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A NOTE ON INVARIANT SUBSPACES FOR OPERATOR ALGEBRAS

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ABSTRACT. Various operator relationships are shown to be equivalent to the existence of an invariant subspace for an algebra of operators.

1. Introduction

Although many of the results and techniques found in this note apply in a Banach space setting, we shall focus our attention to that of a complex Hilbert space $\mathcal{H}$, and bounded linear operators on $\mathcal{H}$. The most celebrated unsolved problem in the field of (Hilbert space) operator theory is whether for every bounded operator one can find a proper closed linear subspace which is invariant for that operator. An apparently more difficult problem is the transitive algebra problem:

Given a weakly closed algebra of operators $\mathcal{A}$ on $\mathcal{H}$ for which there does not exist a proper (closed, linear) subspace of $\mathcal{H}$ invariant under all the operators of $\mathcal{A}$, must $\mathcal{A}$ be the algebra of all operators on $\mathcal{H}$?

An algebra of operators without common proper invariant subspace is said to be transitive. Burnside's classical linear algebra result supplies an affirmative answer to the preceding question when $\mathcal{H}$ is finite dimensional. In 1963 W. B. Arveson showed in [1] how Burnside's techniques could be modified so as to characterize transitivity in terms of densely defined unbounded operators commuting with (all of the members of) the algebra. His method proved quite useful in establishing results of the form.

If $\mathcal{A}$ is a transitive algebra containing —, then $\mathcal{A}$ is the ring of all operators.

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Here, of course, the name of the game is to fill in — with specific operator entities; e.g.

- a maximal abelian von Neumann algebra ([1]);
- the unilateral shift ([1]);
- a strictly cyclic operator ([3]);

and in conjunction with Lomonosov's result cited below

- a compact operator ([5] and [6]).

Invariant subspace investigations took a different tack after the appearance of V. Lomonosov's innovative use of compact operators. This result and several of its consequences will be presented in detail later in this note. We shall bring together several statements shown to be equivalent to the assertion of the existence of a proper invariant subspace for an operator algebra. Some, and perhaps most, of these are well known to operator theory practitioners, some are perhaps novel views of the former, and a few may be classified as new. Proofs will be provided for some of the middle group and all of the last.

All operator algebras herein encountered are assumed to be closed in the weak operator topology and all such operators are assumed to be bounded and linear. The term invariant subspace will be reserved for proper closed linear subspaces of the underlying Hilbert space.

2. Notation and preliminaries

Let $\mathcal{H}$ be a complex Hilbert space. $\mathcal{L}(\mathcal{H})$ is the algebra of all bounded linear operators on $\mathcal{H}$. If only one space is involved in a specific setting we will simply write $\mathcal{L}$ for $\mathcal{L}(\mathcal{H})$. $\mathcal{Q}$ is the set of quasinilpotent operators. For each positive integer k, $\mathcal{N}_k = \{T : T^k = 0\}$. A subspace is *hyperinvariant* for A if it is invariant for every operator in $(A)'$, the commutant of A. More generally, a subspace is *invariant* for a set of operators if it is invariant for each member of that set. The term *projection* will be reserved for a self adjoint idempotent operator. I is the identity operator. For an operator T and a set of operators $\mathcal{S}, A\mathcal{S} = \{AS : S \in \mathcal{S}\}$; *etc.*

RESULTS. We present collections of properties of an algebra equivalent to asserting the existence of a nontrivial invariant subspace for the algebra. For the sake of clarity of presentation we divide the collection into three parts. The first of these is related to V. Lomonosov's now classic result [4]; see also [2]:

If $\mathcal{B}$ is a transitive algebra and K is a non zero compact operator, then there is an operator $B \in \mathcal{B}$ such that $BK \notin \mathcal{Q}$.

These explicit groupings do not include the two most commonly employed restatements that are based on, effectively, displaying the invariant subspace:

The algebra $\mathcal{A}$ has an invariant subspace if and only if for some nozero vector $x, \mathcal{A}x$ is not dense in $\mathcal{H}$;

The algebra $\mathcal{A}$ has an invariant subspace if and only if for some projection $P \neq 0$ or $I, (I-P)\mathcal{A}P = 0$.

Both of these will be employed below on numerous occasions.

PROPOSITION 1. *Let $\mathcal{A}$ be a unital algebra of operators. The following are equivalent:*

a) *$\mathcal{A}$ has a non trivial invariant subspace.*

b) *There is a non zero compact operator K such that $\mathcal{A}K \subset \mathcal{Q}$.*

c) *There is a rank one operator K such that $\mathcal{A}K \subset \mathcal{Q}$.*

d) *There is a rank one operator K such that $\mathcal{A}K \subset \mathcal{N}_2$.*

Proof. The implication (b) $\Rightarrow$ (a) is a restatement of what is frequently referred to as Lomonosov's Lemma

Certainly (d) $\Rightarrow$(c) $\Rightarrow$ (b). Suppose (a) holds. We may then choose non zero vectors u and v such that $\mathcal{A}u \perp v$. Let $A \in \mathcal{A}$. Then

$$(A(u \otimes v))^2 = (Au \otimes v)^2 =< Au, v > u \otimes v = 0,$$

closing the circle of implications. □

Before establishing the second group of equivalences we exhibit some elementary properties and examples relating spaces of nilpotent operators and invariant subspaces. As mentioned above, the definition of the existence of a proper invariant subspace could be taken as follows: An algebra $\mathcal{A}$ has a proper invariant subspace if and only if there is a projection P other than 0 or I satisfying $(I-P)\mathcal{A}P = 0$. The equivalences to be shown shortly may be viewed as a loosening of the projection requirement.

LEMMA 2. *Let $\mathcal{M}$ be a linear space of operators that contains the identity operator, and suppose that $(MT)^2 = 0$ for every M in $\mathcal{M}$. Then $TMT = 0$ for every M in $\mathcal{M}$.*

Proof. Let $M \in \mathcal{M}$ and let ρ be a member of the resolvent set for M. Then $M - \rho I \in M$ so that

$$(M - \rho I)T(M - \rho I)T = 0.$$

Since $M - \rho I$ is invertible we have $T(M - \rho I)T = 0$. But the identity operator is in $\mathcal{M}$, so that $T^2 = 0$. This in turn shows that $TMT = 0$. □

REMARK. The condition

$$TMT = 0$$

for a linear space $\mathcal{M}$ does not automatically lead to a proper invariant subspace for $\mathcal{M}$. To see this, consider the following example: Let

$$T = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and let $\mathcal{M}$ consist of all three by three matrices with 0 in the lower left corner. Then $\mathcal{M}$ is precisely the set of matrices M for which $TMT = 0$. Since

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

the smallest algebra containing $\mathcal{M}$ is the full matrix ring; consequently $\mathcal{M}$ can have no proper invariant subspace.

We now show that multiplicative structure in this context does indeed lead to invariant subspaces.

LEMMA 3. *Suppose $T \neq 0$ and $\mathcal{M}$ is a unital linear space of operators for which $T\mathcal{M}T = 0$. Then every algebra contained in $\mathcal{M}$ has a proper invariant subspace.*

Proof. Let $\mathcal{A}$ be an algebra contained in $\mathcal{M}$, and let x be a vector for which $Tx \neq 0$. Then $\mathcal{A}Tx$ is invariant for $\mathcal{A}$. If $\mathcal{A}Tx = \{0\}$ then the one dimensional space $\mathbb{C}Tx$ is invariant for $\mathcal{A}$. Otherwise, $\mathcal{A}Tx \subset \ker T$, so that the closure of $\mathcal{A}Tx$ is a proper invariant subspace for $\mathcal{A}$. □

PROPOSITION 4. *Let $\mathcal{A}$ be a unital algebra of operators. The following are equivalent:*

a) *$\mathcal{A}$ has a non trivial invariant subspace.*

b) *There is a non zero operator T such that $\mathcal{A}T \subset \mathcal{N}_2$.*

c) *There is a non zero operator T such that $T\mathcal{A}T = \{0\}$.*

Proof. The preceding proposition shows that (a) implies (b) (indeed, T may even be taken to be a rank one operator), and the preceding two lemmas show that (b) $\Rightarrow$ (c) $\Rightarrow$ (a). □

The next result is separated from the previous chain of equivalences solely for ease of exposition. A bit of terminology is needed first:

DEFINITION. Let $\mathcal{A}$ be an algebra of operators on the Hilbert space $\mathcal{H}$. The finite ordered set $\mathcal{T} = \{T_1, \cdots, T_N\}$ of non zero operators is defined to be a *nil set* for $\mathcal{A}$ if

$$\{A_1, A_2, \cdots, A_N\} \subset \mathcal{A} \Rightarrow (A_1T_1)(A_2T_2)\cdots(A_NT_N) = 0.$$

The displayed material above will be paraphrased as $(\mathcal{A} : \mathcal{T}) = 0$. Note that a statement equivalent to the assertion of the existence of a proper invariant subspace for an algebra $\mathcal{A}$ is that there exists a projection $P \neq 0$ or I such that $A \in \mathcal{A} \Rightarrow (I - P)AP = 0$; that is to say $\{\mathcal{A} : \{I - P, P\}\} = 0$. In fact, we need not be so specific about the nil set:

PROPOSITION 5. *Suppose that $\mathcal{A}$ is a unital algebra with a nil set $\mathcal{T}$ as above. Then $\mathcal{A}$ has a proper invariant subspace.*

Proof. Let M be the smallest nonnegative integer such that $\{T_1, \cdots, T_{M+1}\}$ is a nil set for A. Since we may choose $A_1 = I \in \mathcal{A}$, we have, for all choices of $A_2, \cdots, A_{M+1}$ in $\mathcal{A}$

$$T_1(A_2T_2)(A_3T_3)\cdots(A_{M+1}T_{M+1}) = 0.$$

Let $\mathcal{B}$ be the linear span of all operators of the form

$$(A_2T_2)(A_3T_3)\cdots(A_{M+1}T_{M+1}); \qquad \{A_2, \cdots A_{M+1}\} \subset \mathcal{A}.$$

The minimality condition on M guarantees that there is a vector x for which $\mathcal{B}x \neq \{0\}$ (since the displayed material above involves only M members of $\mathcal{A}$).

Now, for $A, A_2, \cdots, A_{M+1} \in \mathcal{A}$, since $\mathcal{A}$ is an algebra,

$$(AA_2T_2)\cdots(A_{M+1}T_{M+1}) \in \mathcal{B}.$$

Thus $\mathcal{A}\mathcal{B}x \subset \mathcal{B}x$. However, $T_1\mathcal{B}x = \{0\}$, so

$$\{0\} \neq \mathcal{B}x \subset \ker T_1 \neq \mathcal{H};$$

showing that the closure of $\mathcal{B}x$ is a proper invariant subspace for $\mathcal{A}$. □

PROPOSITION 6. *The following are equivalent*

a) *$\mathcal{A}$ has a proper invariant subspace.*

b) *$\mathcal{A}$ has a nil set.*

c) *$\mathcal{A}$ has a nil set $\{T, T\}$.*

d) *$\mathcal{A}$ has a nil set $\{T, T\}$, where T is a rank one operator.*

Proof. We need only verify the implication (a) $\Rightarrow$ (d), and this was established in Proposition 1. □

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