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On fuzzy semi-topogenous orders

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Abstract

We investigate the properties of fuzzy semi-topogenous orders. We study the relationship among fuzzy supra topologies, fuzzy supra interior operators and fuzzy semi-topogenous orders. We give examples of them.

Keywords : Fuzzy semi-topogenous orders, Fuzzy supra-interior operators, Fuzzy supra topologies.

1. Introduction and Preliminaries

Csaszar[3] introduced the concept of a syntopogenous structure to develop to the three main structures of topologies, proximities and uniformities. Katsaras and Petalas [11] extended them to the theory of fuzzy sets. Katsaras [4-11] has developed in many directions. Abd El-Monsef and Ramadan [1] defined some of their properties and studied the concept of fuzzy supra topological spaces. Kim and Ko [12] investigated fuzzy semi-topogenous orders and fuzzy supra topologies.

In this paper, we investigate the properties of fuzzy semi-topogenous orders fuzzy supra topologies and fuzzy interior operators. We study the relationship between them and give examples of them.

Thought this paper, let X be a nonempty set, $I=[0,1]$ and I^X the family of all fuzzy subsets of X. For $\alpha\in I$, $\overline{\alpha}(x)=\alpha$ for all $x\in X$. For a subset A of X, χ_A is a characteristic function of A.

Definition 1.1([1,2]) A subset τ of I^X is called *a fuzzy supra topology* on X if it satisfies the following conditions:

(O1) $\overline{0}, \overline{1}\in\tau$,

(O2) $\bigvee_{i\in\Gamma}\mu_i\in\tau$ for any $\mu_i\in\tau$.

A fuzzy supra topology τ is called a *fuzzy topology* if it satisfies

(O3) $\mu_1\wedge\mu_2\in\tau$ for any $\mu_1,\mu_2\in\tau$.

The pair (X,τ) is called a *fuzzy* (resp. *supra*) *topological space.*

Definition 1.2([1,2]) A function $\mathrm{int}: I^X\to I^X$ is called *a fuzzy supra interior operator* on X if it satisfies the following conditions:

(I1) $\mathrm{int}(\overline{1})=\overline{1}$.

(I2) $\mathrm{int}(\lambda)\le\lambda$ for each $\lambda\in I^X$.

(I3) If $\lambda_1\le\lambda_2$, then $\mathrm{int}(\lambda_1)\le\mathrm{int}(\lambda_2)$ where $\lambda_1,\lambda_2\in I^X$.

A fuzzy supra interior operator int is called a *fuzzy interior operator* if it satisfies

(I) $\mathrm{int}(\lambda_1\wedge\lambda_2)=\mathrm{int}(\lambda_1)\wedge\mathrm{int}(\lambda_2)$, where $\lambda_1,\lambda_2\in I^X$.

A fuzzy supra interior operator int is called *topological* if it satisfies

(T) $\mathrm{int}(\mathrm{int}(\lambda))=\mathrm{int}(\lambda)$ for each $\lambda\in I^X$.

Theorem 1.3([1,2]) Let (X,τ) be a fuzzy (resp. supra) topological space. We define a function $\mathrm{int}_\tau: I^X\to I^X$ as follows:

$$\mathrm{int}_\tau(\lambda)=\bigvee\{\mu\in I^X \mid \mu\le\lambda, \mu\in\tau\}$$

Then int_τ is a topological fuzzy (resp. supra) interior operator on X.

Let $\ll$ be a binary relation on X. The facts that $(\lambda,\mu)\in\ll$ and $(\lambda,\mu)\notin\ll$ are denoted by $\lambda\ll\mu$ and $\lambda\overline{\ll}\mu$, respectively.

Definition 1.4([11]) A binary relation $\ll$ on I^X is called a *fuzzy semi-topogenous order* on X if it satisfies:

(T1) $\overline{1}\ll\overline{1}$ and $\overline{0}\ll\overline{0}$,

(T2) if $\lambda\ll\mu$, then $\lambda\le\mu$, where $\lambda,\mu\in I^X$.

(T3) if $\lambda\le\lambda_1\ll\mu_1\le\mu$, then $\lambda\ll\mu$, where $\lambda_1,\lambda_2,\mu_1,\mu_2\in I^X$.

Let $\ll$ be a fuzzy semi-topogenous order on X. We define $\lambda\ll^s\mu$ iff $(\overline{1}-\mu)\ll(\overline{1}-\lambda)$, where $\lambda,\mu\in I^X$. Then $\ll^s$ is a fuzzy semi-topogenous order on X.

Definition 1.5 ([11]) A fuzzy semi-topogenous order $\ll$ is called:

(1) *symmetric* if $\ll=\ll^s$, that is,

(T4) $\lambda\ll\mu$ iff $(\overline{1}-\mu)\ll(\overline{1}-\lambda)$

(2) *fuzzy topogenous* if for any $\lambda,\lambda_1,\lambda_2,\mu,\mu_1,\mu_2\in I^X$,

(T5) $\lambda_1\vee\lambda_2\ll\mu$ iff $\lambda_1\ll\mu$, $\lambda_2\ll\mu$

(T6) $\lambda\ll\mu_1\wedge\mu_2$ iff $\lambda\ll\mu_1$, $\lambda\ll\mu_2$.

(3) *perfect* if, for any $\{\mu,\lambda_i \mid i\in\Gamma\}\subset I^X$,

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(T7) $\bigvee_{i\in\Gamma}\lambda_i \ll \mu$ iff $\lambda_i \ll \mu$, for all $i\in\Gamma$.

(4) *biperfect* if it is perfect and any $\{\lambda, \mu_i \mid i\in\Gamma\}\subset I^X$,

(T8) $\lambda \ll \bigwedge_{i\in\Gamma}\mu_i$ iff $\lambda \ll \mu_i$, for all $i\in\Gamma$.

Definition 1.6 ([11]) Let $\langle\langle_1$ and $\langle\langle_2$ be fuzzy semi-topogenous orders on X. $\langle\langle_1$ is finer than $\langle\langle_2$ ($\langle\langle_2$ is *coarser than* $\langle\langle_1$) if $\lambda\langle\langle_1\mu$ for all $\lambda\langle\langle_2\mu$.

Definition 1.7 ([11]) A fuzzy topogenous order $\ll$ on X is called a *fuzzy topogenous structure* if it satisfies the following condition:

(T) $\ll\circ\ll$ is finer than $\ll$ where $\ll\circ\ll$ is defined by, for any $\lambda,\mu\in I^X$, $\lambda(\ll\circ\ll)\mu$ iff there exists $\rho\in I^X$ such that $\lambda\ll\rho$ and $\rho\ll\mu$.

The pair $(X,\ll)$ is called a *fuzzy topogenous space*.

Theorem 1.8 ([12]) Let $\ll$ be a fuzzy semi-topogenous order on X. We define the mapping $\mathrm{int}_{\ll}: I^X\to I^X$ is as follows: for each $\lambda\in I^X$

$$\mathrm{int}_{\ll}(\lambda)=\bigvee\{\mu\in I^X \mid \mu\ll\lambda\}.$$

Then we have the following properties:

(1) $\mathrm{int}_{\ll}$ is a fuzzy supra interior operator.

(2) If $\ll$ satisfies (T6), then $\mathrm{int}_{\ll}$ is a fuzzy interior operator.

(3) If $\ll$ satisfies (T5), then $\mathrm{int}_{\ll}$ is a fuzzy interior operator.

(4) If $\ll\circ\ll$ is finer than $\ll$, then, for each $\lambda\in I^X$, $\mathrm{int}_{\ll}(\mathrm{int}_{\ll}(\lambda))=\mathrm{int}_{\ll}(\lambda)$.

(5) If $\ll$ is a fuzzy topogenous structure, then $\mathrm{int}_{\ll}$ is a topological fuzzy interior operator.

Theorem 1.9 ([12]) Let $\ll$ be a fuzzy semi-topogenous order on X. Define a fuzzy topology on X by $\tau_{\mathrm{int}_{\ll}}=\{\lambda\in I^X \mid \mathrm{int}_{\ll}(\lambda)=\lambda\}$.

Then:

(1) $\tau_{\mathrm{int}_{\ll}}$ is a fuzzy supra topology on X induced by $\mathrm{int}_{\ll}$.

(2) If $\ll$ satisfies (T6), then $\tau_{\mathrm{int}_{\ll}}$ is a fuzzy topology on X.

(3) If $\ll$ is perfect, then $\lambda\in\tau_{\mathrm{int}_{\ll}}$ iff $\lambda\ll\lambda$ for each $\lambda\in I^X$.

Theorem 1.10 ([12]) Let int be a fuzzy supra interior operator on X. Define a binary relation $\langle\langle_{\mathrm{int}}$ as

$$\lambda\langle\langle_{\mathrm{int}}\mu \text{ iff } \lambda\leq \mathrm{int}(\mu).$$

Then :

(1) $\langle\langle_{\mathrm{int}}$ is a perfect fuzzy semi-topogenous order on X such that $\mathrm{int}_{\langle\langle_{\mathrm{int}}}(\lambda)=\mathrm{int}(\lambda)$ for each $\lambda\in I^X$.

(2) If int is a fuzzy interior operator on X, then $\langle\langle_{\mathrm{int}}$ is a fuzzy topogenous order on X.

(3) If $\mathrm{int}(\mathrm{int}(\lambda))=\mathrm{int}(\lambda)$ for each $\lambda\in I^X$, then $(\langle\langle_{\mathrm{int}}\circ\langle\langle_{\mathrm{int}})$ is finer than $\langle\langle_{\mathrm{int}}$.

(4) If int be a topological fuzzy interior operator on X, then $\langle\langle_{\mathrm{int}}$ is a fuzzy topogenous structure on X.

(5) If $\ll$ is a fuzzy semi-topogenous order, then $\langle\langle_{\mathrm{int}_{\ll}}$ is finer than $\ll$.

(6) If $\ll$ is a perfect fuzzy semi-topogenous order, then $\ll=\langle\langle_{\mathrm{int}_{\ll}}$.

2. The properties of fuzzy semi-topogenous orders

Theorem 2.1 Let τ be a fuzzy supra topology on X. Define a relation

$$\lambda\langle\langle_{\tau}\mu \text{ iff there exists } \rho\in\tau \text{ such that } \lambda\leq\rho\leq\mu.$$

Then:

(1) $\langle\langle_{\tau}$ is a perfect fuzzy semi-topogenous order on X.

(2) If τ be a fuzzy topology on X, then $\langle\langle_{\tau}$ is a perfect fuzzy topogenous order on X.

(3) $\langle\langle_{\tau}=\langle\langle_{\mathrm{int}_{\tau}}$.

Proof (1) (T1) Since $\overline{0},\overline{1}\in\tau$, we have $\overline{1}\langle\langle_{\tau}\overline{1}$ and $\overline{0}\langle\langle_{\tau}\overline{0}$.

(T2) If $\lambda\langle\langle_{\tau}\mu$, there exists $\rho\in\tau$ such that $\lambda\leq\rho\leq\mu$.

(T3) It follows from the definition of $\langle\langle_{\tau}$.

(T7) Since $\lambda_i\leq\bigvee_{i\in\Gamma}\lambda_i$ for $i\in\Gamma$, by (T3)implies $\lambda_i\langle\langle_{\tau}\mu$, for all for $i\in\Gamma$.

Let $\lambda_i\langle\langle_{\tau}\mu$, for all for $i\in\Gamma$. For each $i\in\Gamma$, there exists $\rho_i\in\tau$ such that $\lambda_i\leq\rho_i\leq\mu$ X. It implies $\bigvee_{i\in\Gamma}\lambda_i\leq\bigvee_{i\in\Gamma}\rho_i\leq\mu$ and $\bigvee_{i\in\Gamma}\rho_i\in\tau$. Hence $\bigvee_{i\in\Gamma}\lambda_i\langle\langle_{\tau}\mu$.

(2) We only show that $\lambda\langle\langle_{\tau}\mu_1$ and $\lambda\langle\langle_{\tau}\mu_2$ implies $\lambda\langle\langle_{\tau}\mu_1\wedge\mu_2$

Let $\lambda\langle\langle_{\tau}\mu_1$ and $\lambda\langle\langle_{\tau}\mu_2$. Then there exist $\rho_1,\rho_2\in\tau$ such that $\lambda\leq\rho_1\leq\mu_1$, $\lambda\leq\rho_2\leq\mu_2$

Since $(\rho_1\wedge\rho_2)\in\tau$ and $\lambda\leq(\rho_1\wedge\rho_2)\leq(\mu_1\wedge\mu_2)$.

we have $\lambda\langle\langle_{\tau}(\mu_1\wedge\mu_2)$.

(3) Let $\lambda\langle\langle_{\mathrm{int}_{\tau}}\mu$. Since $\lambda\leq\mathrm{int}_{\tau}(\mu)\leq\mu$ and $\mathrm{int}_{\tau}(\mu)\in\tau$, then $\lambda\langle\langle_{\tau}\mu$. Let $\lambda\langle\langle_{\tau}\mu$. Then there exists $\rho\in\tau$ such that $\lambda\leq\rho\leq\mu$. Since $\mathrm{int}_{\tau}(\rho)=\rho$, $\lambda\langle\langle_{\mathrm{int}_{\tau}}\rho\leq\mu$ implies $\lambda\langle\langle_{\mathrm{int}_{\tau}}\mu$.

Example 2.2 Let $X=\{x,y,z\}$ be a set. Define a fuzzy supra topology $\tau=\{\overline{0},\overline{1},\chi_{\{x,y\}},\chi_{\{y,z\}}\}$.We obtain

$$\lambda\langle\langle_{\tau}\mu \text{ iff } \begin{cases}\lambda=\overline{0} \text{ or } \mu=\overline{1},\\ \overline{0}\neq\lambda\leq\chi_{\{x,y\}}\ \overline{1}\neq\mu\geq\chi_{\{x,y\}}\\ \overline{0}\neq\lambda\leq\chi_{\{y,z\}}\ \overline{1}\neq\mu\geq\chi_{\{y,z\}}\end{cases}$$

Then $\langle\langle_{\tau}$ is a fuzzy semi-topogenous order but not topogenous order from:

$$\chi_{\{y\}} \ll_\tau \chi_{\{x,y\}}, \quad \chi_{\{y\}} \ll_\tau \chi_{\{y,z\}}$$
$$\chi_{\{y\}} \overline{\ll_\tau} \chi_{\{y\}} = (\chi_{\{x,y\}} \wedge \chi_{\{y,z\}}).$$

From Theorem 1.2, we obtain supra interior operator $\mathrm{int}_\tau : I^X \to I^X$ as follows:

$$\mathrm{int}_\tau(\lambda) = \begin{cases} \overline{1} \text{ if } \lambda = \overline{1}, \\ \chi_{\{x,y\}} \text{ if } \chi_{\{x,y\}} \leq \lambda \neq \overline{1} \\ \chi_{\{y,z\}} \text{ if } \chi_{\{y,z\}} \leq \lambda \neq \overline{1} \\ \overline{0} \quad \text{otherwise} \end{cases}$$

But it is not a fuzzy interior operator because
$\overline{0} = \mathrm{int}_\tau(\chi_{\{x,y\}} \wedge \chi_{\{y,z\}}) \neq \mathrm{int}_\tau(\chi_{\{x,y\}}) \wedge \mathrm{int}_\tau(\chi_{\{y,z\}}) = \chi_{\{y\}}$
We easily show that the property (3) in Theorem 2.1 hold: $\lambda \ll_\tau \mu$ iff $\lambda \ll_{\mathrm{int}_\tau} \mu$ for all $\lambda, \mu \in I^X$, that is, $\ll_\tau = \ll_{\mathrm{int}_\tau}$. Thus $\tau_{\mathrm{int}_\tau} = \{\overline{0}, \chi_{\{x,y\}}, \chi_{\{y,z\}}, \overline{1}\}$ is a supra fuzzy topology but not topology from

$$\chi_{\{x,y\}} \wedge \chi_{\{y,z\}} = \chi_{\{y\}} \notin \tau_{\mathrm{int}_\tau}.$$

Theorem 2.3 Let $\ll$ be a perfect fuzzy semi-topogenous order on X. Define $\lambda \in \tau_\ll$ iff $\lambda \ll \lambda$.
Then (1) $\tau_\ll$ is a fuzzy supra topology on X.
(2) $\tau_\ll = \tau_{\mathrm{int}_\ll}$.
(3) If $\ll$ is perfect fuzzy topogenous order on X, then $\tau_\ll$ is a fuzzy topology on X.
(4) $\ll$ is finer than $\ll_{\tau_\ll}$.
(5) If $\ll \circ \ll$ is finer than $\ll$, then $\ll = \ll_{\tau_\ll}$.

Proof (1) (O1) Since $\overline{1} \ll \overline{1}$ and $\overline{0} \ll \overline{0}$, then $\overline{0}, \overline{1} \in \tau_\ll$.
(O2)For all $i \in \Gamma$, $\lambda_i \in \tau_i$ iff for all $i \in \Gamma$, $\lambda_i \ll \lambda_i$
$\Rightarrow$ for all $i \in \Gamma$, , $\lambda_i \ll \bigvee_{i\in\Gamma} \lambda_i$ (by (T3))
iff $\bigvee_{i\in\Gamma} \lambda_i \ll \bigvee_{i\in\Gamma} \lambda_i$ ($\ll$ is perfect)
iff $(\bigvee_{i\in\Gamma} \lambda_i) \in \tau_\eta$.
(2) From Theorem 1.9(3), $\lambda \in \tau_\ll$ iff $\lambda \ll \lambda$

$$\text{iff } \lambda = \mathrm{int}_\ll(\lambda) \text{ iff } \lambda \in \tau_{\mathrm{int}_\ll}.$$

(3) We only show the condition (O3).

$$\lambda_1, \lambda_2 \in \tau_\ll \text{ iff } \lambda_1 \ll \lambda_1, \ \lambda_2 \ll \lambda_2$$
$$\Rightarrow \lambda_1 \wedge \lambda_2 \ll \lambda_1, \ \lambda_1 \wedge \lambda_2 \ll \lambda_2$$

iff $\lambda_1 \wedge \lambda_2 \ll \lambda_1 \wedge \lambda_2$ ($\ll$ is topogenous)
iff $\lambda_1 \wedge \lambda_2 \in \tau_\ll$.
(4) Let $\lambda \ll_{\tau_\ll} \mu$. Then there exists $\rho \in \tau_\ll$ with $\lambda \leq \rho \leq \mu$. Since $\rho \in \tau_\ll$ iff $\rho \ll \rho$ and $\lambda \leq \rho \leq \mu$, by(T3), then $\lambda \ll \mu$. Thus, $\ll$ is finer than $\ll_{\tau_\ll}$.
(5) We show that $\ll_{\tau_\ll}$ is finer than $\ll$ from (4).
Let $\lambda \ll \mu$. Then $\lambda \leq \mathrm{int}_\ll(\mu) \leq \mu$.

Since $\mathrm{int}_\ll(\mathrm{int}_\ll(\mu)) = \mathrm{int}_\ll(\mu)$from Theorem 1.8(4), we have $\mathrm{int}_\ll(\mu) \in \tau_{\mathrm{int}_\ll}$. Since $\tau_{\mathrm{int}_\ll} = \tau_\ll$ from (2), we have $\mathrm{int}_\ll(\mu) \in \tau_\ll$. By Theorem 2.1, $\lambda \ll_{\tau_\ll} \mu$.

Theorem 2.4 Let τ be a fuzzy supra topology on X. Then $\tau_{\ll_\tau} = \tau_{\ll_{\mathrm{int}_\tau}} = \tau$.

Proof By Theorem 2.1, $\ll_\tau$ is a perfect fuzzy semi-topogenous order on X and $\ll_\tau = \ll_{\mathrm{int}_\tau}$. Then, by Theorem 2.3, $\tau_{\ll_\tau} = \tau_{\ll_{\mathrm{int}_\tau}}$. Let $\lambda \in \tau_{\ll_{\mathrm{int}_\tau}}$. Then, by definition of $\tau_{\ll_{\mathrm{int}_\tau}}$, $\lambda \ll_{\mathrm{int}_\tau} \lambda$ .Thus $\lambda \ll_\tau \lambda$. By definition of $\ll_\tau$, $\lambda \in \tau$ So, $\tau_{\ll_{\mathrm{int}_\tau}} \subset \tau$.
Let $\lambda \in \tau$. Then $\mathrm{int}_\tau(\lambda) = \lambda$. By Theorem 1.10, it implies $\lambda \ll_{\mathrm{int}_\tau} \lambda$ iff $\lambda \in \tau_{\ll_{\mathrm{int}_\tau}}$. Thus $\tau \subset \tau_{\ll_{\mathrm{int}_\tau}}$. Hence $\tau = \tau_{\ll_{\mathrm{int}_\tau}}$. It completes the proof.

Example 2.5 Let $X = \{x, y, z\}$ be a set. Define a relation $\ll$ on I^X as follows:

$$\lambda \ll \mu \text{ iff } \begin{cases} \lambda = \overline{0} \text{ or } \mu = \overline{1}, \\ \overline{0} \neq \lambda \leq \chi_{\{y\}} \ \ \overline{1} \neq \mu \geq \chi_{\{x,y\}} \\ \overline{0} \neq \lambda \leq \chi_{\{y\}} \ \ \overline{1} \neq \mu \geq \chi_{\{y,z\}} \end{cases}$$

Then $\ll$ is a perfect fuzzy semi-topogenous order but not topogenous order from:

$$\chi_{\{y\}} \ll \chi_{\{x,y\}}, \ \chi_{\{y\}} \ll \chi_{\{y,z\}}$$
$$\chi_{\{y\}} \overline{\ll} \chi_{\{y\}} = (\chi_{\{x,y\}} \wedge \chi_{\{y,z\}}).$$

From Theorem 1.8, we can obtain a fuzzy supra interior operator $\mathrm{int}_\ll : I^X \to I^X$ as follows:

$$\mathrm{int}_\ll(\lambda) = \begin{cases} \overline{1} \text{ if } \lambda = \overline{1}, \\ \chi_{\{y\}} \text{ if } \chi_{\{x,y\}} \leq \lambda \neq \overline{1} \\ \chi_{\{y\}} \text{ if } \chi_{\{y,z\}} \leq \lambda \neq \overline{1} \\ \overline{0} \quad \text{otherwise} \end{cases}$$

But it is not a fuzzy interior operator from

$$\overline{0} = \mathrm{int}_\ll(\chi_{\{x,y\}} \wedge \chi_{\{y,z\}})$$
$$\neq \mathrm{int}_\ll(\chi_{\{x,y\}}) \wedge \mathrm{int}_\ll(\chi_{\{y,z\}}) = \chi_{\{y\}}.$$

We obtain $\tau_\ll = \tau_{\mathrm{int}_\ll} = \{\overline{0}, \overline{1}\}$
Moreover, by Theorem 2.3(5), $\ll \circ \ll$ is not finer than $\ll$ from the fact that

$$\chi_{\{y\}} \overline{\ll \circ \ll} \chi_{\{x,y\}} \text{ and } \chi_{\{y\}} \ll \chi_{\{x,y\}}.$$

Then $\ll$ is finer than $\ll_{\tau_\ll}$ from the fact that

$$\lambda \ll_{\tau_\ll} \mu \text{ iff } \lambda = \overline{0} \text{ or } \mu = \overline{1}.$$

Example 2.6 Let X and τ be given as in Example 2.2. From Theorem 2.4, we can obtain

$$\tau_{\ll_\tau} = \tau_{\ll_{\text{int}}} = \tau = \{\overline{0}, \overline{1}, \chi_{\{x,y\}}, \chi_{\{y,z\}}\}$$

Theorem 2.7 Let (X, int) be a fuzzy supra interior space. Define a subset τ_{int} of I^X as

$$\tau_{\text{int}} = \{\lambda \mid \text{int}(\lambda) = \lambda\}.$$

Then:

(1) τ_{int} is a fuzzy supra topology on X induced by *int*.

(2) If (X, int) is a fuzzy interior space, τ_{int} is a fuzzy topology on X.

(3) $\tau_{\text{int}} = \tau_{\ll_{\text{int}}}$.

(4) $\text{int}_{\tau_{\text{int}}} \leq \text{int}$.

(5) $\ll_{\text{int}}$ is finer than $\ll_{\tau_{\text{int}}}$.

(6) If (X, int) is topological, $\ll_{\text{int}} = \ll_{\tau_{\text{int}}}$ and $\text{int}_{\tau_{\text{int}}} = \text{int}$.

Proof (1) and (2) are similarly prove as in Theorem 1.9.

(3) Let $\lambda \in \tau_{\text{int}}$ with $\text{int}(\lambda) = \lambda$. It implies $\lambda \ll_{\text{int}} \lambda$ iff $\lambda \in \tau_{\ll_{\text{int}}}$. Thus, $\tau_{\text{int}} \subset \tau_{\ll_{\text{int}}}$.

Let $\lambda \in \tau_{\ll_{\text{int}}}$. Then $\lambda \ll_{\text{int}} \lambda$. It implies $\lambda \leq \text{int}(\lambda)$. Thus, $\lambda \in \tau_{\text{int}}$. So $\tau_{\ll_{\text{int}}} \subset \tau_{\text{int}}$.

(4) Suppose there exists $\lambda \in I^X$ such that

$$\text{int}_{\tau_{\text{int}}}(\lambda) \nleq \text{int}(\lambda).$$

Then there exists $x \in X$ such that

$$\text{int}_{\tau_{\text{int}}}(\lambda)(x) > \text{int}(\lambda)(x).$$

By the definition of $\text{int}_{\tau_{\text{int}}}$, there exists $\mu \in \tau_{\text{int}}$ with $\text{int}(\mu) = \mu \leq \lambda$ such that

$$\text{int}_{\tau_{\text{int}}}(\lambda)(x) \geq \mu(x) > \text{int}(\lambda)(x).$$

Since $\text{int}(\mu) \leq \text{int}(\lambda)$, it is a contradiction.

Hence $\text{int}_{\tau_{\text{int}}} \leq \text{int}$.

(5) Let $\lambda \ll_{\tau_{\text{int}}} \mu$. There exists $\rho \in \tau_{\text{int}}$ with $\text{int}(\rho) = \rho$ such that $\lambda \leq \rho \leq \mu$. Since $\lambda \leq \rho = \text{int}(\rho)$, we have $\lambda \ll_{\text{int}} \rho$ implies $\lambda \ll_{\text{int}} \mu$. Hence $\ll_{\text{int}}$ is finer than $\ll_{\tau_{\text{int}}}$.

(6) Let $\lambda \ll_{\text{int}} \mu$. Then $\lambda \leq \text{int}(\mu) \leq \mu$. Since (X, int) is topological, $\text{int}(\mu) \in \tau_{\text{int}}$ Thus, $\lambda \ll_{\tau_{\text{int}}} \mu$

Hence $\ll_{\tau_{\text{int}}}$ is finer than $\ll_{\text{int}}$. Since (X, int) is topological, $\text{int}(\text{int}(\lambda)) = \text{int}(\lambda) \leq \lambda$.

It implies $\text{int}_{\tau_{\text{int}}}(\lambda) \geq \text{int}(\lambda)$. By (4), $\text{int}_{\tau_{\text{int}}} = \text{int}$.

Example 2.8 Let $X = \{x, y, z\}$ be a set. Define a fuzzy supra interior operator $\text{int}: I^X \to I^X$ as follows:

$$\text{int}(\lambda) = \begin{cases} \overline{1} & \text{if } \lambda = \overline{1}, \\ \chi_{\{x,y\}} & \text{if } \chi_{\{x,y\}} \leq \lambda \neq \overline{1} \\ \chi_{\{y,z\}} & \text{if } \chi_{\{y,z\}} \leq \lambda \neq \overline{1} \\ \overline{0} & \text{otherwise} \end{cases}$$

But it is not a fuzzy interior operator because $\overline{0} = \text{int}(\chi_{\{x,y\}} \wedge \chi_{\{y,z\}}) \neq \text{int}(\chi_{\{x,y\}}) \wedge \text{int}(\chi_{\{y,z\}}) = \chi_{\{y\}}$. Thus τ_{int} is a supra fuzzy topology but not a topology from the fact that

$$\tau_{\text{int}} = \{\overline{0}, \chi_{\{x,y\}}, \chi_{\{y,z\}}, \overline{1}\}.$$

Since $\ll$ is topological, we have $\ll_{\text{int}} = \ll_{\tau_{\text{int}}}$ as follows:

$$\lambda \ll_{\text{int}} \mu \text{ iff } \begin{cases} \lambda = \overline{0} \text{ or } \mu = \overline{1}, \\ \overline{0} \neq \lambda \leq \chi_{\{x,y\}} \quad \overline{1} \neq \mu \geq \chi_{\{x,y\}} \\ \overline{0} \neq \lambda \leq \chi_{\{y,z\}} \quad \overline{1} \neq \mu \geq \chi_{\{y,z\}} \end{cases}$$

Furthermore, $\tau_{\text{int}} = \tau_{\ll_{\text{int}}}$ and $\text{int} = \text{int}_{\tau_{\text{int}}}$.

Example 2.9 Let $X = \{x, y, z\}$ be a set. Define a fuzzy supra interior operator $\text{int}: I^X \to I^X$ as follows:

$$\text{int}(\lambda) = \begin{cases} \overline{1} & \text{if } \lambda = \overline{1}, \\ \chi_{\{x\}} & \text{if } \chi_{\{x,y\}} \leq \lambda \neq \overline{1} \\ \overline{0} & \text{otherwise} \end{cases}$$

We obtain $\tau_{\text{int}} = \tau_{\ll_{\text{int}}} = \{\overline{0}, \overline{1}\}$. Since

$$\chi_{\{x\}} = \text{int}(\chi_{\{x,y\}}) \neq \text{int}(\text{int}(\chi_{\{x,y\}})) = \overline{0}$$

then int is not topological. We have $\ll_{\text{int}} \neq \ll_{\tau_{\text{int}}}$ as follows:

$$\lambda \ll_{\text{int}} \mu \text{ iff } \begin{cases} \lambda = \overline{0} \text{ or } \mu = \overline{1}, \\ \overline{0} \neq \lambda \leq \chi_{\{x\}} \quad \overline{1} \neq \mu \geq \chi_{\{x,y\}} \end{cases}$$

and

$$\lambda \ll_{\tau_{\text{int}}} \mu \text{ iff } \lambda = \overline{0} \text{ or } \mu = \overline{1},$$

Furthermore, $\text{int} \neq \text{int}_{\tau_{\text{int}}}$ from the fact that

$$\text{int}_{\tau_{\text{int}}}(\lambda) = \begin{cases} \overline{1} & \text{if } \lambda = \overline{1}, \\ \overline{0} & \text{otherwise} \end{cases}$$

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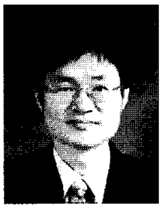

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