Aspects on Nonuniqueness and Instability Inherent in Inverse Scattering Problems

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Abstract

The nonuniqueness of a mathematically rigorous solution to 2-dimensional inverse scattering problems is explained in a limiting view of the numerical calculations based on the spectral-domain moment method. It is illustrated that its theoretical uniqueness cannot be assured even by performing additional measurements of the scattered fields not only along multiple lines but also with angular/frequency-diversities. In a real situation, however, computational error and measurement noise are inevitable. Those limitations render it meaningless to controvert the existence of a theoretically rigorous solution. Hence the most practical issue is how to remedy the instability of its practically approximate solution.

Key words : Inverse Scattering, Imaging, Uniqueness, Instability, Moment Method.

Ⅰ. Introduction

Electromagnetic scattering problems are divided broadly into two categories. One is forward scattering problem to predict the waves disturbed by a known constitution of objects. The other is inverse scattering problem to estimate the location, geometry, and internal characteristics of unknown objects from the measurement of those scattered fields. One of striking distinctions between forward and inverse scattering problems is the uniqueness of those solutions. In usual, forward scattering problems can be solved uniquely under general conditions realizable in a practical situation. But the uniqueness posed in the inverse scattering problem is still in a controversy. One of counterevidences against the uniqueness posed in inverse scattering problems has been considered the existence of nonradiating sources[1],[2]. In consequence, the present issue is what kinds of prior informations and/or constraints need to eliminate unreal solutions generated from nonradiating sources[3],[4].

At the core of difficulties to clear up the argued point of the nonuniqueness is no method to solve inverse scattering problems rigorously. Several analysis tools are available to solve forward scattering problems rigorously. However, those applicabilities to 2- and 3-dimensional inverse scattering problems are hampered because the effects of refraction and diffraction cannot be accounted exactly. It makes conventional inverse scattering algorithms mostly rely on heuristic assumptions to the actual wave motion through the scattered media. Typical examples are straight-line propagation of high-frequency waves[5], Born and Rytov approximations of the induced field over slightly refractive media[6],[7], physical optics approximation for conducting scatterers[8], etc. Hence, in spite of providing practically approximate solutions, conventional inversion schemes cannot play an adequate role on criticizing the nonuniqueness of its theoretically rigorous solution.

It is well recognized that numerical techniques as the finite element method[9], moment method[10], boundary element method[11], etc. are now established well as leading tools in the treatment of forward scattering problems. Some investigators have turned their attention to the extension of the moment method procedures to inverse scattering problems. Those numerical results have shown that complex permittivity distributions on 2- and 3-dimensional inhomogeneous dielectric objects could be reconstructed very accurately[12]~[16]. In recent, an alternative algorithm has been developed by employing the moment method procedure in the spectral domain[17]~[20]. The inverse scheme can be derived rigorously from the integral equation governing the wave motion in inhomogeneous dielectric objects under only one assumption. It is the approximation of induced source distribution over each discretized cell to pulse basis. The above approximation is also considered exact if the size of each discretized cell decreases infinitely

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small. Hence, the spectral-domain inverse scheme becomes adequate to deal with the nonuniqueness of its theoretically rigorous solution.

Based on the spectral-domain moment method procedure, numerical aspects on two canonical issues posed in inverse scattering problem are presented here. The first issue is the nonuniqueness of its theoretically rigorous solution to the inverse scattering problem in an ideal situation. In a real situation, however, a theoretically rigorous solution to inverse scattering problem is meaningless because measurement error, noise, and computational error are inevitable. It leads us to concern the second issue, which is the instability of a practically approximate solution to the same inverse scattering problem[21]. For convenience, the reconstruction of 2-dimensional permittivity distributions over an inhomogeneous dielectric cylinder is treated here.

Ⅱ. Nonlinear Equation to Inverse Scattering Problem

Consider 2-dimensional electromagnetic scattering by a dielectric cylinder as shown in Fig. 1. Relative dielectric constant $\varepsilon(x, y)$ is distributed over its arbitrary cross-section S. The region outside the cylinder, V is assumed free-space with wavenumber k_0. When an E-polarized field $u_i(x, y)$ is incident on the cylinder, the total field $u(x, y)$ satisfies the following integral equations as

$$u(x,y) = u_i(x,y) + k_0^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J_s(x',y')G(x,y;x',y';k_0)\cdot dx'dy', \text{ in } S+V \quad (1)$$

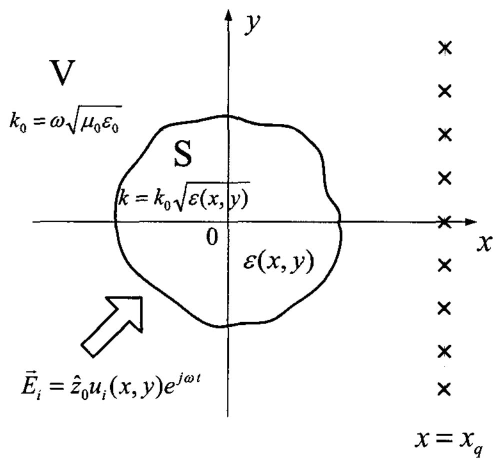

Fig. 1. Two-dimensional electromagnetic scattering by an inhomogeneous dielectric cylinder with arbitrary cross section.

where J_s denotes the induced current distribution as

$$J_s(x,y) = [\varepsilon(x,y)-1]u(x,y), \text{ in } S+V \quad (2)$$

In (1), G denotes 2-dimensional free-space Green's function. It should be noted that (1) and (2) are valid not only on V but also on S.

Our inverse scattering problem is how to reconstruct $\varepsilon(x, y)$ on S exactly from the known scattered field $u_s(x, y)$ on V. In this case, the incident field $u_i(x, y)$ is known on S and V. From measurement outside the dielectric cylinder, $u(x, y)$ is also known on V. Hence, as initial data, $u_s(x, y)$ on V is obtained by $u(x, y) - u_i(x, y)$. It should be noted that $u(x, y)$ on S is not known. In a mathematical point, the function $f: u_s(x, y)$ on $V \rightarrow \varepsilon(x, y)$ on S is nonlinear. The nonlinear property can be easily proved because N times of $u_s(x, y)$ on V cannot provide N times of $\varepsilon(x, y)$ on S. In general, there is no way to solve such a nonlinear equation exactly.

Then, how can we solve the nonlinear equation consisting of (1) and (2) in practice? One way is the approximation of the nonlinear equation into a linearized equation under some physical constraints. Under one of those physical constraints, the approximately linearized equation can be solved to obtain a mathematically approximate but practically useful solution. For example, diffraction tomography[6] has been widely used for microwave imaging of dielectric object. It is well recognized that in diffraction tomography, multiple incident fields(angular-diversity) play an important role on obtaining a practical solution uniquely. But its reconstructed image cannot become rigorous in view of a mathematical point because diffraction tomography algorithm itself can be formulated only under such a physical assumption as the Born or Rytov approximation. It leads us to conclude that a theoretically rigorous solution cannot be obtained by solving such an approximately linearized equation instead of the original nonlinear equation.

Let us consider another linearization method. Instead of relating $\varepsilon(x, y)$ with $u_s(x, y)$ directly, we introduce an intermediate parameter, $J_s(x, y)$. In (1), one may find that $J_s(x, y)$ is linearly related to $u_s(x, y)$. Assume that $J_s(x, y)$ can be obtained by solving (1) on V. Applying $J_s(x, y)$ into (1), and then solving (1) on S, one may obtain $u(x, y)$ on S. Then, $\varepsilon(x, y)$ is routinely obtained from $1+J_s(x, y)/u(x, y)$ on S. In the next section, the above linearization process is implemented by using the moment method in the spectral domain.

Ⅲ. Moment Method Procedure in Spectral Domain

Taking the spectral field $U(x, \beta)$ as the Fourier transform of the total field $u(x, y)$ in y, one may rewrite (1) into [17]

$$U(x,\beta) = U_i(x,\beta) + F(x,\beta)I(\beta) \tag{3}$$

where

$$F(x,\beta) = -\frac{jk_o^2}{2\sqrt{k_o^2-\beta^2}}\exp(\mp j\sqrt{k_o^2-\beta^2}\,x) \tag{4}$$

$$I(\beta) = \int\int_S dx'dy'[\varepsilon(x',y')-1]u(x',y')\cdot \exp(\pm j\sqrt{k_o^2-\beta^2}\,x' + j\beta y') \tag{5}$$

In (4) and (5), two algebraic signs in front of $\sqrt{k_o^2-\beta^2}$ are used for $x \geq x'$ (upper) and $x<x'$ (lower).

To calculate (5) numerically, the rectangular region R including S is discretized into MN numbers of rectangular cells uniformly, as shown in Fig. 2. Let R_{mn} designate the mn-th cell with the center at (x_m, y_n) and the area $2d_1 \times 2d_2$. The only assumptions in this derivation are as following;

$$u(x,y) = u(x_m,y_n) = u_{mn}, \quad \text{in } R_{mn} \tag{6a}$$

$$\varepsilon(x,y) = \varepsilon(x_m,y_n) = \varepsilon_{mn}, \quad \text{in } R_{mn} \tag{6b}$$

Substituting (6) into (5), and then integrating over each discretized cell, one may obtain a simple form of $I(\beta)$ as

$$I(\beta) = B(\beta)\sum_{m=1}^{M}\sum_{n=1}^{N}G_{mn}(\beta)I_{mn} \tag{7}$$

where

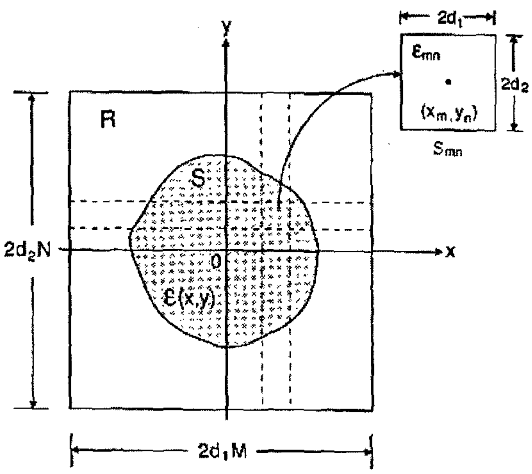

Fig. 2. Discretization of arbitrary cross section into $M \times N$ numbers of rectangular cells.

$$B(\beta) = \begin{cases} \dfrac{2\sin(\sqrt{k_o^2-\beta^2}\,d_1/2)}{\sqrt{k_o^2-\beta^2}/2}\exp(-j\sqrt{k_o^2-\beta^2}\,d_1/2)\dfrac{2\sin(\beta d_2)}{\beta}, & \text{for } x=x_m \\ \dfrac{2\sin(\sqrt{k_o^2-\beta^2}\,d_1)}{\sqrt{k_o^2-\beta^2}}\dfrac{2\sin(\beta d_2)}{\beta}, & \text{for } x\neq x_m \end{cases} \tag{8}$$

$$G_{mn}(\beta) = \begin{cases} \exp(j\sqrt{k_o^2-\beta^2}\,x_m + j\beta y_n) & , \text{for } x \geq x_m \\ \exp(-j\sqrt{k_o^2-\beta^2}\,x_m + j\beta y_n) & , \text{for } x < x_m \end{cases} \tag{9}$$

$$I_{mn} = (\varepsilon_{mn}-1)u_{mn} \tag{10}$$

Substitution of (7) into (3) yields

$$U(x,\beta) = U_i(x,\beta) + F(x,\beta)B(\beta)\sum_{m=1}^{M}\sum_{n=1}^{N}G_{mn}(\beta)I_{mn} \tag{11}$$

Ⅳ. Inverse Scattering Procedure

We now consider the inverse scattering procedure to reconstruct MN numbers of ε_m. For initial data, $U(x_q, \beta)$ is known by taking the Fourier transform of the total field $u(x_q, y)$ measured at $x=x_q$. And $U_i(x_q, \beta)$ is also obtained from the known incident field $u_i(x, y)$ at $(x=x_q)$. After choosing L numbers of discrete β, β_l for l=1, 2 ,···, L, one may obtain the matrix equation as,

$$[Z]\, I=V \tag{12}$$

where I is the column vector of unknown pulse expansion coefficients of the equivalent currents induced in the discretized cells, of which elements are defined by (10). And both elements of the MN by L matrix $[Z]$ and the L column vector V are given by

$$Z_{mn,\,l} = G_{mn}(\beta_l) \tag{13}$$

$$V_l = \frac{U(x_q,\beta_l) - U_i(x_q,\beta_l)}{F(x_q,\beta_l)B(\beta_l)} \tag{14}$$

Then, the reconstruction of MN numbers of ε_m can be implemented by following three steps;

step 1: evaluate I_{mn} by performing inversion of the matrix $[Z]$ in (12),

step 2: substitute I_{mn} into (11), and then calculate u_{mn} by taking the inverse Fourier transform of $U(x, \beta)$ at $(x=x_m)$,

step 3: reconstruct ε_m by calculating the ratio of I_{mn} to u_{mn} in the cell R_{mn}.

In the first step, an inverse source problem must be solved to estimate the induced source distribution over the dielectric object from the scattered field measured outside the object. And the second step looks like to solve a radiation problem.

One of interesting features in (11) is that the effects of measurement location, basis function, and geometry of objects are separated into $F(x, \beta)$, $B(\beta)$ and $G_n(\beta)$, respectively. For example, another basis function may be employed in (6) instead of the pulse basis. Even in this case, all the functions involved in (11) still hold except of changing $B(\beta)$ in (8). Furthermore, the exchanged function of $B(\beta)$ is also expressed in the analytic form for a number of different basis expansions since the integrand in (5) has no singularities over the integration region[18].

The above spectral inverse scattering algorithm is implemented numerically. Fig. 3 illustrates a test geometry of $d_1 = d_2 = 0.05\lambda$ (wavelength), $x_q = 1\lambda$, $M=1$, and $N=L=24$. Simulation results for three different dielectric constant profiles are shown in Fig. 4, where bold lines are original profiles and three different markers denote the reconstructed values at the center of each cell. The reconstruction errors appear negligible,

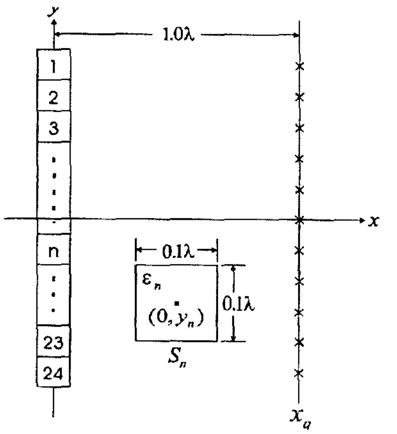

Fig. 3. A test geometry for $d_1=d_2=0.05\lambda$, $x_q=1\lambda$, $M=1$, and $N=24$.

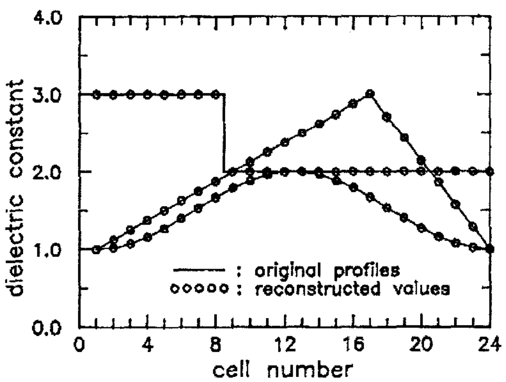

Fig. 4. Three different profiles of relative dielectric constant reconstructed from the scattered fields without noise.

in spite of including the round-off error in numerical analysis and the approximate expansion of the field over the discretized cells by pulse basis.

V. Nonuniqueness and Instability

We now consider the nonuniqueness of a theoretically rigorous solution to the underlying inverse scattering problem in an ideal situation. When the size of each discretized cell is reduced infinitely small, the approximation involved in (6) may be considered exact. It implies that the presented inverse scattering formulation is rigorous if both M and N become infinite. Then, our question is whether the exact dielectric constant profile can be reconstructed uniquely or not by employing such the rigorous inverse scattering formulation. For convenience, and without loss of generality, both of computation error and measurement noise are neglected perfectly. The answer is clearly no because the number MN of total points in S is always larger than the number L of total points along the line $x=x_q$. Although the area S is drastically small, the rank of the matrix in (12) is always less than MN.

The nonuniqueness seems to be remedied by applying some additional treatments. One of intuitive approaches is multiple measurement of the scattered fields at Q number of the $x=x_q$ lines for $q=1, 2, \cdots, Q$, as shown in Fig. 5. If LQ becomes equal to MN, the matrix in (12) becomes square. It seems to provide a unique solution. However, the measurement of the scattered fields at several lines cannot increase the total number of scattering data V_l. From (11)~(14), one may obtain

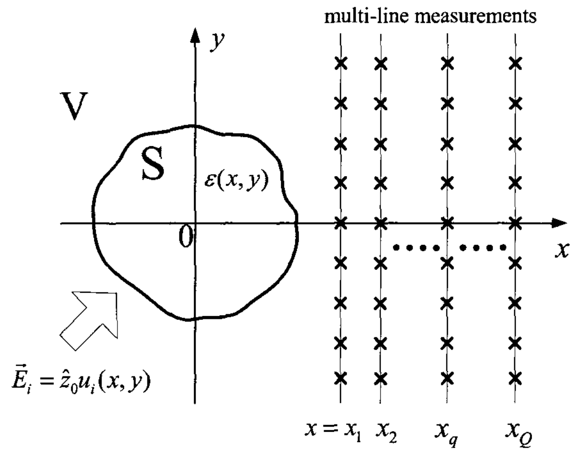

Fig. 5. Multiple measurement of the scattered fields at Q number of the $x=x_q$ lines for $q=1, 2, \cdots, Q$.

$$V_l = \sum_{m=1}^{M} \sum_{n=1}^{N} G_{mn}(\beta_l) I_{mn} \tag{15}$$

Since the right side of (15) is independent of the parameter x_q, V_l in (15) is also independent of x_q.

It leads us to conclude that the multiple-line measurement cannot remedy the underlying nonuniqueness.

Then, one may suggest another approach as frequency- or angular-diversity measurement of the scattered fields. As shown in (3)~(5), an incident field with different frequency or incident angle generates another induced source distribution over S. In a pure theoretical point, T number of different frequencies or incident angles yield T number of different inverse source problems. According to the presented inverse scattering algorithm, the uniqueness of the inverse scattering problem is assured only if its induced source distribution should be obtained uniquely in advance. In consequence, the underlying nonuniqueness cannot be resolved even though frequency- or angular-diversity measurements are performed additionally.

One may claim that if neglecting $J_s(x, y)$ and relating $\varepsilon(x, y)$ to $u_s(x, y)$ directly, the angular-diversity measurements help for solving the inverse scattering problem uniquely. The reason is that $\varepsilon(x, y)$ is independent of the variation of $u_s(x, y)$. In this case, however, our linearization scheme is broken due to loss of the intermediate linearization parameter $J_s(x, y)$. Hence the inverse scattering problem cannot be solved uniquely even if using multiple incident fields.

In practice, however, the nonuniqueness of a rigorous solution to the inverse scattering problem is beside the point. Round-off error in computer calculation and the noise in measurement of scattered field are inevitable in a real situation. Then, our interest turns to the validity of approximate solutions for some finite numbers of M, N, and L. In this case, the uniqueness of an approximate solution is usually assured by taking a finite number of L equal to MN. But the deviation from the rigorous solution becomes a more serious problem. It is called the instability or ill-posedness[21] inherent in inverse scattering problems. For example, when the scattered field in Fig. 4 is contaminated by 1 % Gaussian random noise, the reconstructed image is affected by a great amount of fluctuations, as shown in Fig. 6. To obtain superresolution as fine as 0.1×0.1 wavelength, the higher spectral components of induced current distribution are required. But the exponentially decaying behavior of $G_{mn}(\beta)$ for $|\beta| > k$ in (9) renders the matrix $[Z]$ in (12) unstable[18]. Hence, one may find

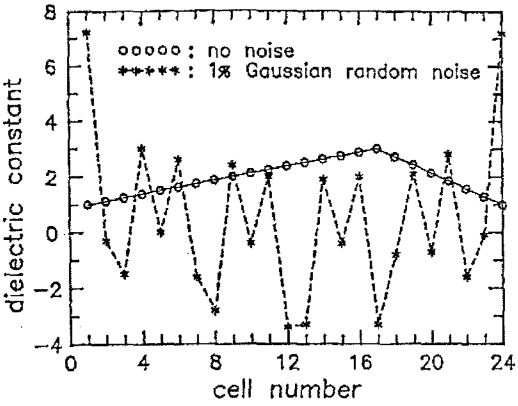

Fig. 6. The relative dielectric constant profile reconstructed from the scattered fields contaminated by 1 % Gaussian random noise.

the optimal point between reducing this instability and enhancing the resolution of reconstructed profiles.

One of methods to mediate this contradiction is called regularization[22], which means a compromise on degradation of its resolution with improvement of the instability. In our previous work[23], it was shown that SVD(singular value decomposition) technique reduced the root-mean-square error from 10^2 to 10^{-4} in reconstructed profile. And we have also found that the Tikhonov regularization norm was effective on suppressing the instability. In the spectral domain, the low-pass filtering[19] and the enlargement of the discretized cell size[20] have revealed more powerful for suppressing the reconstruction error even at the cost of the resolution.

Ⅵ. Conclusions

Two canonical issues posed in the reconstruction of 2-dimensional permittivity profile over an inhomogeneous dielectric cylinder were illustrated by using the moment-method procedure in the spectral domain. The first issue was the nonuniqueness of a theoretically rigorous solution to the inverse scattering problem in an ideal situation. The underlying nonuniqueness was explained by showing that the total number of known scattering data could not exceed the required number of unknowns. It was also shown that the total number of knowns could not increase even from multi-line, angular-diversity, and frequency-diversity measurements. In a mathematical point, there is no general way to solve the 2-dimensional inverse scattering problem exactly and uniquely. This paper illustrated the same conclusion but in a different view, which was a limiting aspect on the numerical calculations by using the moment method in the spectral domain.

The second issue was the instability of a practically approximate solution to the same inverse scattering problem in a real situation including measurement error, noise, computational error, etc. The uniqueness of a practically approximate solution was assured by showing that the total number of knowns could be equal to the finite number of unknowns. One of interesting features in the presented inversion scheme is that a finer resolution of reconstructed images calls for many discretized cells. However, a large amount of error in the reconstructed images can be arisen from even a negligible noise of the scattered field due to the fine discretization of the scatterer. Hence, the most important issue posed in practical inverse scattering problems should be how to improve the accuracy of those practically approximate solutions. It could be implemented by compromising degradation of its resolution with improvement of its instability properly.

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