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ON A FUNCTIONAL CENTRAL LIMIT THEOREM FOR STATIONARY LINEAR PROCESSES GENERATED BY ASSOCIATED PROCESSES

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Abstract. A functional central limit theorem is obtained for a stationary linear process of the form $X_t = \sum_{j=0}^{\infty} a_j \epsilon_{t-j}$, where $\{\epsilon_t\}$ is a strictly stationary associated sequence of random variables with $E\epsilon_t = 0$, $E(\epsilon_t^2) < \infty$ and $\{a_j\}$ is a sequence of real numbers with $\sum_{j=0}^{\infty} |a_j| < \infty$. A central limit theorem for a stationary linear process generated by stationary associated processes is also discussed.

1. Introduction and main results

A finite collection of random variables $\{\epsilon_1, \cdots, \epsilon_m\}$ is said to be associated if for any two coordinatewise nondecreasing functions f_1, f_2 on $\mathbb{R}^m$ such that $\tilde{f_j} = f_j(\epsilon_1, \cdots, \epsilon_m)$ has finite variance for $j = 1, 2$, $cov(\tilde{f_1}, \tilde{f_2}) \geq 0$. An infinite collection of random variables is said to be associated if every finite subcollection of random variables is associated. This definition was introduced by Esary, Proschan and Walkup ([2]) as an extension of the bivariate notion of positive quadrant dependence of Lehmann ([7]). A large amount of papers has been concerned with limit theorems for associated processes (see, for example, Newman ([8], [9]).

Let $\{X_t,\ t \in \mathbb{Z}^+\}$ be a stationary linear process defined on a probability space $(\Omega, \mathcal{F}, P)$ of the form

$$X_t = \sum_{j=0}^{\infty} a_j \epsilon_{t-j}, \tag{1}$$

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where $\{a_j\}$ is a sequence of real numbers with $\sum_{j=0}^{\infty}|a_j| < \infty$ and $\{\epsilon_t\}$ is a strictly stationary process such that $E\epsilon_t = 0$ and $0 < E\epsilon_t^2 < \infty$.

The linear processes are special importance in time series analysis and they arise from a wide variety of contexts (see, e.g., Hannan ([6]) Ch.6). Applications to economics, engineering and physical sciences are extremely broad and a vast amount of literature is devoted to the study of the limit theorems for linear processes under various conditions on ϵ_t. For the linear processes, Fakhre-Zakeri and Lee ([4]) and Fakhre-Zakeri and Farshidi ([3]) established a central limit theorem (CLT) under the iid assumption on ϵ_t and Fakhre-Zakeri and Lee ([5]) proved a functional central limit theorem (FCLT) under the strong mixing condition on ϵ_t.

Let $S_n = \sum_{t=1}^{n} X_t$ and $\tau^2 = \sigma^2(\sum_{j=0}^{\infty} a_j)^2$. Define, for $n \geq 1$, the stochastic process

$$\xi_n(u) = n^{-\frac{1}{2}}\tau^{-1}S_{[nu]},\ u \in [0,1], \tag{2}$$

where $[x]$ is the greatest integer not exceeding x.

In this paper, we establish a CLT (FCLT) for a strictly stationary linear process of the form (1), generated by an associated process $\{\epsilon_t\}$. More precisely, we will prove the following theorems:

THEOREM 1. *Let $\{X_t\}$ be a stationary linear process of the form* (1), *where $\{a_j\}$ is a sequence of constants with $\sum_{j=0}^{\infty}|a_j| < \infty$ and $\{\epsilon_t\}$ is a strictly stationary associated process with $E\epsilon_t = 0$, $0 < E\epsilon_t^2 < \infty$. Assume*

$$0 < \sigma^2 = E\epsilon_1^2 + 2\sum_{t=2}^{\infty} E(\epsilon_1\epsilon_t) < \infty. \tag{3}$$

Then the linear process $\{X_t\}$ fulfills the CLT.

THEOREM 2. *Let $\{X_t\}$ be a stationary linear process of the form* (1) *defined in Theorem 1. If* (3) *fulfilled then the process $\{\xi_n\}$ satisfies the FCLT, that is, the process $\{\xi_n\}$ converges weakly to Wiener measure W on the space of all functions on* $[0,1]$, *which have left hand limits and are continuous from the right.*

2. Proofs

The following lemma needs to prove Theorems 1 and 2 and it is proved by modifying the proof of Lemma 3 in Fakhre-Zakeri and Lee ([5]). Doob's maximal inequality played important role in their proof. However, in our case, Newman and Wrights' maximal inequality $E(\max_{1\leq k\leq n}$

$|\epsilon_1 + \cdots + \epsilon_k|^2) \le n\sigma^2$ (see Theorem 2 of Newman and Wright ([10]) will be used.

LEMMA 1. *Let* $\{\epsilon_t\}$ *be a strictly stationary associated process with* $E\epsilon_t = 0$, $0 < E\epsilon_t^2 < \infty$. *Let* $X_t = \sum_{j=0}^{\infty} a_j \epsilon_{t-j}$, $S_k = \sum_{t=1}^{k} X_t$, $\tilde{X}_t = \left(\sum_{j=0}^{\infty} a_j\right) \epsilon_t$ *and* $\tilde{S}_k = \sum_{t=1}^{k} \tilde{X}_t$, *where* $\{a_j\}$ *is a sequence of real numbers with* $\sum_{j=0}^{\infty} |a_j| < \infty$. *If* (3) *are fulfilled, then*

$$(n^{-\frac{1}{2}}) \max_{1 \le k \le n} |\tilde{S}_k - S_k| \xrightarrow{P} 0. \tag{4}$$

Proof. See Appendix. □

Proof of Theorem 1. As in Lemma 1 set

$$\tilde{X}_t = \sum_{j=0}^{\infty} a_j \epsilon_t$$

and

$$\tilde{S}_n = \sum_{t=1}^{n} \tilde{X}_t = \left(\sum_{j=0}^{\infty} a_j\right) \sum_{t=1}^{n} \epsilon_t.$$

Then

$$\begin{aligned} E(\tilde{X}_t)^2 &= E(\sum_{j=0}^{\infty} a_j \epsilon_t)^2 \\ &= (\sum_{j=0}^{\infty} a_j)^2 E\epsilon_t^2 \\ &\le (\sum_{j=0}^{\infty} |a_j|)^2 E\epsilon_t^2 < \infty, \end{aligned} \tag{5}$$

$$\begin{aligned} E\tilde{X}_1^2 + 2\sum_{t=2}^{\infty} E(\tilde{X}_1\tilde{X}_t) &= (\sum_{j=0}^{\infty} a_j)^2 E\epsilon_1^2 + 2(\sum_{j=0}^{\infty} a_j)^2 \sum_{t=2}^{\infty} E(\epsilon_1\epsilon_t) \\ &= (\sum_{j=0}^{\infty} a_j)^2 \sigma^2 = \tau^2 < \infty \text{ by } (3) \end{aligned} \tag{6}$$

and $\tilde{X}_t{}'s$ are stationary associated process (see [2]). Thus $\{\tilde{X}_t,\ t \in \mathbb{Z}^+\}$ satisfies the CLT by Theorem 12 of [9], that is,

$$n^{-\frac{1}{2}} \tilde{S}_n \xrightarrow{\mathcal{D}} N(0, \tau^2). \tag{7}$$

According Lemma 1 we also have

$$(8) \qquad n^{-\frac{1}{2}}|\tilde{S}_n - S_n| \xrightarrow{P} 0.$$

Hence from (7) and (8) the desired conclusion follows. □

Proof of Theorem 2. Note that $\{\tilde{X}_t\}$ is a stationary associated process and that $\{\tilde{X}_t\}$ satisfies conditions of Theorem 3 of Newman and Wright ([10]) according to (5) and (6). This implies that Theorem 2 holds for the sequence $\{\tilde{\xi}_n\}$, where we define $\tilde{\xi}_n$ as in (2), but $\tilde{S}_{[nu]}$ replacing by $S_{[nu]}$. By Lemma 1 $|\tilde{\xi}_n(u) - \xi_n(u)| \xrightarrow{P} 0$ for all $0 \le u \le 1$. Hence, the desired conclusion follows. □

Appendix

Proof of Lemma 1. Like in the proof of Lemma 3 of [5] we have

$$\begin{aligned} \tilde{S}_k &= \sum_{t=1}^{k}\left(\sum_{j=0}^{k-t} a_j\right)\epsilon_t + \sum_{t=1}^{k}\left(\sum_{j=k-t+1}^{\infty} a_j\right)\epsilon_t \\ &= \sum_{t=1}^{k}\left(\sum_{j=0}^{t-1} a_j\epsilon_{t-j}\right) + \sum_{t=1}^{k}\left(\sum_{j=k-t+1}^{\infty} a_j\right)\epsilon_t. \end{aligned}$$

Thus

$$\begin{aligned} \tilde{S}_k - S_k &= -\sum_{t=1}^{k}\left(\sum_{j=t}^{\infty} a_j\epsilon_{t-j}\right) + \sum_{t=1}^{k}\left(\sum_{j=k-t+1}^{\infty} a_j\right)\epsilon_t \\ &= I + II \ (say). \end{aligned}$$

It suffices to prove

$$(A.1) \qquad n^{-\frac{1}{2}} \max_{1\le k\le n} |I| \xrightarrow{P} 0,$$

and

$$(A.2) \qquad n^{-\frac{1}{2}} \max_{1\le k\le n} |II| \xrightarrow{P} 0.$$

First we have for

$$
\begin{aligned}
& n^{-1} E \max_{1 \le k \le n} \left| \sum_{t=1}^{k} \sum_{j=t}^{\infty} a_j \epsilon_{t-j} \right|^2 \\
&= n^{-1} E \max_{1 \le k \le n} \left| \sum_{j=1}^{\infty} \sum_{t=1}^{j \wedge k} a_j \epsilon_{t-j} \right|^2 \\
&\le n^{-1} \left(\sum_{j=1}^{\infty} |a_j| \left\{ E \max_{1 \le k \le n} \left| \sum_{t=1}^{j \wedge k} \epsilon_{t-j} \right|^2 \right\}^{\frac{1}{2}} \right)^2 \\
(A.3) \qquad & \qquad \text{(by Minkowski's inequality)} \\
&\le n^{-1} \left(\sum_{j=1}^{\infty} |a_j| \sigma(j \wedge n)^{\frac{1}{2}} \right)^2 \\
& \qquad \text{(by (3) and Theorem 2 of [10])} \\
&= \left(\sum_{j=1}^{\infty} |a_j| \sigma((j \wedge n)/n)^{\frac{1}{2}} \right)^2 \\
& \qquad \text{(by the dominated convergence theorem)} \\
&= o(1).
\end{aligned}
$$

Hence (A.1) is proved by Markov inequality. To prove (A.2) write

$$II = II_{k1} + II_{k2},$$

where

$$II_{k1} = a_1 \epsilon_k + a_2(\epsilon_k + \epsilon_{k-1}) + \cdots + a_k(\epsilon_k + \cdots + \epsilon_1)$$

and

$$II_{k2} = (a_{k+1} + a_{k+2} + \cdots)(\epsilon_k + \cdots + \epsilon_1),$$

and let $\{p_n\}$ be a sequence of positive integers such that

$$(A.4) \qquad p_n \to \infty \text{ and } p_n/n \to 0 \text{ as } n \to \infty.$$

Then

$$(A.5)\quad \begin{aligned} &n^{-\frac{1}{2}} \max_{1\le k\le n} |II_{k2}| \\ &\le \left(\sum_{j=0}^{\infty}|a_j|\right) n^{-\frac{1}{2}} \max_{1\le k\le p_n} |\epsilon_1+\cdots+\epsilon_k| \\ &\quad + \left(\sum_{j>p_n}|a_j|\right) n^{-\frac{1}{2}} \max_{1\le k\le n} |\epsilon_1+\cdots+\epsilon_k| \\ &= III + IV \text{ (say)}. \end{aligned}$$

It follows from (3) and (A.4) that

$$\begin{aligned} &\left(\sum_{j=0}^{\infty}|a_j|\right)^2 n^{-1}E \max_{1\le k\le p_n} |\epsilon_1+\cdots+\epsilon_k|^2 \\ &\le \left(\sum_{j=0}^{\infty}|a_j|\right)^2 \sigma^2(p_n/n) = o(1) \end{aligned}$$

by Theorem 2 of Newman and Wright ([10]) and thus $III \xrightarrow{P} 0$ by Markov inequality. Similarly, by assumption $\sum_{j=0}^{\infty}|a_j| < \infty$ and Theorem 2 of Newman and Wright ([10])

$$\begin{aligned} &\left(\sum_{j>p_n}|a_j|\right)^2 n^{-1}E \max_{1\le k\le n} |\epsilon_1+\cdots+\epsilon_k|^2 \\ &\le \left(\sum_{j>p_n}|a_j|\right)^2 \sigma^2 = o(1) \end{aligned}$$

and thus $IV \xrightarrow{P} 0$ by Markov inequality. Hence, $n^{-\frac{1}{2}}\max_{1\le k\le n}|II_{k2}| \xrightarrow{P} 0$. It remains to show that $L_n = n^{-\frac{1}{2}}\max_{1\le k\le n}|II_{k1}| \xrightarrow{P} 0$. For each $m \ge 1$, define $II_{k1,m} = b_1\epsilon_k + b_2(\epsilon_k+\epsilon_{k-1}) + \cdots + b_k(\epsilon_k+\cdots+\epsilon_1)$, where $b_k = a_k$ for $k \le n$ and $b_k = 0$ otherwise and let $L_{n,m} = n^{-\frac{1}{2}}\max_{1\le k\le n}|II_{k1,m}|$. Then

$$(A.6)\qquad L_{n,m} \le n^{-\frac{1}{2}}(|a_1|+\cdots+|a_m|)(|\epsilon_1|+\cdots+|\epsilon_m|) \xrightarrow{P} 0$$

as $n \to \infty$ for each m, and

$$(A.7) \qquad |L_{n,m} - L_n| \le n^{-\frac{1}{2}} \max_{1 \le k \le n} \left| \sum_{i=1}^{k} (a_i - b_i)(\epsilon_k + \cdots + \epsilon_{k-i+1}) \right| .$$

Since

$$\left| \sum_{i=1}^{k} (a_i - b_i)(\epsilon_k + \cdots + \epsilon_{k-i+1}) \right| = \begin{cases} 0, & k \le m \\ \left| \sum_{i=m+1}^{k} a_i(\epsilon_k + \cdots + \epsilon_{k-i+1}) \right|, & \text{otherwise,} \end{cases}$$

the right-hand side of $(A.7)$

$$\begin{aligned}
&\le n^{-\frac{1}{2}} \max_{m<k\le n} \left(\sum_{i=m+1}^{k} |a_i||\epsilon_k + \cdots + \epsilon_{k-i+1}| \right) \\
&\le n^{-\frac{1}{2}} \max_{m<k\le n} \sum_{i=m+1}^{k} |a_i| \max_{m<i\le k} |\epsilon_k + \cdots + \epsilon_{k-i+1}| \\
(A.8) \qquad &\le n^{-\frac{1}{2}} \sum_{i>m} |a_i| \max_{m<k\le n} \max_{m<i\le k} (|\epsilon_1 + \cdots + \epsilon_k| + |\epsilon_1 + \cdots + \epsilon_{k-i}|) \\
&\le n^{-\frac{1}{2}} \sum_{i>m} |a_i| \Big(\max_{m<k\le n} |\epsilon_1 + \cdots + \epsilon_k| \\
&\quad + \max_{m<k\le n} \max_{m<i\le k} |\epsilon_1 + \cdots + \epsilon_{k-i}| \Big) \\
&\le n^{-\frac{1}{2}} \sum_{i>m} |a_i| \left(\max_{1\le j\le n} |\epsilon_1 + \cdots + \epsilon_j| + \max_{1\le j\le n} |\epsilon_1 + \cdots + \epsilon_j| \right) \\
&= 2n^{-\frac{1}{2}} \sum_{i>m} |a_i| \max_{1\le j\le n} |\epsilon_1 + \cdots + \epsilon_j|.
\end{aligned}$$

Therefore, by Theorem 2 of Newman and Wright ([10]) it follows from (A.6), (A.8) and Markov inequality that for any $\delta > 0$,

$$\begin{aligned}
&\lim_{m\to\infty} \lim_{n\to\infty} supP(|L_{n,m} - L_n| > \delta) \\
\le \; &\lim_{m\to\infty} 2^2\delta^2 \left(\sum_{j>m} |a_j| \right)^2 \lim_n sup\, n^{-1} E \max_{1\le j\le n} |\epsilon_1 + \cdots + \epsilon_j|^2
\end{aligned}$$

$$\begin{aligned} &\leq \sigma \lim_{m\to\infty} \delta^2 \cdot 2^2 \left(\sum_{j>m} |a_j| \right)^2 \quad ((3) \text{ and Theorem 2 of [10]} \\ (A.9) \qquad &= 0 \quad \left(\text{by assumption } \sum_{j=0}^{\infty} |a_j| < \infty \right). \end{aligned}$$

In view of (A.6) and (A.9) it follows from Theorem 4.2 of Billingsley ([1], p.25) that $L_n \xrightarrow{P} 0$ and thus (A.2) is proved. The proof of lemma now completes. □

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