

TOPOLOGICAL OPERATIONS OF ITERATED STAR-COVERING PROPERTIES

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ABSTRACT. In this paper, we study topological operations of iterated star covering properties, i.e., subspaces, products, and images and preimages of iterated starcompact spaces.

1. Introduction

Let X be a set and \mathcal{U} a collection of subsets of X . For any nonempty subset A of X , let $\text{st}(A, \mathcal{U}) = \text{st}^1(A, \mathcal{U}) = \bigcup\{U \in \mathcal{U} : A \cap U \neq \emptyset\}$ and $\text{st}^{n+1}(A, \mathcal{U}) = \text{st}(\text{st}^n(A, \mathcal{U}), \mathcal{U})$ for all $n \in \mathbb{N}$. In particular, $\text{st}^0(A, \mathcal{U}) = A$. We write $\text{st}^n(x, \mathcal{U})$ instead of $\text{st}^n(\{x\}, \mathcal{U})$.

DEFINITION 1.1. [1, 7] Let $n \in \mathbb{N}$.

(1) A space X is called *n-starcompact* if, for every open cover \mathcal{U} of X , there exists a finite subset F of X such that $\text{st}^n(F, \mathcal{U}) = X$.

(2) A space X is called *$n\frac{1}{2}$ -starcompact* if, for every open cover \mathcal{U} of X , there exists a finite subcollection \mathcal{V} of \mathcal{U} such that $\text{st}^n(\bigcup \mathcal{V}, \mathcal{U}) = X$.

A space X is called *countably compact* if every countable open cover of X has a finite subcover. A space X is *pseudocompact* if every continuous real-valued function on X is bounded. It is well-known that countable compactness is equivalent to 1-starcompactness in the class of Hausdorff spaces [3]. Also, if $n \in \tilde{\mathbb{N}} = \mathbb{N} \cup \{n + \frac{1}{2} : n \in \mathbb{N}\}$ and $n \geq 3$, then every n -starcompact regular space is $2\frac{1}{2}$ -starcompact [7], and $2\frac{1}{2}$ -starcompactness is equivalent to pseudocompactness in the class of Tychonoff spaces [7].

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DEFINITION 1.2. [7] A space X is said to be (n, k) -starcompact if for every open cover \mathcal{U} of X there is an n -starcompact subspace A of X such that $st^k(A, \mathcal{U}) = X$. For the sake of unification, a compact space is called $\frac{1}{2}$ -starcompact.

The above definition has appeared in [7], but no thorough investigation has been done so far. We can obtain the following lemma from the definition.

LEMMA 1.3. [7]

- (1) Every (n, k) -starcompact space is $(n + k)$ -starcompact for $n \in \tilde{\mathbb{N}}$ and $k \in \mathbb{N}$.
- (2) Every (n_1, k) -starcompact space is (n_2, k) -starcompact for $n_1, n_2 \in \tilde{\mathbb{N}}$ with $n_1 \leq n_2$ and $k \in \mathbb{N}$.
- (3) Every (n, k_1) -starcompact space is (n, k_2) -starcompact for $n \in \tilde{\mathbb{N}}$ and $k_1, k_2 \in \mathbb{N}$ with $k_1 \leq k_2$.

Most of (n, k) -starcompact properties which were introduced in Definition 1.2 have been distinguished in [5]. For example, it was proved that the Isbell-Mrówka space is a 2-starcompact Tychonoff space which is not $(1\frac{1}{2}, 1)$ -starcompact, and the modification of Tree's example is a $(2, 2)$ -starcompact Tychonoff space which is not $(2, 1)$ -starcompact [5].

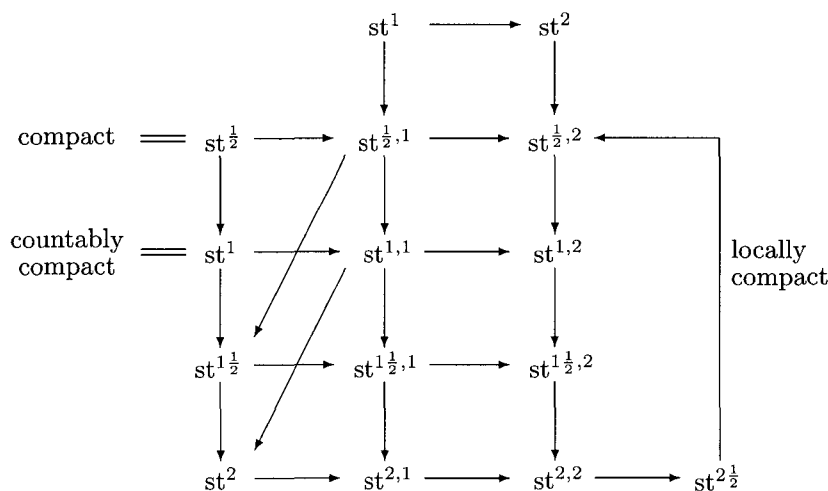


Diagram 1. (In the class of regular spaces)

In this paper, we shall study topological operations of iterated star covering properties, i.e., subspaces, products, and images and preimages of iterated starcompact spaces. In section 2, we give examples of iterated starcompact spaces, one of which is not hereditary to regular closed subspaces, and the other is not hereditary to closed G_δ subspaces even in the class of Hausdorff spaces if $n + k \geq 2$. Section 3 is devoted to products of iterated starcompact spaces. We prove a product theorem which is equivalent to $(1, n)$ -starcompactness. We ask a question about iterated starcompactness of product spaces. Section 4 consists of images and preimages. We show that a continuous image of a (n, k) -starcompact space is (n, k) -starcompact. We also show that the open perfect preimage of an $(n\frac{1}{2}, k)$ -starcompact space is $(n\frac{1}{2}, k)$ -starcompact. We investigate conditions under which preimages of (n, k) -starcompact spaces are (n, k) -starcompact for any $n, k \in \mathbb{N}$.

2. Subspaces

In this section, we shall discuss hereditary properties of subspaces of spaces having (n, k) -starcompact properties. It is well-known that countable compactness is hereditary to closed subspaces and pseudocompactness is not hereditary to closed subspaces in general. Even closed subspaces of an (n, k) -starcompact space may not be (n, k) -starcompact if $n + k \geq 1\frac{1}{2}$. For instance, one can check easily that the Tychonoff plank $T = (\omega_1 + 1) \times (\omega + 1) \setminus \{(\omega_1, \omega)\}$ is $(\frac{1}{2}, 1)$ -starcompact and has an infinite closed discrete subspace.

It is also well-known that pseudocompactness is hereditary to regular closed subspaces (A subset A of a space X is *regular closed* if $A = Cl_X Int_X A$). It is strange that this is not the case with $1\frac{1}{2}$ -starcompactness (See Example 9 in Section 2.3 of [7]). However, it is still unknown whether 2-starcompactness is preserved by regular closed sets [7]. In the following example, we show that (n, k) -starcompact properties are not hereditary to regular closed subspaces in general.

EXAMPLE 2.1. Let $\Psi = \omega \cup \mathcal{R}$ be the Isbell-Mrówka space with $|\mathcal{R}| = \mathfrak{c}$ and let D be a discrete space with $|D| = |\mathcal{R}|$ and $D \cap \mathcal{R} = \emptyset$. Let $N_{\mathfrak{c}^+}(D) = \beta D \times (\mathfrak{c}^+ + 1) \setminus [(\beta D \setminus D) \times \{\mathfrak{c}^+\}]$, and $i : \mathcal{R} \rightarrow D$ a bijection. Define the quotient space X of $N_{\mathfrak{c}^+}(D) \cup \Psi$ by identifying $r \in \mathcal{R}$ with $i(r)$ for each $r \in \mathcal{R}$. Then X is a $(\frac{1}{2}, 1)$ -starcompact Tychonoff space which has a regular closed subspace homeomorphic to Ψ . It is proved that Ψ is not $(1\frac{1}{2}, 1)$ -starcompact in Example 2.1 of [5].

Therefore, $(\frac{1}{2}, 1)$ -starcompactness, $(1, 1)$ -starcompactness, and $(1\frac{1}{2}, 1)$ -starcompactness are not hereditary to regular closed subspaces.

A space X is ω -starcompact [1] provided that for every open cover \mathcal{U} of X there exist $k \in \mathbb{N}$ and a finite subset F of X such that $\text{st}^k(F, \mathcal{U}) = X$. Note that for any $n \in \tilde{\mathbb{N}}$ and $k \in \mathbb{N}$, an (n, k) -starcompact space is ω -starcompact.

In Example 3.5 of [5], X is a $(1, 2)$ -starcompact Hausdorff space, and Y_1 is regular closed in X , which is not ω -starcompact. Therefore, $(n, 2)$ -starcompactness, $(1, k)$ -starcompactness and (i, j) -starcompactness are not hereditary to regular closed subspaces in the class of Hausdorff spaces whenever $n \geq 1$, $k \geq 2$ and $(i \geq 3, j \geq 0)$.

In [7], Matveev asked whether $1\frac{1}{2}$ -starcompactness is preserved by closed G_δ -sets (in particular, zero-sets). The answer is no under some set-theoretic assumption [10]. The following example shows that (n, k) -starcompactness is not hereditary to closed G_δ subspaces in the class of Hausdorff spaces if $n + k \geq 2$.

EXAMPLE 2.2. Let $T = ((\omega_1 + 1) \times (\omega + 1)) \setminus \{(\omega_1, \omega)\}$ be the Tychonoff plank and \mathcal{R} be a maximal almost disjoint family of infinite subsets of ω with $|\mathcal{R}| = \mathfrak{c}$. Let $X = T \cup \mathcal{R}$. Topologize X as follows:

- i) T is open in X ;
- ii) a basic neighborhood of $A \in \mathcal{R}$ has the form $\{A\} \cup ((\alpha, \omega_1] \times (A \setminus F))$, where $\alpha < \omega_1$ and F is a finite subset of ω . Equivalently, X can be considered as a quotient space of $T \cup \Psi$ ($\Psi = \omega \cup \mathcal{R}$ is the Mrówka space) which identifies ω with $\{\omega_1\} \times \omega$. Then X is Hausdorff, but not regular. Since $\omega_1 \times (\omega + 1)$ is a dense countably compact subspace of X , X is $(1, 1)$ -starcompact. But, $Y = (\omega_1 \times \{\omega\}) \cup \mathcal{R}$ is a closed G_δ subspace which is not ω -starcompact. \square

3. Products

It is well-known that the product of a countably compact (respectively, a pseudocompact space) and a compact space is countably compact (respectively, pseudocompact). The situation is more difficult with the group of n -starcompactness properties where $n \in \mathbb{N}$.

QUESTION 1. [7] *Is the product $X \times Y$ 2-starcompact provided X is 2-starcompact and Y is compact?*

In [1], it is shown that if X is $n\frac{1}{2}$ -starcompact and Y is compact, then $X \times Y$ is $n\frac{1}{2}$ -starcompact. However, the product of two (n, k) -starcompact spaces may not be (n, k) -starcompact for any $n \in \tilde{\mathbb{N}}, k \in \mathbb{N}$. Indeed, if X and Y are countably compact Tychonoff spaces in Novák's example of [9], then $X \times Y$ has an infinite discrete clopen subset.

A space X is called *totally countably compact* if any infinite subset of X contains an infinite subset whose closure is compact. It is clear that every totally countably compact space is countably compact. A Hausdorff space X is a k -space if for each $A \subseteq X$, A is closed in X provided that the intersection A with any compact subspace Z of X is closed in Z . It is known that every countably compact k -space is totally countably compact. The following theorem is known by N. Noble in 1969.

THEOREM 3.1. [8] *The product $X \times Y$ of a totally countably compact space X and a countably compact space Y is countably compact.*

THEOREM 3.2. *The following are equivalent for a space X and $n \in \mathbb{N}$:*

- (1) X is $(1, n)$ -starcompact;
- (2) $X \times Y$ is $(1, n)$ -starcompact for every totally countably compact space Y ;
- (3) $X \times Y$ is $(1, n)$ -starcompact for every compact space Y .

Proof. (1) \rightarrow (2) Let \mathcal{U} be an open cover of $X \times Y$ and let $\pi : X \times Y \rightarrow X$ be the projection mapping onto X . Then $\mathcal{V} = \{\pi(U) : U \in \mathcal{U}\}$ is an open cover of X . Since X is $(1, n)$ -starcompact, there exists a countably compact subspace A of X such that $\text{st}^n(A, \mathcal{V}) = X$. Then $B = A \times Y$ is countably compact by Theorem 3.1 and $\text{st}^n(B, \mathcal{U}) = X \times Y$.

(2) \rightarrow (3) It is straightforward.

(3) \rightarrow (1) We can prove this by taking Y as a singleton. \square

COROLLARY 3.1. *The following are equivalent for a Hausdorff space X and $n \in \mathbb{N}$:*

- (1) X is $(1, n)$ -starcompact;
- (2) $X \times Y$ is $(1, n)$ -starcompact for every countably compact k -space Y ;
- (3) $X \times Y$ is $(1, n)$ -starcompact for every compact space Y .

THEOREM 3.3. [4] *If X is a $(\frac{1}{2}, 1)$ -starcompact space and Y is a compact space, then $X \times Y$ is $(\frac{1}{2}, 1)$ -starcompact.*

Comparing with Theorem 3.3, the situation is much more difficult with the group of (n, k) -starcompactness properties for $n \geq 2$.

QUESTION 2. Let X be a (n, k) -starcompact space for $n \geq 2$ and Y be a compact space. Is $X \times Y$ (n, k) -starcompact?

4. Images and preimages

It is known [1] that a continuous image of a n -starcompact space is n -starcompact. We show that a continuous image of a (n, k) -starcompact space is (n, k) -starcompact.

THEOREM 4.1. Let $n \in \tilde{\mathbb{N}}$, $k \in \mathbb{N}$. If $f : X \rightarrow Y$ is a continuous mapping from an (n, k) -starcompact space X onto Y , then Y is (n, k) -starcompact.

Proof. Let \mathcal{U} be an open cover of Y . Then $\mathcal{V} = \{f^{-1}(U) : U \in \mathcal{U}\}$ is an open cover of X . Thus, there exists an n -starcompact subspace A of X such that $\text{st}^k(A, \mathcal{V}) = X$. By a result, Theorem 2.4.1 in [1], $f(A)$ is n -starcompact. For any $y \in Y$, choose $x \in f^{-1}(y)$. Then there exists $\{f^{-1}(U_1), \dots, f^{-1}(U_k)\}$ such that $x \in f^{-1}(U_1)$, $f^{-1}(U_i) \cap f^{-1}(U_{i+1}) \neq \emptyset$ for each $i < k$, and $f^{-1}(U_k) \cap A \neq \emptyset$. It follows that $y \in U_1$, $U_i \cap U_{i+1} \neq \emptyset$ for each $i < k$, and $U_k \cap f(A) \neq \emptyset$. Therefore, $y \in \text{st}^k(f(A), \mathcal{U})$ and Y is (n, k) -starcompact. \square

Before turning to preimages, we first recall that a continuous mapping $f : X \rightarrow Y$ is *perfect* if it is a closed mapping and all fibers $f^{-1}(y)$ are compact subsets of X . It is well-known that a perfect preimage of a countably compact space or a pseudocompact space is countably compact or pseudocompact [2]. An open perfect preimage of an $n\frac{1}{2}$ -starcompact space is $n\frac{1}{2}$ -starcompact [7]. We show that the same is true for $(n\frac{1}{2}, k)$ -starcompactness.

THEOREM 4.2. Let $f : X \rightarrow Y$ be an open perfect mapping from a space X onto a space Y and let $n, k \in \mathbb{N}$. If Y is $(n\frac{1}{2}, k)$ -starcompact, then X is $(n\frac{1}{2}, k)$ -starcompact.

Proof. Let \mathcal{U} be an open cover of X . For each $y \in Y$, since $f^{-1}(y)$ is compact, there exists a finite subcollection $\mathcal{U}_y \subseteq \mathcal{U}$ such that $f^{-1}(y) \subseteq \bigcup \mathcal{U}_y$ and each element of \mathcal{U}_y meets $f^{-1}(y)$. Since f is open, $O_y = \bigcap f(\mathcal{U}_y)$ is an open neighborhood of y . It follows from the continuity and closedness of f that there exists an open neighborhood V_y of y such that $V_y \subseteq O_y$ and $f^{-1}(V_y) \subseteq \bigcup \mathcal{U}_y$. Then $\mathcal{V} = \{V_y : y \in Y\}$ is an open cover of Y . Since Y is $(n\frac{1}{2}, k)$ -starcompact, there exists an $n\frac{1}{2}$ -starcompact subspace A of Y such that $\text{st}^k(A, \mathcal{V}) = Y$. One can

check easily that $f|_{f^{-1}(A)} : f^{-1}(A) \rightarrow A$ is open and perfect. By a result, Theorem 62 of [7], $f^{-1}(A)$ is $n_{\frac{1}{2}}$ -starcompact. Let $x \in X$ and let $f(x) = y$. Since $\text{st}^k(A, \mathcal{V}) = Y$, there exists a finite subcollection $\{V_{y_1}, \dots, V_{y_k}\}$ of \mathcal{V} such that $y \in V_{y_1}$, $V_{y_i} \cap V_{y_{i+1}} \neq \emptyset$ for each $i < k$, and $V_{y_k} \cap A \neq \emptyset$. By the construction of V_z , $V_z \cap V_t \neq \emptyset$ implies that for each $U \in \mathcal{U}_z$, there exists $U' \in \mathcal{U}_t$ such that $U \cap U' \neq \emptyset$. So we can find a finite subcollection $\{U_1, \dots, U_k\}$ of \mathcal{U} such that $x \in U_1$, $U_i \cap U_{i+1} \neq \emptyset$ for $i < k$, and $U_k \cap f^{-1}(A) \neq \emptyset$. Therefore, $x \in \text{st}^k(f^{-1}(A), \mathcal{U})$ and X is $(n_{\frac{1}{2}}, k)$ -starcompact. \square

Since it is still unknown [7] that an open perfect preimage of a 2-starcompact space is 2-starcompact, we do not know yet whether an open perfect preimage of an (n, k) -starcompact space is (n, k) -starcompact for any $n, k \in \mathbb{N}$ ($n \geq 2$). Of course, we note that the affirmative answer to this question would imply the affirmative answer to Question 2. It is, however, true for $(1, 1)$ -starcompactness and it can be generalized to $(1, k)$ -starcompactness for $k \in \mathbb{N}$ as in the following theorems.

THEOREM 4.3. *If $f : X \rightarrow Y$ is an open perfect mapping from a Hausdorff space X onto a $(1, 1)$ -starcompact space Y , then X is $(1, 1)$ -starcompact.*

Proof. Let \mathcal{U} be an open cover of X and let $y \in Y$. Since $f^{-1}(y)$ is compact, there is a finite subcollection \mathcal{U}_y of \mathcal{U} such that $f^{-1}(y) \subseteq \bigcup \mathcal{U}_y$ and each element of \mathcal{U}_y meets $f^{-1}(y)$. Since f is open, $O_y = \bigcap f(\mathcal{U}_y)$ is an open neighborhood of y . By continuity and closedness of f , there exists an open neighborhood V_y of y such that $V_y \subseteq O_y$ and $f^{-1}(V_y) \subseteq \bigcup \mathcal{U}_y$. Then $\mathcal{V} = \{V_y : y \in Y\}$ is an open cover of Y . By assumption, there exists a countably compact subspace A of Y such that $\text{st}(A, \mathcal{V}) = Y$. Let $B = f^{-1}(A)$. Then B is countably compact. We claim that $\text{st}(B, \mathcal{U}) = X$. Let $x \in X$. Then there exists $V_y \in \mathcal{V}$ such that $f(x) \in V_y$ and $V_y \cap A \neq \emptyset$. By the construction of V_y , there exists $U \in \mathcal{U}_y$ such that $x \in U$. Since $V_y \subseteq f(U)$, $U \cap f^{-1}(A) \neq \emptyset$. So, $x \in \text{st}(B, \mathcal{U})$. Therefore, X is $(1, 1)$ -starcompact. \square

As a generalization of Theorem 4.3, we have the following result.

THEOREM 4.4. *If $f : X \rightarrow Y$ is an open perfect mapping from a Hausdorff space X onto a $(1, k)$ -starcompact space Y ($k \in \mathbb{N}$), then X is $(1, k)$ -starcompact.*

Instead of “perfect”, one can only demand “ Y is an n -starcompact space with all fibers k -starcompact”.

THEOREM 4.5. *Let $n \in \mathbb{N}$ and $k \in \tilde{\mathbb{N}}$. If $f : X \rightarrow Y$ is an open mapping onto an n -starcompact space Y such that all fibers are k -starcompact, then X is (k, n) -starcompact.*

Proof. Let \mathcal{U} be an open cover of X . Since f is open, $\mathcal{V} = f(\mathcal{U})$ is an open cover of Y . Because Y is n -starcompact, there exists a finite subset F of Y such that $\text{st}^n(F, \mathcal{V}) = Y$. Then $A = f^{-1}(F)$ is k -starcompact.

Claim : $\text{st}^n(A, \mathcal{U}) = X$

Let $x \in X$ and let $f(x) = y$. then there exists a finite subcollection $\{V_1, \dots, V_n\}$ of \mathcal{V} such that $y \in V_1$, $V_i \cap V_{i+1} \neq \emptyset$ for each $i < n$, and $V_n \cap F \neq \emptyset$. Denote $V_i = f(U_i)$ for each i . Then $x \in U_1$, $U_i \cap U_{i+1} \neq \emptyset$ for each $i < n$, and $U_n \cap A \neq \emptyset$. Thus $x \in \text{st}^n(A, \mathcal{U})$.

Therefore X is (k, n) -starcompact. □

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