韓國經營科學會誌
第28卷 第4號
2003年 12月

혼합 얼랑 확률변수의 극한치

강 성 열*

Extreme Values of Mixed Erlang Random Variables

Sungyeol Kang*

■ Abstract ■

In this paper, we examine the limiting distributional behaviour of extreme values of mixed Erlang random variables. We show that, in the finite mixture of Erlang distributions, the component distribution with an asymptotically dominant tail has a critical effect on the asymptotic extreme behavior of the mixture distribution and it converges to the Gumbel extreme-value distribution. Normalizing constants are also established. We apply this result to characterize the asymptotic distribution of maxima of sojourn times in $M/M/s$ queuing system. We also show that Erlang mixtures with continuous mixing may converge to the Gumbel or Type Ⅱ extreme-value distribution depending on their mixing distributions, considering two special cases of uniform mixing and exponential mixing.

Keyword : Extreme Values, Erlang, Finite Mixture Distribution, Continuous Mixing

1. Introduction

In a communication system, a typical concern is the delay times of message units in the system. The grade of service for the system may be controlled by keeping the longest delay of a group of message units below a given limit. In such a situation, there arises a need of investigating the distributional behavior of the maxima.

Suppose X_i represent the random time in the system of message i, $i=1,2,\cdots,n$. The maxima of the random times in the system may be given by $M_n=\max\{X_1, X_2, \cdots, X_n\}$, where X_1, X_2,

논문접수일 : 2003년 11월 10일 논문게재확정일 : 2003년 11월 24일
* 홍익대학교 경영정보학과 교수

$\cdots, X_n$ are independent random variables with distribution F. The distribution F is a *finite mixture distribution* if it is of the form $F(x) = \sum_{i=1}^{m} c_i F_i(x)$, where *mixing parameters* c_i are nonzero real numbers that sum to 1. F is a continuous mixture if it is defined by $F(x) = \int_{\Theta} F_\theta(x) dH(\theta)$, where $\{F_\theta : \theta \in \Theta\}$ is a set of component probability distributions and the index θ is governed by the *mixing distribution* H. The aim is to determine the type of a limiting extreme-value distribution G and normalizing constants $a_n > 0, b_n$ such that

$$P\{(M_n - b_n)/a_n \le x\} = F^n(a_n x + b_n) \to G(x)$$

at each continuity point x of G.

The random times in the system can be often represented by mixed Erlang random variables. So we address the asymptotic extreme behavior of mixtures of Erlang distributions.

Here is an overview of this study. In Section 2, for finite mixtures of Erlang distributions, we give criteria to determine the component distribution with an asymptotically dominant tail, type of extreme-value distribution G, and normalizing constants $a_n > 0$ and b_n. We use the fact that the component distribution with an *asymptotically dominant* tail in a finite mixture distribution has a crucial effect on the asymptotic extreme behavior of the finite mixture. We then relate the tail behavior of the waiting time distributions in $M/M/s$ queuing system to those of finite mixture distributions of Erlang random variables and characterize the asymptotic extreme behavior of the waiting times in the system. In Section 3, we consider a few Erlang mixtures with continuous mixing and determine the extreme-value distributions of the continuous mixtures. In the last section, we summarize our study.

2. Extreme values of Erlang mixtures with finite mixing

2.1 Convergence of the maxima of mixed Erlang random variables

We assume that $M_n = \max\{X_1, X_2, \cdots, X_n\}$, where $X_1, X_2, \cdots, X_n$ are independent random variables with distribution F. The focus is on determining the distribution of M_n for large n. This involves finding sequences of *normalizing constants* $a_n > 0$, b_n and a non-degenerate distribution function G such that

$$P\{(M_n - b_n)/a_n \le x\} = F^n(a_n x + b_n) \to G(x) \tag{2.1}$$

for each continuity point x of G. In this case, F belongs to the *domain of attraction (for maxima)* of G, denoted by $F \in \mathbf{D}(G)$. For standard results of extreme value theory, see for instance [1, 6].

Extreme-value distributions. A classical result of the extreme value theory is that if (2.1) holds, then the limiting distribution G must be one of the following three extreme value types,

$$\Lambda(x) = \exp(-e^{-x}), \ -\infty < x < \infty ;$$

(Gumbel distribution)

$$\Phi_\alpha(x) = \begin{cases} 0, & x \le 0, \\ \exp(-x^{-\alpha}), & x > 0, \ \text{for some } \alpha > 0; \end{cases}$$

(Type II distribution)

$$\Psi_\alpha(x)=\begin{cases}\exp(-(-x)^\alpha), & x\le 0, \text{ for some } \alpha>0\\ 1, & x>0\end{cases}$$

(Type III distribution)

Convergence criterion. Expression (2.1) holds if and only if

$$n(1-F(a_n x+b_n)) \to -\log G(x), \text{ as } n\to\infty \quad (2.2)$$

for each x such that $G(x)>0$.

We use the following result on the convergence of M_n under the assumption that the distribution of the X_n is the finite mixture $F(x) = \sum_{i=1}^{m} c_i F_i(x)$, where $c_i \neq 0$ and $\sum_{i=1}^{m} c_i = 1$. The limiting behavior of the maximum M_n is governed by the asymptotic behavior of the tail of the distribution F, and this tail is critically influenced by (or *equivalent* to) the tail of an asymptotically dominant component function. Here a component function F_d in the finite mixture $F=\sum_{i=1}^{m} c_i F_i(x)$, is said to have an *asymptotically dominant* tail if

$$\lim_{x\to x_0}(1-F_i(x))/(1-F_d(x))=r_i$$

for each $i\in I\equiv\{1,2,\cdots,m\}$ and $x_0=\sup\{x \mid F(x)<1\}$, where $r_d=1$ and $0\le r_i\le 1$ for $i\neq d$.

Lemma 1. (Kang and Serfozo [3]) Suppose, in the mixture $F=\sum_{i=1}^{m} c_i F_i(x)$, F_d has an asymptotically dominant tail such that $\gamma\equiv\sum_{i=1}^{m} c_i r_i$ and $F_d\in \mathbf{D}(G)$ with normalizing constants $\hat{a}_n, \hat{b}_n$. Then

$$P\{M_n\le a_n x+b_n\}=F^n(a_n x+b_n) \to G(x),$$

with the normalizing constants

$$\begin{aligned} a_n &= \hat{a}_n,\ b_n = \hat{b}_n+\hat{a}_n\log\gamma, && \text{if } G=\Lambda,\\ a_n &= \gamma^{1/\alpha}\hat{a}_n,\ b_n=\hat{b}_n=0, && \text{if } G=\Phi_\alpha,\\ a_n &= \gamma^{-1/\alpha}\hat{a}_n,\ b_n=\hat{b}_n=0, && \text{if } G=\Psi_\alpha. \end{aligned}$$

Using the convergence criterion (2.2) and Lemma 1, we can obtain the following result which enables us to determine the limiting extreme-value distributions of finite mixtures of Erlangs.

Proposition 2. Let F be the distribution function given by

$$F=\sum_{i=1}^{m} c_i F_i(x), \quad (2.3)$$

where $c_i\neq 0$, $\sum_{i=1}^{m} c_i=1$, and F_i is an Erlang distribution with parameters (η_i, θ_i). Then $F\in \mathbf{D}(\Lambda)$ with the normalizing constants

$$\begin{aligned} a_n &= 1/\theta_d,\\ b_n &= \theta_d^{-1}[\log n+(\eta_d-1)\log\log n\\ &\quad -\log(\eta_d-1)!+\log c_d], \end{aligned}$$

where the component d is chosen such that

$$\begin{aligned} &\theta_d\le\theta_i \quad \text{for } i\in I\\ &\eta_d>\eta_i \quad \text{for } i\in\{j \mid \theta_d=\theta_j, j\in I\}. \end{aligned} \quad (2.4)$$

Proof. The component function F_d has an asymptotically dominant tail since, with the condition (2.4),

$$\begin{aligned} \lim_{x\to\infty}\frac{1-F_i(x)}{1-F_d(x)} &= \lim_{x\to\infty}\frac{f_i(x)}{f_d(x)}\\ &= \lim_{x\to\infty}\frac{\theta_i(\theta_i x)^{\eta_i-1}e^{-\theta_i x}/(\eta_i-1)!}{\theta_d(\theta_d x)^{\eta_d-1}e^{-\theta_d x}/(\eta_d-1)!}\\ &= \frac{\theta_i^{\eta_i}(\eta_d-1)!}{\theta_d^{\eta_d}(\eta_i-1)!}\lim_{x\to\infty} x^{-(\eta_d-\eta_i)}e^{-(\theta_i-\theta_d)x}\\ &= 0, \text{ for any } i\neq d. \end{aligned}$$

And, it follows directly by the convergence criterion (2.2) that $F_d \in \mathbf{D}(\Lambda)$ with the normalizing constants

$$\hat{a}_n = 1/\theta_d,$$
$$\hat{b}_n = \theta_d^{-1}[\log n + (\eta_d - 1)\log\log n - \log(\eta_d - 1)!].$$

Hence, by Lemma 1, $F \in \mathbf{D}(\Lambda)$ with the normalizing constants $a_n = \hat{a}_n$, $b_n = \hat{b}_n + \hat{a}_n \log c_d$.

We can see that, in determining the extreme-value distribution G and the normalizing constants a_n, b_n for a finite mixture distribution with an asymptotically dominant component, we need just the mixing parameters c_i and the asymptotically dominant tail ratios r_i of *asymptotically persistent* (i.e., $r_i > 0, i \neq d$) components, in addition to the asymptotically dominant component term. For a finite mixture of Erlang distributions, there are no asymptotically persistent components in the mixture. Thus we can ignore all the other terms but the asymptotically dominant one in characterizing its asymptotic extreme behavior. This is a nice simplification for asymptotic computations.

2.2 Application to the maxima of waiting times in $M/M/s$ queue

We now apply our previous results on extreme values of mixed Erlang random variables to characterize the asymptotic limiting behavior of maxima of waiting times in the $M/M/s$ queueing system. Suppose that F is the distribution function of the sojourn time in the queuing system and $\{X_n, n \geq 1\}$ is a sequence of independent and identically distributed random variables from the distribution F. Our main interest is in determining the limiting extreme-value distribution of the maximum of $X_1, X_2, \cdots, X_n$.

To investigate the asymptotic extreme behavior of sojourn times, we first need to characterize the asymptotic behavior of the upper tail of the sojourn time distribution F. We relate the tail behavior of the sojourn time distributions to those of finite mixture distributions.

Let the random variable X represent time spent waiting in the system in equilibrium and $f(x)$ denote its probability density function. Then, in the case with $\lambda \neq (s-1)\mu$,

$$f(x) = \frac{\lambda - s\mu + \mu W_q(0)}{\lambda - (s-1)\mu}\mu e^{-\mu x} + \frac{(1 - W_q(0))\mu}{\lambda - (s-1)\mu}(s\mu - \lambda)e^{-(s\mu-\lambda)x}, \quad \text{for } x > 0, \tag{2.5}$$

where $W_q(0) = 1 - \frac{s(\lambda/\mu)^s}{s!(s-\lambda/\mu)} p_0$, and

$$p_0 = \left(\sum_{n=0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s \left(\frac{s\mu}{s\mu - \lambda}\right) \right)^{-1}$$

(see [2], p. 91).

From expression (2.5), we observe the following :

- The probability density function of the waiting times in the $M/M/s$ system, represented by (2.5), may be viewed as a mixture of two component exponential density functions, that is, $f(x) = c_1 f_1(x) + c_2 f_2(x)$, with component functions
  $f_1(x) = \mu e^{-\mu x}$, $f_2(x) = (s\mu - \lambda)e^{-(s\mu-\lambda)x}$
  and mixing parameters
  $$c_1 = \frac{\lambda - s\mu + \mu W_q(0)}{\lambda - (s-1)\mu}, \quad c_2 = \frac{(1 - W_q(0))\mu}{\lambda - (s-1)\mu}.$$
- If $\lambda < (s-1)\mu$, then $\mu = \min\{\mu, s\mu - \lambda\}$ and so $f_1(x)$ is asymptotically dominant with its

mixing parameter c_1 positive. In this case, c_2 is negative.

- If $\lambda > (s-1)\mu$, then $s\mu - \lambda = \min\{\mu, s\mu - \lambda\}$ and so $f_2(x)$ is asymptotically dominant with its mixing parameter c_2 positive. In this case, c_1 is negative.

And, in the case with $\lambda = (s-1)\mu$, $f(x)$ is given by

$$f(x) = W_q(0)\mu e^{-\mu x} + \frac{(1-W_q(0))(s\mu-\lambda)}{\mu}\mu(\mu x)e^{-\mu x}, \quad \text{for } x > 0, \tag{2.6}$$

where $W_q(0)$ is given in (2.5).
From the expression (2.6), we observe the following :

- The probability density function of the waiting times in the $M/M/s$ system, represented by (2.6), may be viewed as a mixture of two component Erlang density functions, that is, $f(x) = c_1 f_1(x) + c_2 f_2(x)$, with component functions $f_1(x) = \mu e^{-\mu x}, f_2(x) = \mu(\mu x)e^{-\mu x}$ and mixing parameters $c_1 = W_q(0), c_2 = (1-W_q(0))(s\mu-\lambda)/\mu$.
- The component density function $f_2(x)$ is asymptotically dominant. Here, both mixing parameters c_1 and c_2 are positive.

Now, with the observations above, Proposition 2 and Lemma 1, we have the following results on the asymptotic extreme behavior of waiting times in the $M/M/s$ system.

Proposition 3. Suppose that the independent sequence $\{X_n, n \ge 1\}$ is from the distribution (2.5) or (2.6) appropriately, which represents the waiting times in the $M/M/s$ system. Let M_n be the maximum of $X_1, X_2, \ldots, X_n$. Then

$$P\{M_n < a_n x + b_n\} \to \exp(-\gamma e^x), \quad -\infty < x < \infty,$$

where a_n, b_n are defined as follows :

(i) If $\lambda < (s-1)\mu$, $a_n = 1/\mu$, $b_n = \log n/\mu$, and $\gamma = (\lambda - s\mu + \mu W_q(0))/(\lambda - (s-1)\mu)$.

(ii) If $\lambda > (s-1)\mu$, $a_n = 1/(s\mu - \lambda)$, $b_n = \log n/(s\mu - \lambda)$, and $\gamma = ((1-W_q(0))\mu)/(\lambda - (s-1)\mu)$.

(iii) If $\lambda = (s-1)\mu$, $a_n = 1/\mu$, $b_n = (\log n + \log\log n - \log 2)/\mu$, and $\gamma = (1-W_q(0))(s\mu-\lambda)/\mu$.

Note that when $s = 1$, (ii) in Proposition 3 readily reduces to

$$P\{M_n < (x + \log n)/(\mu - \lambda)\} \to \exp(-e^x), \quad -\infty < x < \infty,$$

which is what we expect for the $M/M/1$ system.

The sojourn times in the system of n consecutive customers are actually correlated and therefore the independent assumption is not exact. However, we know that, if the traffic intensities are quite low or the *target* customers are separated significantly from one another, the correlation of the sojourn times is negligible. For an instance of this situation, consider a multiplexer of high-speed communication networks. The traffic stream of the multiplexer may come from lots of various connections. Then we may envisage that the data units for a *target* connection are to be *sparsely interspersed* in the merged traffic stream. For such situations in which the correlation tends to diminish, we can apply the asymptotic distributional results on M_n for approximately describing the extreme values of sojourn times for a group of n customers. A numerical study has been made with an acyclic three node queueing network to illustrate that such an approximation is quite effective [4].

3. Extreme values of Erlang mixtures with continuous mixing

3.1 In case of uniform mixing

For the following result, we consider the convergence of M_n under the assumption that the distribution of X_n is the *mixture of Erlang distributions with continuous mixing* whose mixing distribution is uniform on $[l, u]$.

Proposition 4. Suppose F is a continuous mixture of this form

$$F(x)=\int_l^u F_\theta(x)/(u-l)\,d\theta,\ x>0 \qquad (3.1)$$

where F_θ is an Erlang distribution with parameters (η, θ), and η is fixed.

(i) If $l=0$, then $F\in \mathbf{D}(\Phi_1)$ with the normalizing constants $a_n=\eta n/u,\ b_n=0$.

(ii) If $l>0$, then $F\in \mathbf{D}(\Lambda)$ with the normalizing constants $a_n=1/l$,
$b_n=(\log n\gamma+(\eta-2)\log\log n\gamma)/l$,
where $\gamma=l/((\eta-1)!(u-l))$.

We have shown previously that in a finite mixture, the asymptotically dominant component is critical in the asymptotic extreme behavior of the mixture distribution. The limiting extreme distribution of the finite mixture of Erlang distributions is determined by its asymptotically dominant component distribution and its mixing parameter. This kind of approach, however, is not applicable to mixtures with continuous mixing. Instead, we will use next two lemmas for studying extremes of continuously-mixed Erlang random variables.

Lemma 5. (Villasenor [7]) Let F be a distribution function with right endpoint $x_0=\infty$. Suppose that, for some constants $\alpha>0$, $\omega>0$, β and ξ,

$$\lim_{x\to\infty}(\omega x+\xi)^\beta\, e^{(\omega x+\xi)^\alpha}(1-F(x))=\gamma,\ 0<\gamma<\infty.$$

Then $F\in\mathbf{D}(\Lambda)$ with normalizing constants

$$a_n=\left(\alpha\omega\,(\log n\gamma)^{(\alpha-1)/\alpha}\right)^{-1},$$

$$b_n=\frac{(\log n\gamma)^{1/\alpha}}{\omega}-\frac{\beta\log\log n\gamma}{\alpha^2\omega\,(\log n\gamma)^{(\alpha-1)/\alpha}}-\frac{\xi}{\omega}.$$

Lemma 6. (Lamperti [5]) Let F be a distribution function with right endpoint $x_0=\infty$. Suppose that for some constant $\alpha>0$

$$\lim_{x\to\infty}x^\alpha(1-F(x))=\gamma,\ 0<\gamma<\infty. \qquad (3.2)$$

Then $F\in\mathbf{D}(\Phi_\alpha)$ with normalizing constants $a_n=(\gamma n)^{1/\alpha},\ b_n=0$.

The assumption (3.2) means that the distribution F is tail equivalent to a Pareto distribution $G(x)=1-x^{-\alpha},\ \alpha>0, x\ge 1$ with a ratio approaching to γ. In this respect, we may interpret the relation $F\in\mathbf{D}(\Phi_\alpha)$ above as a logical consequence of this tail equivalence to the Pareto distribution.

Proof of Proposition 4. The tail function of F is

$$1-F(x) = \int_l^u (1-F_\theta(x))/(u-l)\,d\theta$$
$$= \frac{1}{u-l}\int_l^u \sum_{k=0}^{\eta-1} \frac{e^{-\theta x}(\theta x)^k}{k!}\,d\theta$$
$$= \frac{1}{u-l}\sum_{k=0}^{\eta-1}\frac{x^k}{k!}\int_l^u e^{-\theta x}\,\theta^k d\theta$$
$$= \frac{1}{u-l}\sum_{k=0}^{\eta-1}\frac{x^k}{k!}\left[-e^{-\theta x}\sum_{i=0}^{k}\frac{k!}{(k-i)!}\frac{\theta^{k-i}}{x^{i+1}}\right]_{\theta=l}^{u}$$
$$= \frac{1}{u-l}\sum_{k=0}^{\eta-1}x^k\left(e^{-lx}\sum_{i=0}^{k}\frac{l^{k-i}}{(k-i)!x^{i+1}} - e^{-ux}\sum_{i=0}^{k}\frac{u^{k-i}}{(k-i)!x^{i+1}}\right), \text{ for } x>0.$$

If $l=0$, then

$$\lim_{x\to\infty} x(1-F(x)) = \lim_{x\to\infty}\frac{1}{u}\sum_{k=0}^{\eta-1}\left(1-e^{-ux}x^{k+1}\sum_{i=0}^{k}\frac{u^{k-i}}{(k-i)!x^{i+1}}\right) = \eta/u \tag{3.3}$$

Thus, assertion (i) follows by Lemma 6.
If $l>0$, then

$$\lim_{x\to\infty}(lx)^{2-\eta}e^{lx}(1-F(x))$$
$$= \lim_{x\to\infty}\frac{1}{u-l}\sum_{k=0}^{\eta-1}\left(x^{2-\eta+k}\sum_{i=0}^{k}\frac{l^{2-\eta+k-i}}{(k-i)!x^{i+1}} - e^{-(u-l)x}(lx)^{2-\eta}\sum_{i=0}^{k}\frac{u^{k-i}}{(k-i)!x^{i+1}}\right)$$
$$= \lim_{x\to\infty}\frac{1}{u-l}\left(\frac{l}{(\eta-1)!}+\sum_{i=1}^{\eta-1}\frac{l^{1-i}}{(\eta-1+i)!x^i} + \sum_{k=0}^{\eta-2}\sum_{i=0}^{k}\frac{l^{2-\eta+k-i}}{(k-i)!x^{\eta-1-k+i}} - e^{-(u-l)x}(lx)^{2-\eta}\sum_{i=0}^{k}\frac{u^{k-i}}{(k-i)!x^{i+1}}\right)$$
$$= \frac{l}{(\eta-1)!\,(u-l)}. \tag{3.4}$$

Hence, by Lemma 5, we have assertion (ii).

It is interesting that the domain of attraction of F is dependent on whether l is equal to 0 or not. Note that the extreme value distribution of the mixture F of the component distribution F_θ under the mixing distribution H may be not of the same type as that of the component distribution F_θ in this case : if $l=0$, then $F\in \mathbf{D}(\Phi_1)$ even though $F_\theta \in \mathbf{D}(\Lambda)$. Compare this result with the fact that a finite mixture of Erlang distributions belongs to the domain of attraction (for maxima) of Λ.

The relation (3.3) implies that the mixture of Erlang distributions, F given by (3.1), with uniform mixing on $[0,u]$ is tail equivalent to a Pareto distribution $G(x)=1-x^{-1}$, $x\geq 1$ with a ratio approaching to η/u. If F is the mixture with uniform mixing on $[l,u]$ and $l>0$, we can see from the relation (3.4) that, for $\eta\geq 2$, F is tail equivalent to an Erlang distribution with parameters $(\eta-1,l)$ with a ratio approaching to $l/((\eta-1)!(u-l))$.

If in (3.1) F_θ is exponential with rate θ, Proposition 5 reduces to the following.

Corollary 7. Suppose F is the continuous mixture of exponential distributions with uniform mixing defined by

$$F(x)=\int_l^u (1-e^{-\theta x})/(u-l)\,d\theta,\ x>0.$$

(i) If $l=0$, then $F\in\mathbf{D}(\Phi_1)$ with the normalizing constants $a_n=n/u$, $b_n=0$.

(ii) If $l>0$, then $F\in\mathbf{D}(\Lambda)$ with the normalizing constants $a_n=1/l$, $b_n=(\log n\gamma - \log\log n\gamma)/l$, where $\gamma=l/(u-l)$.

3.2 In case of exponential mixing

Now, we assume that the distribution of X_n is the *mixture of Erlang distributions with continuous mixing* whose mixing distribution is exponential with rate λ. We then obtain the following result on the convergence of M_n.

Proposition 8. Suppose F is a mixture distribution with exponential mixing of this form

$$F(x) = \int_0^\infty F_\theta(x)\lambda e^{-\lambda\theta}d\theta, \quad x > 0, \qquad (3.5)$$

where F_θ is an Erlang distribution with parameters (η, θ). Then, $F \in \mathrm{D}(\Phi_1)$ with the normalizing constants $a_n = \eta\lambda n$, $b_n = 0$.

Proof. The tail function of F is

$$\begin{aligned} 1 - F(x) &= \int_0^\infty \sum_{k=0}^{\eta-1} \frac{e^{-\theta x}(\theta x)^k}{k!} \lambda e^{-\lambda\theta} d\theta \\ &= \lambda \sum_{k=0}^{\eta-1} \frac{x^k}{k!} \int_0^\infty e^{-(x+\lambda)\theta}\theta^k d\theta \\ &= \frac{\lambda}{x+\lambda} \sum_{k=0}^{\eta-1} \left(\frac{x}{x+\lambda}\right)^k, \text{ for } x > 0 \end{aligned}$$

Then

$$\begin{aligned} \lim_{x\to\infty} x(1-F(x)) &= \lim_{x\to\infty} \frac{x\lambda}{x+\lambda} \sum_{k=0}^{\eta-1} \left(\frac{x}{x+\lambda}\right)^k \\ &= \eta\lambda. \end{aligned} \qquad (3.6)$$

The assertion follows by Lemma 6.

Note from relation (3.6) that the mixture (3.5) of Erlang distributions with exponential mixing is tail equivalent to a Pareto distribution $G(x) = 1 - x^{-1}$, $x \ge 1$ with a ratio approaching to $\eta\lambda$.

4. Conclusion

In this study, we have examined the limiting distributional behaviour of extreme values of mixed Erlang random variables. We have shown that the distribution of the maxima of independent random variables converges to a Gumbel extreme-value distribution when the distributions of the variables are finite mixtures of Erlang distributions. The key idea is that the limiting distribution is determined by one component distribution among the mixtures whose tail dominates the other tails. We have applied these results to approximate the limiting distribution of the maxima of waiting times in $M/M/s$ queuing system. This application of the Gumbel extreme-value distribution to the maxima of sojourn times is quite effective in case either the traffic to the system is light or the arrivals of target units are sparsely interspersed. We have also found that the extreme-value distributions of Erlang mixtures with continuous mixing are not always of the same type as those of their component distributions, unlike to the case of finite mixtures. We have shown that, for two special cases of uniform mixing and exponential mixing, they may converge to the Gumbel or Type II extreme-value distribution depending on their mixing distributions.

REFERENCES

[1] Galambos, J., *The Asymptotic Theory of Extreme Order Statistics*, Roberte E. Krieger Publishing Co., Malabar, 1987.

[2] Gross, D. and C. M. Harris, *Fundamentals of Queueing Theory,* John Wiley & Sons, New York, 1985.

[3] Kang, S. and R.F. Serfozo, "Extreme Values of Phase-type and Mixed Random Variables with Parallel-processing Exam-

ples," *Journal of Applied Probability*, Vol. 36(1999), pp.194-210.

[4] Kang, S. and R.F. Serfozo, "Maxima of Sojourn Times in Acyclic Jackson Queueing Networks," *Computers and Operations Research*, Vol.24, No.11(1997), pp.1085-1095.

[5] Lamperti, J., *Probability : A Survey of the Mathematical Theory*, W.A. Benjamin, Inc., New York, 1966.

[6] Resnick, S.I., *Extreme Values, Regular Variations, and Point Processes*, Springer-Verlag, New York, 1987.

[7] Villasenor, J., *On Univariate and Bivariate Extreme Value Theory*, Thesis for Ph.D., Iowa State University, 1976.