Bull. Korean Math. Soc. **40** (2003), No. 4, pp. 699–701

THE DUAL OF A FORMULA OF VISKOV

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ABSTRACT. This minipaper offers a formula which is dual to that of Viskov [5]. While Viskov's can be thought of as a rising formula for Laguerre polynomials, ours is precisely the lowering one. Besides documenting the formula, which seems to be missing, we want to provide a (rather elementary) operator theory argument instead of making crude calculations. In other words, the annihilation and creation *operators* are confronted with lowering and rising *formulae*; they are often failed to be distinguished.

The Laguerre polynomials $L_n^{(\alpha)}$, $n = 0, 1, \ldots$, $\alpha > -1$, can be given as

$$L_n^{(\alpha)}(x) = \frac{1}{n!} x^{-\alpha} \mathrm{e}^x \frac{d^n}{dx^n}[x^{n+\alpha}\mathrm{e}^{-x}], \quad n = 1, 2, \ldots, \quad L_0^{(\alpha)} = 1.$$

The formula proven in [5] and generalizing that of [2] is

$$(1) \quad (n+1)L_{n+1}^{(\alpha)}(x)x = [-xD^2 - (\alpha + 1 - 2x)D + (\alpha + 1 - x)]L_n^{(\alpha)}(x),$$

for $n = 0, 1, \ldots$ with the shorthand notation $D \overset{\mathrm{df}}{=} \frac{\mathrm{d}}{\mathrm{d}x}$.

The dual formula

The formula is (with convention $L_{-1}^{(\alpha)} = 0$)

$$(2) \quad (n+\alpha)L_{n-1}^{(\alpha)}(x) = [-xD^2 - (\alpha+1)D + 1]L_n^{(\alpha)}(x), \quad n = 0, 1, \ldots.$$

To *prove* it consider the Hilbert space $\mathcal{L}^2([0, +\infty), x^\alpha \mathrm{e}^{-x}\mathrm{d}x)$ and denote by R the operator which appears in the right hand side of (1), that is more precisely

$$R \overset{df}{=} -xD^2 - (\alpha + 1 - 2x)D + (\alpha + 1 - x), \quad \mathcal{D}(R) \overset{df}{=} \mathbb{C}[X].$$

Received October 2, 2002.
2000 Mathematics Subject Classification: Primary 33C45; Secondary 47A05.
Key words and phrases: Laguerre polynomials, raising and lowering formulae, weighted shift operator.

If $\langle\cdot,-\rangle$ stands for the inner product in $\mathcal{L}^2([0,+\infty),x^\alpha e^{-x}dx)$, integration by parts gives us for $f,g\in\mathcal{D}(R)$

$$\langle xDf,g\rangle = fx^{\alpha+1}e^{-x}\bar{g}|_0^{+\infty} - \int_0^{+\infty}(f((\alpha+1)-x)x^\alpha e^{-x}\bar{g}+fx^{\alpha+1}e^{-x}\bar{g}')dx.$$

Since $\alpha>-1$, we infer that $\mathbb{C}[X]\subset\mathcal{D}((xD)^*)$ and

$$(xD)^* = -(\alpha+1)+x(1-D). \tag{3}$$

Now we can perform further calculations as follows (using (3) a couple of times and the fact that multiplication by x in the Hilbert space in question is a symmetric operator)

$$\begin{aligned}
&\langle Rf,g\rangle\\
=&\ -\langle (xD+\alpha+1)Df,g\rangle + 2\langle xDf,g\rangle + (\alpha+1)\langle f,g\rangle - \langle xf,g\rangle\\
=&\ -\langle Df,x(1-D)g\rangle + 2\langle xDf,g\rangle + (\alpha+1)\langle f,g\rangle - \langle xf,g\rangle\\
=&\ -\langle xDf,(1-D)g\rangle + 2\langle xDf,g\rangle + (\alpha+1)\langle f,g\rangle - \langle f,xg\rangle\\
=&\ -\langle f,(-(\alpha+1)+x(1-D))(1-D)g\rangle\\
&+\ 2\langle f,(-(\alpha+1)+x(1-D))g\rangle + (\alpha+1)\langle f,g\rangle - \langle f,xg\rangle\\
=&\ \langle f,[-xD^2-(\alpha+1)D+1]g\rangle.
\end{aligned}$$

Hence

$$R^* = -xD^2-(\alpha+1)D+1. \tag{4}$$

Set $l_n^{(\alpha)} \stackrel{df}{=} \sqrt{\frac{n!}{\Gamma(n+\alpha+1)}}L_n^{(\alpha)}$. Then (1) reads as

$$R\,l_n^{(\alpha)} = \sqrt{(n+1)(n+\alpha+1)}\,l_{n+1}^{(\alpha)}. \tag{5}$$

Due to orthogonality

$$\int_0^\infty L_m^{(\alpha)}(x)L_n^{(\alpha)}(x)x^\alpha e^{-x}dx = \frac{\Gamma(n+\alpha+1)}{n!}\delta_{m,n}, \quad m,n=0,1,\ldots,$$

$\{l_n^{(\alpha)}\}_{n=0}^\infty$ is an orthonormal basis in $\mathcal{L}^2([0,+\infty),x^\alpha e^{-x}dx)$. Thus the operator R act as an unbounded weighted shift with respect to $\{l_n^{(\alpha)}\}_{n=0}^\infty$ with the weights $\{\sqrt{(n+1)(n+\alpha+1)}_n\}_{n=0}^\infty$. Consequently, the adjoint R^* must necessarily be a backward weighted shift, that is

$$R^*\,l_n^{(\alpha)} = n\sqrt{n(n+\alpha)}\,l_{n-1}^{(\alpha)}.$$

Going back to the Laguerre polynomials $\{L_n^{(\alpha)}\}_{n=0}^\infty$ we get immediately by (4) the wanted formula (2). □

Back to Viskov's formula

For the reader to enjoy more the paper we propose a brisk *proof* of (1). It uses the well know formula

$$[1 - D]L_{n-1}^{(\alpha)} = -DL_n^{(\alpha)}$$

as well as the Laguerre differential equation

$$[xD^2 + (\alpha + 1 - x)D]L_n^{(\alpha)} = -nL_n^{(\alpha)}$$

and goes as follows

$$\begin{aligned} &[-xD^2 - (\alpha + 1 - 2x)D + (\alpha + 1 - x)]L_n^{(\alpha)} \\ &= [xD + (\alpha + 1 - x)][1 - D]L_n^{(\alpha)} = [xD + (\alpha + 1 - x)][-D]L_{n+1}^{(\alpha)} \\ &= (n + 1)L_{n+1}^{(\alpha)}. \end{aligned}$$

□

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