*Journal of the Korean Statistical Society (2003),* **32**: *4, pp 401–409*

# SITE-DEPENDENT IRREGULAR RANDOM WALK ON NONNEGATIVE INTEGERS

MOKHTAR H. KONSOWA[1] AND HASSAN M. OKASHA[2]

ABSTRACT

We consider a particle walking on the nonnegative integers and each unit of time it makes, given it is at site k, either a jump of size m distance units to the right with probability $p_k$ or it goes back (falls down) to its starting point 0, a retaining barrier, with probability $q_k = 1 - p_k$. This is a Markov chain on the integers $mZ^+$. We show that if $q_k$ has a nonzero limit, then the Markov chain is positive recurrent. However, if $q_k$ speeds to 0, then we may get transient Markov chain. A critical speeding rate to zero is identified to get transience, null recurrence, and positive recurrence. Another type of random walk on $Z^+$ is considered in which a particle moves m distance units to the right or 1 distance unit to left with probabilities $p_k$ and $q_k = 1 - p_k$, respectively. A necessary condition to having a stationary distribution and positive recurrence is obtained.

*AMS 2000 subject classifications.* Primary 60J10; Secondary 05C05.
*Keywords.* Irregular random walk, transient, null recurrent, positive recurrent.

## 1. INTRODUCTION

The theory and applications of random walks are ubiquitous in the modern probability literature and random walks form perhaps the simplest and most important examples of stochastic processes-random phenomena unfolding with time. The connection between random walks and electric networks has been recognized for some time. See the colorful book by Doyle and Snell (1984). The regular random walk is a sequence $S_n = \sum_{k=1}^{n} X_k$ of partial sums of independent identically distributed random variables $X_1, X_2, \ldots$ The nearest neighbor random walk on the lattice of integers is well known. See, for example, Kemeny *et al.* (1976) and Doyle and Snell (1984). A state $x$ of a Markov chain is called recurrent

Received October 2002; accepted July 2003.
[1]Mathematics Department, Helwan University, Cairo, Egypt
[2]Mathematics Department, Al-Azhar University, Cairo, Egypt

if the return time to $x$ is finite with probability 1 and transient otherwise. A Markov chain is recurrent (transient) if all the states are recurrent (transient). If the mean return time of a recurrent state is also finite, the state is called positive recurrent and the Markov chain is positive recurrent if all the states are positive recurrent. The nearest neighbor random walk on a graph $G$ is a Markov chain on the set of vertices of $G$ and the transitions are made from a given vertex to one of its neighbors. A Markov chain is said to be irreducible if each vertex is accessible from each other vertex. Konsowa and Mitro (1991) observed that the type (transience or recurrence) of the nearest neighbor random walk on a random spherically symmetric tree (all vertices of the same distance from the root of the tree have the same degree) is the same as the type of the walk in a random environment on the lattice of integers.

Two types of irregular random walks on nonnegative integers are considered in this paper. In the first type, we consider a particle that makes, given it is at site $k$, a jump of size $m$ distance units to the right with probability $p_k$ or makes a backward jump to 0 with probability $q_k = 1 - p_k$. In that case the size of the jump the particle makes to 0 depends on its distance from 0 and as such these jump sizes are neither independent nor identically distributed random variables. Let $S_j$ denote the site of the particle at time $j$, then we call $S_j$ a site-dependent irregular random walk ($SDIRW$) on the nonnegative integers $mZ^+$. This is a type of markov chains on the integers $mZ^+$. The state 0 is considered to be a retaining barrier, that is, $p(0,0) > 0$. A second type of $SDIRW$ on $Z^+$ is obtained from the first type by restricting the size of the backward jump to be always a unit distance. This type will be denoted by $(m,1)SDIRW$. If the probabilities $p_k$ in the first type are assumed to be equal to a constant $p$, then the $SDIRW$ on $mZ^+$ will be positive recurrent regardless of the value of $m$ and $p$. Whilst, we show that if the sequence $q_k$ speeds fast enough to zero, we may obtain transient $SDIRW$ and less speeding to zero may yield null or positive recurrence.

For the second type a necessary condition to having a positive recurrent $(m,1)SDIRW$ is obtained. The probability generating function is used to explore the case of positive recurrence of our Markov chains. The probability generating function $\Pi(s)$ of a probability distribution $\boldsymbol{\pi} = (\pi_x : x \geq 0)$ is

$$\Pi(s) = \sum_{k=0}^{\infty} \pi_k s^k.$$

We recall that the nonnegative vector $\boldsymbol{\pi} = (\pi_x : x \geq 0)$ is a stationary distribution for a Markov chain with transition matrix $\mathbf{P}$ if its components are summing to 1 and $\boldsymbol{\pi}\mathbf{P} = \boldsymbol{\pi}$. We need the following theorem that may be found in Resnick (1994).

THEOREM 1.1. *An irreducible Markov chain has a stationary distribution iff it is positive recurrent.*

## 2. SITE-DEPENDENT IRREGULAR RANDOM WALK (*SDIRW*)

We first consider the case in which the probabilities $p_k$ are all equal to some constant $p$, $0 < p < 1$. Consider a Markov chain on the integers $mZ^+$ with transition probabilities defined by, for $i \geq 0$,

$$\begin{cases} P(i, i+m) = p, \\ P(i, 0) = 1 - p. \end{cases} \tag{2.1}$$

Obviously, this is an irreducible Markov chain on the integers $mZ^+$ with 0 as a retaining barrier. This Markov chain is positive recurrent regardless of the values of $p$ and $m$. This follows immediately from Theorem 1.1 and the fact that the vector $\boldsymbol{\pi}$ whose components are

$$\begin{cases} \pi_0 = 1 - p, & \\ \pi_{km} = p^k \pi_0, & \text{if} \quad k \geq 1, \\ \pi_j = 0, & \text{if} \quad j \neq km, \quad k \geq 0 \end{cases}$$

is the staionary distribution for the irreducible aperiodic Markov chain defined by (2.1).

We turn our attention now to the case of varying jump probabilities by considering a sequence $\{p_k\}$. Hence, we have a Markov chain on $Z^+$ with transition probabilities:

$$\begin{cases} p\big(km, (k+1)m\big) = p_{km}, \\ p(km, 0) = 1 - p_{km}. \end{cases} \tag{2.2}$$

This is an irreducible on the integers $mZ^+$ with 0 as a retaining barrier. If $T_0$ denotes the time of the first return to 0 and $m_0 = E(T_0)$, then

$$P(T_0 = n+1) = p_0\, p_m\, p_{2m} \cdots p_{(n-1)m}\, (1 - p_{nm})$$

and

$$P(T_0 < \infty) = (1 - p_0) + \lim_{N\to\infty} \sum_{n=1}^{N} p_0\, p_m\, p_{2m} \cdots p_{(n-1)m}\,(1 - p_{nm})$$
$$= 1 - \lim_{N\to\infty} \prod_{k=0}^{N} p_{km}.$$

Thus the *SDIRW* defined by equations (2.2) is recurrent iff

$$\lim_{N\to\infty} \prod_{k=1}^{N} p_{km} = 0,$$

or equivalently iff

$$\sum_{k=1}^{\infty} (1 - p_{km}) = \infty. \tag{2.3}$$

The following proposition follows immediately.

PROPOSITION 2.1. *If the two sequences $\{p_k\}$ and $\{\bar{p}_k\}$ are such that $\bar{p}_k \leq p_k$ and the SDIRW defined by equations (2.2) and corresponding to $\{p_k\}$ is recurrent, then so is the one corresponding $\{\bar{p}_k\}$.*

The following theorem shows that we may get transient, null recurrent, or positive recurrent *SDIRW* depending on how fast $q_n = 1 - p_n$ goes to 0.

THEOREM 2.2. *Let $p_0$ and $p_1$ be arbitrary two real numbers between 0 and 1 and for $n \geq 2$, $p_n = 1 - n^{-\alpha}$, $\alpha > 0$. Then the SDIRW of equations (2.2) is transient if $\alpha > 1$, null recurrent if $\alpha = 1$, and positive recurrent if $\alpha < 1$.*

PROOF. The transience when $\alpha > 1$ and recurrence when $\alpha \leq 1$ follow immediately from equation (2.3). To prove the positive recurrence for $\alpha < 1$ we need to make sure of the finiteness of the mean recurrence time $m_0$. It can easily be shown that

$$m_0 = (1 - p_0) + \sum_{n=2}^{\infty} n\, p_0\, p_m\, p_{2m} \cdots p_{(n-2)m}\,(1 - p_{(n-1)m}). \tag{2.4}$$

Let $b_n = n\, p_0\, p_m\, p_{2m} \cdots p_{(n-2)m}\,(1 - p_{(n-1)m})$. Then

$$
\begin{aligned}
\frac{b_{n+1}}{b_n} &= \frac{n+1}{n}\cdot p_{(n-1)m}\cdot\frac{1-p_{nm}}{1-p_{(n-1)m}}\\
&= \frac{n+1}{n}\left\{1-\frac{1}{(n-1)^\alpha m^\alpha}\right\}\left(\frac{1}{n^\alpha m^\alpha}\right)\left\{\frac{1}{(n-1)^\alpha m^\alpha}\right\}^{-1}\\
&= \frac{n+1}{n}\left\{\frac{m^\alpha(n-1)^\alpha-1}{m^\alpha n^\alpha}\right\}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
n\left(1-\frac{b_{n+1}}{b_n}\right) &= n\left\{1-\frac{n+1}{n}\left(\frac{m^\alpha(n-1)^\alpha-1}{m^\alpha n^\alpha}\right)\right\}\\
&= \frac{m^\alpha n^{\alpha+1}-(n+1)m^\alpha(n-1)^\alpha+(n+1)}{m^\alpha n^\alpha}.
\end{aligned}
$$

Consequently,

$$
\lim_n n\left(1-\frac{b_{n+1}}{b_n}\right) = \begin{cases}\infty & \text{if } \alpha<1\\ \dfrac{1}{m} & \text{if } \alpha=1.\end{cases}
$$

Raabe's test of convergence assures the positive recurrence for $\alpha < 1$ and the null recurrence for $\alpha = 1$ and $m > 1$. While Gauss' test is used to assure the null recurrence for $\alpha = 1$ and $m = 1$. □

Note that if $\alpha > 1$, then $\sum_k q_k < \infty$ and the Borel-Cantelli lemma assures that only finitely many jumps are made to zero and hence the transience must occur. The following proposition is straightforward.

PROPOSITION 2.3. *If $p_k \to p < 1$, then the SDIRW defined by equations (2.2) is positive recurrent.*

PROOF. All that is needed to show $m_0$, of equation (2.4), is finite is to assure the finiteness of

$$
\sum_{n=2}^{\infty} n \prod_{k=0}^{n-2} p_{km}.
$$

Since $p_k \to p < 1$, then for $\epsilon = (1-p)/2$, there exists $n_0$ such that for $k > n_0$,

$$
p_k < \frac{1+p}{2} < 1.
$$

We show that $\sum_{n=n_0}^{\infty} n \prod_{k=0}^{n-2} p_{km} < \infty$. This is equivalent to show

$$\sum_{l=0}^{\infty}(n_0 + l) \prod_{k=0}^{n_0+l-2} p_{km} < \infty.$$

Now

$$\begin{aligned}\sum_{l=3}^{\infty} l \prod_{k=0}^{n_0} p_{km} \prod_{k=n_0+1}^{n_0+l-2} p_{km} &< \sum_{l=3}^{\infty} l \prod_{k=n_0+1}^{n_0+l-2} p_{km} \\ &< \sum_{l=3}^{\infty} l \left(\frac{1+p}{2}\right)^{l-2} \\ &< \infty,\end{aligned}$$

which completes the proof. □

## 3. $(m,1)$SITE-DEPENDENT IRREGULAR RANDOM WALK ($(m,1)SDIRW$)

We study in this section the second type of our site-dependent irregular random walk on $Z^+$. The $(m,n)SDIRW$ can be formulated by the following transition probabilities. For $0 < p_k < 1$,

$$\begin{cases} p(k, k+m) = p_k, & k \geq 0, \\ p(k, k-n) = 1 - p_k, & k \geq n, \\ p(k, 0) = 1 - p_k, & k < n, \\ p(i, k) = 0, & k \neq i+m,\ i-n,\ 0. \end{cases} \tag{3.1}$$

This $(m,n)SDIRW$ is not necessarily irreducible on $Z^+$ and to make it so we restrict $m$ and $n$ to be relatively prime; that is, their greatest common divisor is 1. We introduce the following straightforward lemma.

LEMMA 3.1. *The $(m,n)SDIRW$ defined by (3.1) is irreducible if and only if $m$ and $n$ are relatively prime.*

PROOF. Suppose that the $(m,n)SDIRW$ is irreducible. Then the state 1 is reachable. In which case there are two integers $k$ and $l$ such that $mk - nl = 1$, which assures that $m$ and $n$ are relatively prime. Conversely, suppose that $m$ and $n$ are relatively prime. Then there are two integers $k$ and 1 such that $mk - nl = 1$

and for any positive integer $j$, $mkj - nlj = j$. That is the state $j$ is reached by making $kj$ steps to the right and $lj$ steps to the left. Whence, the chain is irreducible □

We only consider the case where the size of each backward jump is a unit distance. The transition probabilities are formulated as follows:

$$\begin{cases} p(k, k+m) = p_k, & k \geq 0, \\ p(k, k-1) = 1 - p_k, & k \geq 1, \\ p(0,0) = 1 - p_0. \end{cases} \tag{3.2}$$

This is an $(m, 1)SDIRW$ on $Z^+$. The irreducibility of this Markov chain follows from Lemma 3.1. The following theorem gives a necessary condition for the positive recurrence and generalizes somehow Theorem 3.1 of Konsowa and Okasha (2000).

THEOREM 3.2. *Consider the $(m, 1)SDIRW$ defined by (3.2). A necessary condition for a probability distribution $\boldsymbol{\pi}$ to be a stationary distribution is*

$$\sum_{k=0}^{\infty} \pi_k p_k = \frac{1 - \pi_0 q_0}{m+1}, \quad q_0 = 1 - p_0.$$

PROOF. The system $\boldsymbol{\pi P} = \boldsymbol{\pi}$ yields,

$$\pi_0 = \pi_0 q_0 + \pi_1 q_1, \tag{3.3}$$

$$\pi_j = \pi_{j+1} q_{j+1}, \qquad 0 < j \leq m-1, \tag{3.4}$$

$$\pi_j = \pi_{j-m} p_{j-m} + \pi_{j+1} q_{j+1}, \qquad j \geq m. \tag{3.5}$$

Multiplying both sides of equation (3.5) by $s^{j-m}$ and summing over $j$ yield:

$$\sum_{j=m}^{\infty} \pi_j \, s^{j-m} = \sum_{j=m}^{\infty} \pi_{j-m} \, p_{j-m} \, s^{j-m} + \sum_{j=m}^{\infty} \pi_{j+1} \, q_{j+1} \, s^{j-m}.$$

Setting $K(s) = \sum_{k=0}^{\infty} \pi_k \, p_k \, s^k$, we obtain

$$\begin{aligned} & s^{-m} \left\{ \Pi(s) - \sum_{j=0}^{m-1} \pi_j s^j \right\} \\ = \; & K(s) + s^{-m+1} \sum_{j=m}^{\infty} \pi_{j+1} s^{j+1} (1 - p_{j+1}) \\ = \; & K(s) + s^{-m+1} \left[ \left\{ \Pi(s) - \sum_{k=0}^{m} \pi_k s^k \right\} - \left\{ K(s) - \sum_{k=0}^{m} \pi_k p_k s^k \right\} \right]. \end{aligned}$$

Hence,

$$s\left\{\Pi(s)-\sum_{j=0}^{m-1}\pi_j\,s^j\right\} = K(s)\,s^{m+1}+\Pi(s)-\sum_{k=0}^{m}\pi_k\,s^k-K(s)+\sum_{k=0}^{m}\pi_k\,p_k\,s^k$$
$$= K(s)\,s^{m+1}+\Pi(s)-K(s)-\sum_{k=0}^{m}\pi_k\,s^k\,q_k.$$

Thus,

$$\begin{aligned} s\Pi(s) &= s\pi_0+s^2\pi_1+s\sum_{k=2}^{m-1}\pi_k s^k+K(s)s^{m+1}+\Pi(s)-K(s)-\sum_{k=0}^{m}\pi_k s^k q_k \\ &= s\pi_0+s^2\pi_1+K(s)s^{m+1}+\Pi(s)-K(s)+s\sum_{k=2}^{m-1}\pi_k s^k \\ &\quad -\pi_0 q_0-\pi_1 s q_1-s\sum_{k=2}^{m}\pi_k s^{k-1} q_k. \end{aligned} \tag{3.6}$$

Applying (3.4) to the most right term of (3.6) we obtain

$$s\,\Pi(s)=s\,\pi_0+K(s)\,s^{m+1}+\Pi(s)-K(s)-\pi_0\,q_0-\pi_1\,s\,q_1.$$

It follows then from equation (3.2) that

$$s\Pi(s)=K(s)s^{m+1}-K(s)+\Pi(s)+\pi_0 q_0(s-1).$$

Consequently,

$$\Pi(s)=\frac{K(s)(s^{m+1}-1)+\pi_0 q_0(s-1)}{s-1}.$$

Taking the limit as $s\to 1$ we obtain what was to be proved. □

## 4. CONCLUDING RESULTS

In this paper we define two types of irregular random walks on the set of nonnegative integers. For the first one, we consider a particle that makes a jump from a site $k$ to a site $k+m$ with probability $p_k$ and falls down to the state 0 with probability $q_k = 1-p_k$. For the second type, we consider $(m,1)SDIRW$ that describes the walk of a particle on the set of nonnegative integers in a way that it makes a jump of size $m$ to the right with probability $p_k$ and a jump of unit distance to the left with probability $q_k = 1-p_k$. We conclude:

1. For the first type, if $q_k$ speeds fast enough to zero, the walk tends to be transient and the nonzero limit of $q_k$ ensures positive recurrence. Whereas, moderate speed yields null recurrence. Moreover, if $p_k$ equals some constant $p$, then the walk is positive recurrent regardless of the value $p$ and the jump size $m$;

2. For the senond type, we give a sufficient condition for the existence of the stationary distribution which in turn ensures the positive recurrence of the walk.

## REFERENCES

DOYLE, P. G. AND SNELL, J. L. (1984). "Random walks and electrical networks", *The Carus Mathematical Monographs*, Vol. 22, The Mathematical Association of America, Washington.

KEMENY, J. G., SNELL, J. L. AND KNAPP A. W. (1976). *Denumerable Markov Chains*, Springer-Verlag, New York.

KONSOWA, M. H. AND MITRO, J. (1991). "The type problem for random walks on trees", *Journal of Theoretical Probability*, **4**, 535–550.

KONSOWA, M. H. AND OKASHA, H. M. (2000). "Irregular random walks on nonnegative integers", *Kyungpook Mathematical Journal*, **40**, 431–436.

RESNICK, S. (1994). *Adventures in Stochastic Processes*, Birkhauser, Boston.