퍼지 및 지능시스템 학회 2003, Vol. 13, No. 5, pp. 596-599

Nonquadratic Stability Condition of Continuous Fuzzy Systems

Euntai Kim and Minkee Park*

Dept. of Electrical and Electronic Engr., Yonsei University

* Dept. of Electronic and Information Engr., Seoul National University of Technology

Abstract

In this paper, a new asymptotic stability condition of continuous fuzzy system is proposed. The new stability condition considers the nonquadratic stability by using the p-matrix measure. Later the relationship of the suggested stability condition and the well-known stability condition is discussed and it is shown in a rigorous manner that the proposed criterion includes the conventional conditions.

Key Words : Common Lyapunov function, quadratic stability, Common similarity transform matrix.

I. INTRODUCTION

Since the inception of the fuzzy logic by Zadeh in 1965, fuzzy logic has found extensive application for a wide variety of industrial systems and consumer products. Especially, the fuzzy approaches to control have emerged in recent years as one of the most promising research platforms and has attracted the attention of many control researchers.

Despite the much success and popularity in the industrial applications, however, the stability of the fuzzy system remains to be an open problem and to be further addressed. In this paper, the stability condition of continuous fuzzy systems is discussed. The model of the fuzzy system considered in this paper is that proposed by Takagi and Sugeno [1].

In the recent literatures, some notable works regarding the stability of the T-S (Takagi-Sugeno) fuzzy system have been reported by many control theorists in [2-4]. Tanaka et al. have provided a sufficient condition for the stability of the discrete and continuous fuzzy systems in [3] and the references therein. Narendra attempted to handle the similar problem without finding a common Lyapunov matrix for the continuous fuzzy systems [4]. Recently, many fuzzy control theorists tackled the problem by using convex optimization technique called LMI (linear matrix inequalities) [5-8]. For example, Wang et al. suggested the analysis and design methodology of T-S fuzzy system based on LMI [5]. Ma et al. [6] and Tseng et al. [7] proposed the dynamic output feedback control scheme for T-S system by adopting fuzzy observer system. Cao et al. addressed the input saturation problem for T-S fuzzy system [8]. In all of the aforementioned works, Lyapunov function candidate was $V=\mathbf{x}^T\mathbf{P}\mathbf{x}$, that is, the quadratic stability or stabilizability was considered.

접수일자 : 2003년 3월 3일
완료일자 : 2003년 8월 25일

In this paper, a new asymptotic stability condition of continuous fuzzy system is proposed. This work is motivated by [10] which addresses the stability of the discrete-time system. The new stability condition considers the nonquadratic stability by using the p-matrix measure. Later the relationship of the suggested stability condition and the well-known quadratic stability condition reported in [3] is discussed and it is shown in a rigorous manner that the proposed criterion includes the conventional conditions.

The rest of the paper is organized as follows: following the introduction, T-S fuzzy system is reviewed and the quadratic stability condition presented in [3] is explained briefly in Section II. Section III is the main results of this paper. New nonquadratic stability condition is presented and some remarks are made including the relationship with stability condition reported in [3]. Finally some conclusions are drawn in Section IV.

II. CONTINUOUS T-S FUZZY SYSTEM AND ITS QUADRATIC STABILITY CONDITION

The fuzzy system suggested by Takagi and Sugeno in 1985 [1] can represent a general class of nonlinear systems. It is based on "fuzzy partition" of input space and it can be viewed as the expansion of piecewise linear partition. The fuzzy system is of the following IF-THEN form:

$$R^i: \text{ If } x_1(t) \text{ is } M_1^i \text{ and } x_2(t) \text{ is } M_2^i, \cdots x_n \text{ is } M_n^i, \text{ then } \dot{\mathbf{x}}=\mathbf{A}_i\mathbf{x} \tag{1}$$

where $\mathbf{x}^T(t)=[x_1(t) \quad x_2(t) \quad \cdots \quad x_n(t)]$;

$R^i(i=1,2,\cdots,c)$ denotes the ith fuzzy rule and $M_1^i, M_2^i, \cdots, M_n^i$ are fuzzy variables. The I/O form of the fuzzy system of (1) is represented as in (2).

$$\dot{x} = \frac{\sum_{i=1}^{c} w^i A_i x}{\sum_{i=1}^{c} w^i} \quad where \quad w^i = \prod_{j=1}^{n} M_j^i(x_j) \tag{2}$$

From (1) and (2), it is noted that T-S fuzzy system approximates a nonlinear system with a combination of several linear systems by decomposing fuzzily the whole input space into several partial spaces and representing each input-output space with each linear equation.

A sufficient condition derived by Tanaka under which the fuzzy system of (1) is quadratically stable is as follows:

Theorem 1 [3]

The equilibrium of a fuzzy system of (1) or (2) is quadratically stable in the large if there exists a common positive definite matrix P such that

$A_i^T P + P A_i < 0$ for all $A_i \in \Omega_A$

where $\Omega_A \equiv \{A_1, A_2, \cdots, A_c\}$

III. A CRITERION FOR THE ASYMPTOTIC STABILITY

In this section, a criterion for the asymptotic stability of the continuous fuzzy system of (1) is presented.

Theorem 2

If there exists a nonsingular similarity transform $S \in R^{n \times n}$ and a matrix measure $\mu_p(\cdot)$ such that

$\mu_p(S^{-1} A_i S) < 0$, for all $A_i \in \Omega_A$

where $\mu_p(\cdot)$ is the p-matrix measure corresponding to the induced p-norm $\|\cdot\|_p$, then there exists a common Lyapunov function for all $A_i \in \Omega_A$ and the continuous fuzzy system represented by (1) is asymptotically stable.

Proof

Assume that S is the nonsingular similarity transform and $Q \equiv S^{-1}$. Define $V(x(t)) = \|Qx(t)\|_p^2$ where $\|\cdot\|_p$ is the p-norm for vector. In this paper, $\|\cdot\|_p$ will denote the p-norm and the induced p-norm for vectors and matrices, respectively, and $\mu_p(\cdot)$ will denote the corresponding p-matrix measure [9]. Since S is nonsingular, $V(x) \neq 0$ when $V(x) \neq 0$ and $V(x(t)) = \|Qx(t)\|_p^2$ can be a common Lyapunov function candidate for all $\dot{x}(t) = A_i x(t)$, $A_i \in \Omega_A$. For a system represented by

$$\dot{x}(t) = A_i x(t) \tag{3}$$

the time derivative of $V(x(t))$ is

$$\frac{dV(x(t))}{dt} = 2\|Qx(t)\|_p \frac{d}{dt}\|Qx(t)\|_p$$

To analyze the above time derivate, consider the following Taylor's series:

$$x(t+\Delta t) = x(t) + \dot{x}(t)\Delta t + o(\Delta t) \tag{4}$$

where $o(\Delta t)$ represents the higher order terms and $\lim_{\Delta t \to 0} \frac{o(\Delta t)}{\Delta t} = 0$

By premultiplying (4) by Q and using (3)

$$Qx(t+\Delta t) = Qx(t) + QA_i \Delta t\, x(t) + Qo(\Delta t)$$
$$= (I + QA_i Q^{-1}\Delta t) Qx(t) + Qo(\Delta t)$$

The p-norm of $Qx(t+\Delta t)$ is bounded as follows:

$$\|Qx(t+\Delta t)\|_p \le \|I + QA_i Q^{-1}\Delta t\|_p \; \|Qx(t)\|_p + \|Qo(\Delta t)\|_p$$
$$= \|I + S^{-1}A_i S\Delta t\|_p \; \|Qx(t)\|_p + \|Qo(\Delta t)\|_p \tag{5}$$

From (5),

$$\frac{d}{dt}\|Qx(t)\| = \lim_{\Delta t \to 0} \frac{\|Qx(t+\Delta t)\|_p - \|Qx(t)\|_p}{\Delta t}$$
$$\le \lim_{\Delta t \to 0} \left\{ \frac{\|I + S^{-1}A_i S\Delta t\|_p - 1}{\Delta t} \|Qx(t)\|_p + \frac{\|Qo(\Delta t)\|_p}{\Delta t} \right\}$$
$$= \mu_p(S^{-1}A_i S)\|Qx(t)\|_p$$

where $\mu_p(\cdot)$ is the p-matrix measure corresponding to $\|\cdot\|_p$ [9].

Therefore, the time derivative of $V(x(t))$ is represented as follows:

$$\frac{dV(x(t))}{dt} = 2\mu_p(S^{-1}A_i S)\|Qx(t)\|_p^2.$$

By the premise assumption of this theorem,

$$\frac{dV(x(t))}{dt} \le 0.$$

Therefore, if the premise of the Theorem is satisfied, $V(x(t)) = \|Qx(t)\|_p^2$ is a common Lyapunov function for all $\dot{x}(t) = A_i x(t)$, $A_i \in \Omega_A$, and the continuous fuzzy system of (1) is asymptotically stable. ■

Remark 1

(1) The p-matrix measure is defined as in (6) and can be thought of as the directional derivative of the induced p-norm function $\|\cdot\|_p$, as evaluated at the identity matrix I in the direction of the given matrix.

$$\mu_p(A) = \lim_{\varepsilon \to 0} \frac{\|I + \varepsilon A\|_p - 1}{\varepsilon} \tag{6}$$

Thus, the definitions of the matrix measure functions depend on p and the measure functions corresponding to the norms $\|\cdot\|_1$, $\|\cdot\|_2$, and $\|\cdot\|_\infty$ are as follows [9]:

for $x = (x_i)$ and $A = (a_{ij})$

	Norm	Matrix Measure
$p=\infty$	$\lVert x\rVert_\infty=\max_i\lvert x_i\rvert$	$\mu_\infty(A)=\max_i(a_{ii}+\sum_{j\neq i}\lvert a_{ij}\rvert)$
$p=1$	$\lVert x\rVert_1=\sum_{i=1}^{n}\lvert x_i\rvert$	$\mu_1(A)=\max_j(a_{jj}+\sum_{i\neq j}\lvert a_{ij}\rvert)$
$p=2$	$\lVert x\rVert_2=\left(\sum_{i=1}^{n}\lvert x_i\rvert^2\right)^{1/2}$	$\mu_2(A)=\lambda_{\max}(\frac{A+A^T}{2})$

(2) Theorem 2 can be interpreted as follows: If there exists a new transformed state space $y=Qx$ such that the norm of the response of all the subsystems of the fuzzy system of (1) decreases monotonously in the new transformed domain, the fuzzy system is asymptotically stable.

(3) Although p-norm and the corresponding induced p norm and p-matrix measure are used in Theorem 1, if another vector norm is available and the corresponding induced norm and matrix measure are well-defined, Theorem 2 can be extended to include the newly-defined norm.

(4) In Theorem 2, the common Lyapunov function $V(x(t))=\lVert Qx(t)\rVert_p^2$ is not necessarily quadratic. In other words, when there exists a nonsingular similarity transform $S\in R^{n\times n}$ and a matrix measure $\mu_p(\cdot)$ such that

$$\mu_p(S^{-1}A_iS)<0,\quad \text{for all } A_i\in\Omega_A,$$

the continuous fuzzy system of (1) is guaranteed to be just asymptotically stable and not necessarily quadratically stable. However, when $p=2$, the common Lyapunov function in Theorem 2 can be expressed by a quadratic function and the guaranteed stability by the existence of $S\in R^{n\times n}$ is the quadratic stability. This is expressed in the following Theorem 3.

Theorem 3

The continuous fuzzy system of (1) has a common quadratic Lyapunov function $V(x)=x^TPx$ which guarantees the quadratic stability if and only if there exists a nonsingular similarity transform $S\in R^{n\times n}$ such that

$$\mu_2(S^{-1}A_iS)<0,\quad \text{for all } A_i\in\Omega_A$$

where $\mu_2(\cdot)$ is a 2-matrix measure corresponding to the induced 2-norm (spectral norm).

Proof

($\Leftarrow$) If there exists a nonsingular similarity transform $S\in R^{n\times n}$ such that

$$\mu_2(S^{-1}A_iS)<0,\quad \text{for all } A_i\in\Omega_A,$$

there exists a common Lyapunov function for all subsystems of all $\dot{x}(t)=A_ix(t)$, $A_i\in\Omega_A$, and the common Lyapunov function is $V(x)=\lVert Qx\rVert_2^2$ by Theorem 2. Since the involved norm is the 2-norm (Euclidean norm), the common Lyapunov function clearly becomes the following common quadratic function as in (7) and the fuzzy system of (1) is quadratically stable.

$$V(x)=\lVert Qx\rVert_2^2=x^TQ^TQx=x^TPx \tag{7}$$

where $P=Q^TQ>0$.

($\Rightarrow$) Assume that there exists a common quadratic Lyapunov function $V(x)=x^TPx$ for all the subsystems of $\dot{x}(t)=A_ix(t)$, $A_i\in\Omega_A$. Since P is positive definite, there exists a nonsingular S such that $Q=S^{-1}$ and $P=Q^TQ$. For any subsystem $\dot{x}=A_ix$ comprised in the fuzzy system of (1), the time-derivative of V becomes negative definite. Thus,

$$\frac{dV}{dt}=\frac{d}{dt}(x^TQ^TQx)=\dot{x}^TQ^TQx+x^TQ^TQ\dot{x}$$
$$=x^TA_i^TQ^TQx+x^TQ^TQA_ix$$

By introducing a new variable $y=Qx$,

$$\frac{dV}{dt}=(Sy)^TA_i^TQ^TQ(Sy)+(Sy)^TQ^TQA_i(Sy)$$
$$=y^TS^TA_i^TS^{-1T}y+y^TS^{-1}A_iSy$$
$$=y^T\{(S^{-1}A_iS)^T+S^{-1}A_iS\}y<0$$

Therefore $(S^{-1}A_iS)^T+S^{-1}A_iS<0$ and the statement is equivalent to the fact that

$$\mu_2(S^{-1}A_iS)=\lambda_{\max}\left(\frac{(S^{-1}A_iS)^T+S^{-1}A_iS}{2}\right)<0.$$

Then the proof is completed.■

Remark 2

It can be shown that the stability condition in Theorem 2 for the case of $p=2$ is equivalent to the condition in Theorem 1 originally suggested in [3]. The new stability condition in Theorem 2 is based on the nonquadratic common Lyapunov function and can be thought to include the conventional stability condition in Theorem 1. The equivalence is directly discussed in the following Theorem 4.

Theorem 4

There exists a common positive definite matrix P such that

$$A_i^TP+PA_i<0 \text{ for all } A_i\in\Omega_A$$

if and only if there exists a nonsingular similarity transform $S\in R^{n\times n}$ such that

$$\mu_2(S^{-1}A_iS)<0,\quad \text{for all } A_i\in\Omega_A.$$

Proof

($\Leftarrow$) Let $Q=S^{-1}$. Then,

$$\mu_2(S^{-1}A_iS)=\lambda_{\max}\left(\frac{(S^{-1}A_iS)^T+S^{-1}A_iS}{2}\right)<0$$
$$\Leftrightarrow\ (S^{-1}A_iS)^T+S^{-1}A_iS<0$$
$$\Leftrightarrow\ S^TA_i^TQ^TQS+S^TQ^TQA_iS<0$$

$$\Leftrightarrow \quad S^T(A_i^T P + PA_i)S < 0, \quad P \equiv Q^T Q > 0$$

$$\Leftrightarrow \quad A_i^T P + PA_i < 0$$

($\Rightarrow$) The proof can be done straightforwardly by following the above proof in a reverse order.

IV. CONCLUSION

In this paper, a sufficient condition was suggested under which the continuous T-S fuzzy system is guaranteed to be asymptotically stable in the large. Since the suggested condition employs a common Lyapunov function which is not necessarily quadratic, the condition is considered to include the conventional stability condition of [3].

However, only consequent parts of the fuzzy system of (1) are addressed also in this paper and the researches regarding the effects the premise parts have on the stability of the fuzzy system are recommended.

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저 자 소 개

Euntai Kim was born in Seoul, Korea, in 1970. He received the B.S. (summa cum laude) and the M.S. and the Ph.D. degrees in electronic engineering, all from Yonsei University, Seoul, Korea, in 1992, 1994 and 1999, respectively. From 1999 to 2002, he was a full-time lecturer in the Department of Control and Instrumentation Engineering at Hankyong National University, Kyonggi-do, Korea. Since 2002, he has joined the faculties of the school of electrical and electronic engineering at Yonsei University, where he is currently an assistant professor. In 2003, he visited University of Alberta, Canada, as a visiting researcher. His current research interests include intelligent control and modeling based on fuzzy logic and neural networks, robot vision and intelligent service robot. Dr. Kim is an associate editor of International Journal of Control, Automation and Systems.

Min-Kee Park received the B.S., M.S., and Ph.D. degrees in electronic engineering from Yonsei University, Korea, 1985, 1992, and 1996, respectively. He was a researcher for LG Electronics Inc., Korea, from 1985 to 1990. From 2000 to 2001, he was a Ph. D. Researcher at the Institute of Industrial Science, University of Tokyo, Japan. He is currently an Associate Professor in the Department of Electronic and Information Engineering at Seoul National University of Technology, Seoul, Korea. His current research interests include fuzzy modeling, fuzzy application system, intelligent control.