k(t + \frac{4u\pi}{2k})$ .

### 3.3 Adaptive Fourier Series Representation

Foutz and Lee(2000) also showed that each function  $x(t)$  in  $L^2$  has the unique adaptive Fourier series representation

$$x(t) = \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \sum_{k=-m}^m P_G(x_{m,n} | B_{k,n})(t), \quad (10)$$

where  $x_{m,n}(t)$  is the projection  $P_L(x | L_{m,n})(t)$  of  $x(t)$  onto the space  $L_{m,n}$  with respect to the inner product  $\langle x, y \rangle_L$  of (3), where  $P_G(x_{m,n} | B_{k,n})$  is the unique projection of  $x_{m,n}$  onto the subspace of  $B_{k,n}$  with respect to the inner product  $\langle x, y \rangle_G$  of (9), and where the space  $L_{m,n}$  is the direct sum of the space  $B_{k,n}$ . That is,  $L_{m,n} = B_{-m,n} \oplus \dots \oplus B_{m,n}$ .

The component  $P_G(x_{m,n} | B_{k,n})(t)$  in (10) is periodic with  $2\pi/k$  since it is an element of  $B_{k,n}$ . It is adaptive since it depends on  $x(t)$  through its projection  $x_{m,n}(t)$ , and it may take non-sinusoidal shapes. Expression (10) shows that each element  $x(t)$  in  $L^2$  has the unique adaptive Fourier series approximation  $\sum_{k=-m}^m P_G(x_{m,n} | B_{k,n})(t)$  in  $L_{m,n}$  for every  $m, n$ ; and this approximation becomes precise as  $m$  and  $n$  increase.

In the next section, the adaptive Fourier ANOVA is introduced based on the representation of the adaptive Fourier series in (10).

### 3.4 The Adaptive Fourier ANOVA

The adaptive Fourier ANOVA partition of  $|x_{m,n}(t)| = \langle x_{m,n}, x_{m,n} \rangle_G$  can be expressed as:

$$|x_{m,n}(t)|_G^2 = \sum_{k=-m}^m |P_G(x_{m,n}|B_{k,n})|_G^2 = \sum_{k=-m}^m \frac{1}{2\pi} \int_0^{2\pi} |P_G(x_{m,n}|B_{k,n})|_L^2 dt. \quad (11)$$

The equation (11) implies that adaptive Fourier ANOVA partition decomposes time series into a sum of the periodicity components of  $B_{k,n}$ . Also, the functions in  $B_{k,n}$  contains various general periodic shapes compared with the Fourier functions, the shape of the periodicity in time series data may well expressed as the adaptive Fourier periodic components.

### 3.5 The Generalized Adaptive Fourier ANOVA

To apply the real time series data, the adaptive Fourier ANOVA in (11) can be generalized. The generalized adaptive Fourier ANOVA is useful for investigating the interesting frequencies in time series data.

The generalized adaptive Fourier ANOVA decomposition of the variability in  $x^*(t) = P_L(x|L^*)(t)$  proceed as

$$|x^*|_G^2 = \sum_k |P_G(x^*|B_{k,m})|_L^2 + |P_G(x^*|B_{-k,m})|_L^2, \quad (12)$$

where  $L^* = \sum_k (B_{-k,m} + B_{k,m})$ , and the finite summation is over interesting frequencies  $k$ , and where the dimension  $m$  may depend on the general sharpness of the data set.

This generalized adaptive ANOVA partition is the basis of the spectral analysis for the time series data in Section 4.2.

## 4. Example

### 4.1 Time Series Data Set

The time series  $x(1), x(2), \dots, x(96)$  in Figure 1 is taken from the Korea National Statistical Office. It contains monthly intermediate goods shipping index from January 1986 to December 1993. This discrete time series can be represented as a continuous function  $x(t)$  in

$L^2$  by defining  $x(t) = x(n)$  if  $2\pi(n-1)/96 < t \leq 2\pi(n)/96$ , for  $0 < t \leq 2\pi$ , and for  $n=1, 2, \dots, 96$ . Using this example, we compare the goodness-of-fit of the model between adaptive Fourier model and classical Fourier model.

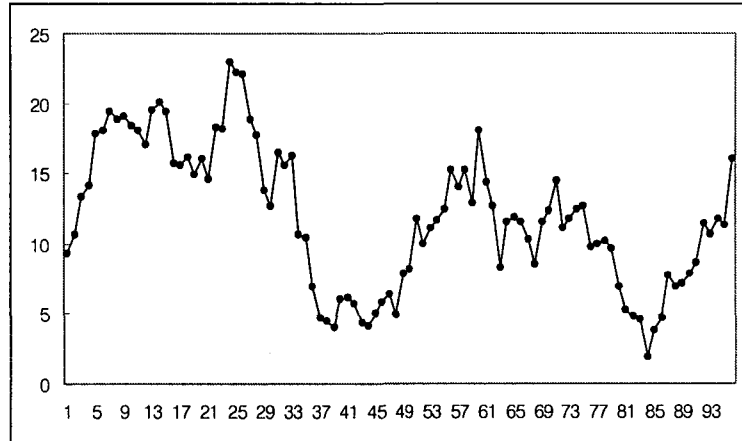


Figure 1 Intermediate Goods Shipping Index Data Set

#### 4.2 Generalized Adaptive Fourier ANOVA in Index Data Set

For analyzing the generalized adaptive Fourier ANOVA in (12), first try to find the suitable value of dimension  $m$ . The dimension  $m$  may be determined by the degree of the sharpness of the data. Two residual statistics, residual sum of squares,  $\sum_t (x(t) - \text{adaptive Fourier fit})^2$  and residual sum of absolute value,  $\sum_t |x(t) - \text{adaptive Fourier fit}|$  were computed at several different values  $m=3, 4, \dots, 15$  in equation (12) as the measure of goodness-of-fit of the model. Table 1 is the summary of the residual statistics of  $m$ . The best choice of  $m$  is 7, because index data set has the smallest residual sum of squares and also smallest residual sum of absolute value at  $m=7$ . Thus, use the dimension  $m=7$  for analyzing the adaptive ANOVA for the intermediate goods shipping index data set. Also, we choose interesting frequencies  $k=1, 2, \dots, 12$  because data shows monthly pattern.

Figure 2 displays the adaptive Fourier ANOVA for  $x(t)$  of (12). In ANOVA equation (12),

$$x^*(t) = P_L(x|L^*)(t) \text{ for } L^* = \sum_{k=1}^{12} (B_{-k,m} + B_{k,m}), \tag{13}$$

where each dimension  $m=7$ .

In other words, frequencies  $k=1, 2, \dots, 12$  are included in the analysis and the subspaces  $B_{k,m}$  and  $B_{-k,m}$  have dimension  $m=7$ . The projection  $x^*(t)$  in (13) represents the proportion  $|x^*|_L^2 / |x|_L^2 = 0.98$  of the total variability in  $x(t)$  attributes to the frequencies  $k=1, 2, \dots, 12$ .

$m$	Residual sum of squares	Residual sum of absolute value
4	168.6304	99.5733
5	191.9535	102.8726
6	128.3676	81.6268
7	102.7097	76.3439
9	142.8979	94.6141
10	161.0454	97.6552

Table 1 Residual analysis for several values of  $m$

Figure 2 show that the dominant periodic components in the data set are at  $k=2$  and  $k=1$ , corresponding to a 48 monthly and 96 monthly index cycle. In next section 4.3, we will examine the significance of the overall periodicity and individual periodicities of the index data set, especially interested in observing dominant periodic component at  $k=2$  and  $k=1$ .

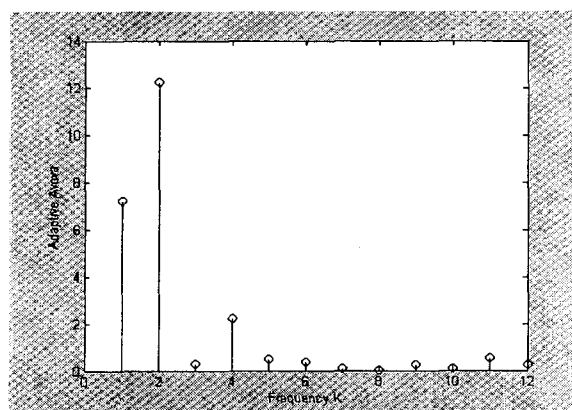


Figure 2 Adaptive Fourier ANOVA for  $x(t)$



### 4.3 Randomization Tests for Periodicities

For analyzing the periodicities of the index data set, randomization tests for overall and individual periodicities are developed in the adaptive Fourier analysis.

#### 4.3.1 Randomization test for overall periodicity

For overall periodicity, test the null hypothesis of randomness against the alternative that there is at least one periodic component in time series data. In the adaptive Fourier analysis, develop an overall test of randomness procedure based on the adaptive Fourier frequencies  $\gamma(k)$  in the expression (14). This procedure is derive from the Kolmogorov-Smirnov test for an overall test of randomness proposed by M. S. Bartlett. See Manly (1994, p 179).

For analyzing the index data set, consider the adaptive Fourier frequencies  $\gamma(k)$ ,

$$\gamma(k) = |P_G(x^*|B_{k,m})|_L^2 + |P_G(x^*|B_{-k,m})|_L^2, \quad (14)$$

for  $m=7$ , and  $k=1, 2, \dots, 12$ .

Now, compute the partial sums  $u_j$ ,

$$u_j = \sum_{k=1}^j \gamma(k) / \sum_{k=1}^{12} \gamma(k).$$

On the null hypothesis the time series being considered consists of independent normal variates from the same distribution. The  $u_j$  values behave like the order statistics of a random sample of 12 observations from a uniform distribution on the range (0, 1). The randomization of the test involves calculating

$$D = \max(D^+, D^-), \quad (15)$$

where  $D^+ = \max\{j/(m-1) - u_j\}$  and  $D^- = \max\{u_j - (j-1)/(m-1)\}$ ,

and comparing  $D$  with the distribution found when the observations in the original index data series are randomized. A significantly large value of  $D$  indicates that at least one periodic component exists. For the index data set, the observed statistic of equation (15) is 0.6779. This is significantly large at the 0.0001 level in comparison with the randomization distribution when it is obtained from 4,999 randomizations of the data and observed value. This implies that there is an evidence that the series is not random and it is necessary to consider the evidence for individual periodicities.

### 4.3.2 Randomization test for individual periodicities

For observing the individual periodicities in index data set, especially, it is worth to test the hypotheses whether there are evidences of dominant periodicities at  $k=2$  and  $k=1$ . In the classical Fourier analysis, usually use the Fisher's test for hidden periodicities (Brockwell & Davis, 1991, p 339) based on the Fourier coefficients  $\beta_k$  in the expression (2).

In the adaptive Fourier analysis, a randomization test for individual periodicities is developed. This randomization test is also based on the adaptive Fourier coefficients  $\gamma(k)$  in (14). The test statistic  $p(k) = \gamma(k) / \sum_{j=1}^{12} \gamma(j)$  is used as the measure of the randomization test for individual periodicities. The  $p(k)$  values are the proportion of the total variance associated with the different period. High  $p(k)$  value indicates important frequency at  $k$ . The estimated significance levels are determined by comparing observed statistic for each  $p(k)$  to the distribution found for this statistic from randomizing the order of the time series data set. An adaptive Fourier analysis for individual periodicities is summarized in Table 2. The significance levels in Table 2 are the values greater than or equal to those observed statistic  $p(k)$  in the randomization distribution approximated by 4,999 randomizations of the data and observed data. The evidence for periodicity in index data set is only one at  $k=2$ , corresponding to a 48-month cycle.

Figure 3 displays the Fourier ANOVA. The dominant periodic components in classical Fourier also show at  $k=2$  and  $k=1$ . In the Fourier analysis, Fisher's statistic for hidden periodicity is 22.12314. This value is significantly large at the 0.001 level and clear evidence that index data is not random and has dominant periodic component at  $k=2$ . This result of the Fourier analysis is consistent to the outcome of the adaptive Fourier analysis.

## 4.4 Goodness-of-Fit Test of the Model

While the periodic component in the classical Fourier ANOVA has a sinusoidal shape, the periodic shape in the adaptive Fourier ANOVA has a more general shapes. The advantage of adaptive Fourier analysis is that the shape of the periodic component at dominant frequency  $k$  may be investigated by plotting

$$P_G(x^*|B_{-k,m})(t) + P_G(x^*|B_{k,m})(t). \quad (16)$$

Frequency $k$	Cycle length	Adaptive Fourier $\gamma(k)$	Observed statistic $p(k)$	Significance level
1	96.0	7.0693	0.2807	0.8564
2	48.0	12.1037	0.4805	0.0200
3	32.0	0.7414	0.0294	0.9986
4	24.0	2.2910	0.0195	0.3189
5	19.2	0.4904	0.0195	0.9940
6	16.0	1.5597	0.0619	0.8388
7	13.7	0.0839	0.0033	0.9998
8	12.0	0.0397	0.0016	0.9998
9	10.7	0.1045	0.0041	0.9998
10	9.6	0.0859	0.0034	1.0000
11	8.7	0.5622	0.0223	0.9056
12	8.0	0.0555	0.0022	0.9994

Table 2 Randomization Test for Individual Periodicities in the Index Data Set

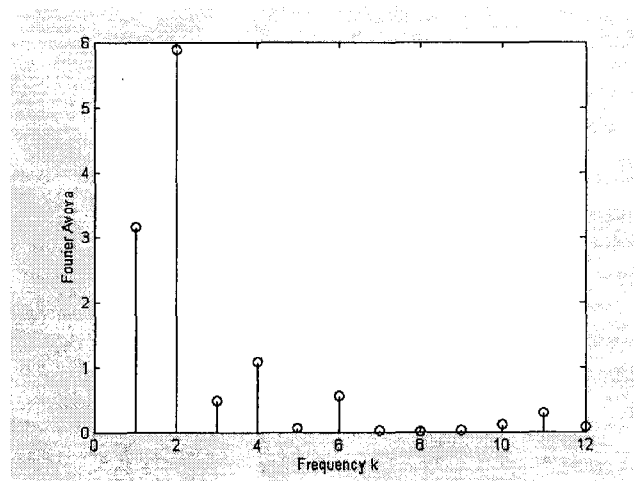


Figure 3 Classical Fourier Anova for  $x(t)$

Figure 4 gives the plots of index data set and adaptive Fourier dominant frequency component at  $k=2$  in (16) and show that  $k=2$  frequency component is not sinusoidal. Figure 5 gives the plots of data set and Fourier frequency component at  $k=2$ . Two Figures

4 and 5 show that the adaptive Fourier frequency component at  $k=2$  does provide a better representation to the frequency component of the index data set.

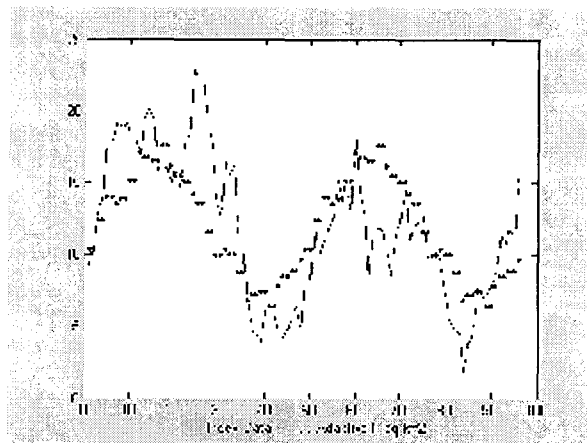


Figure 4 Index Data and  $k=2$  Frequency Component of Adaptive Fourier Analysis

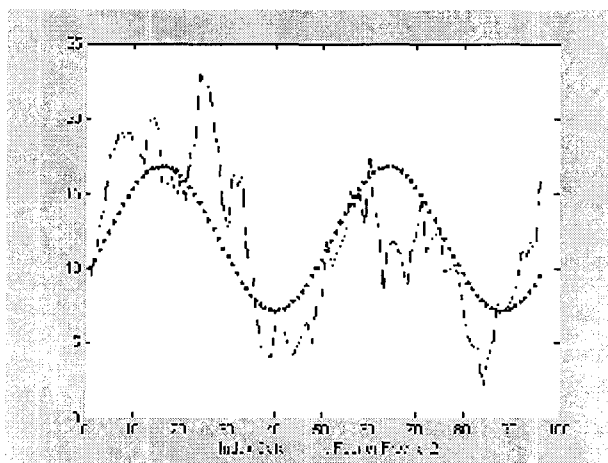


Figure 5 Index Data and  $k=2$  Frequency Component of Fourier Analysis

Two Figures, Figure 6 and Figure 7 display index data with the comparison between, adaptive Fourier fit and classical Fourier fit. For plotting the adaptive Fourier fit, compute the  $P_L(x|L^*)(t)$  in (13), and for plotting the classical Fourier fit, compute the  $\sum_{k=1}^{12} P_L(x|A_k)(t)$  in formula (5). The figures show that the adaptive Fourier fit in Figure 6 provides more close approximation to the index data than Fourier fit. The residual plots in Figure 8 also show that adaptive Fourier fit is better than the Fourier fit.

Table 3 is the summary of residual analysis for measuring the goodness-of-fit of the model between two models, adaptive Fourier model and Fourier model. We again use the two

residual statistics, residual sum of squares and residual sum of absolute value. In both two statistics, adaptive Fourier model has smallest values than the Fourier model, thus adaptive Fourier method is better fit of this intermediate goods index data set.

### 5. Conclusions

Using a goodness-of-fit test of the model, show that the adaptive Fourier method is better fit for representing the intermediate goods index time series data than the classical Fourier method. When we apply the adaptive Fourier method to the additional data sets, where patterns follow a non-sinusoidal shape, expect that adaptive method is more appropriate than the Fourier method. Also, the periodic components in the adaptive Fourier method may take general shapes in  $B_{k,m}$  that include sinusoidal shape.

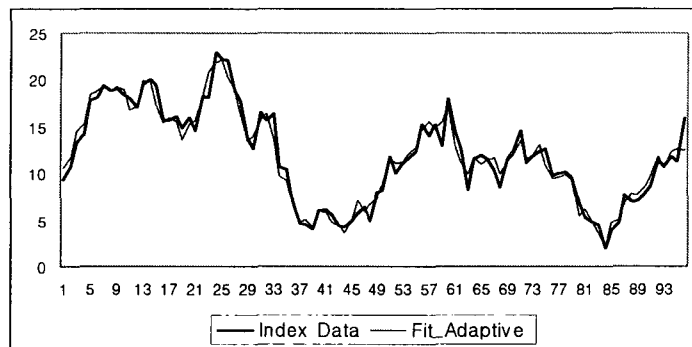


Figure 6 Plots for the Index data and Adaptive Fourier Fit

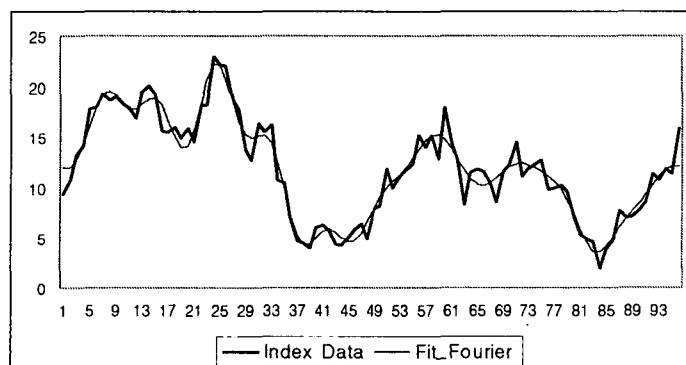


Figure 7 Plots for the Index data and Fourier Fit

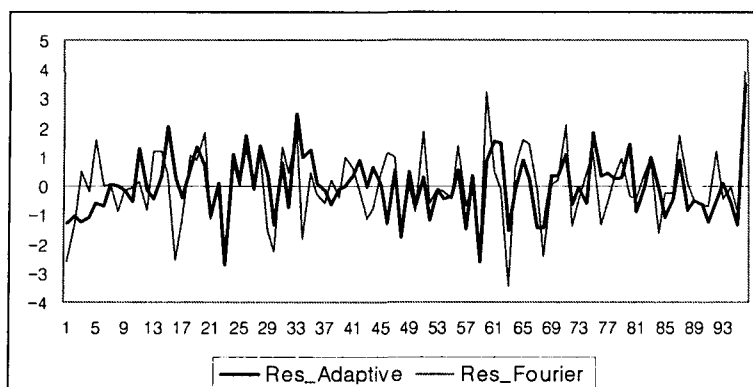


Figure 8 Residual Plots for the Adaptive Fourier Model and Fourier Model

Statistics	Adaptive Method	Fourier Method
Residual sum of squares	102.7097	148.9097
Residual sum of absolute value	76.3439	89.9934

Table 3 Residual Analysis of two methods

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