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A NOTE ON THE SEVERITY OF RUIN IN THE RENEWAL MODEL WITH CLAIMS OF DOMINATED VARIATION

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ABSTRACT. This paper investigates the tail asymptotic behavior of the severity of ruin (the deficit at ruin) in the renewal model. Under the assumption that the tail probability of the claimsize is dominatedly varying, a uniform asymptotic formula for the tail probability of the deficit at ruin is obtained.

1. Model and main result

Throughout this paper, for any $0 \le a < b < \infty$ the integral symbol $\int_a^b$ is understood as $\int_{(a,b]}$ but $\int_a^\infty = \int_{(a,\infty)}$. For a given distribution function (d.f.) F with finite mean μ and support $[0,\infty)$, we denote its tail by $\overline{F}(x) = 1 - F(x)$ and its equilibrium d.f. by

$$F_e(x) = \frac{1}{\mu}\int_0^x \overline{F}(t)\mathrm{d}t, \quad x \ge 0. \tag{1.1}$$

The symbol F^{*n} represents the n-fold convolution of F with F^{*0} being degenerate at 0.

The following renewal risk model has received extensive attention in risk theory; see Embrechts et al. [2], Rolski et al. [9] and Asmussen [1] for reviews. In this model the successive claims, $Z_k, k \ge 1$, form a sequence of independent, identically distributed (i.i.d.) and non-negative random variables (r.v.'s) with common d.f. F and finite mean μ. Their occurrence times, $\sigma_k, k \ge 1$, comprise a renewal process $N(t) = \#\{k \ge 1; \sigma_k \in (0,t]\},\ t \ge 0$, i.e. that the inter-occurrence times $\theta_1 = \sigma_1, \theta_k = \sigma_k - \sigma_{k-1}, k \ge 2$, are i.i.d. non-negative r.v.'s. Suppose

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that the sequences $\{\theta_k, k \geq 1\}$ and $\{Z_k, k \geq 1\}$ are mutually independent. Let $c > 0$ be the gross premium rate. The loss process is then defined by

$$S(t) = \sum_{k=1}^{N(t)} Z_k - ct, \quad t \geq 0, \tag{1.2}$$

where $\sum_{k=1}^{0} Z_k = 0$ by convention. Denote by Z and θ a the generic r.v.'s of $\{Z_k, k \geq 1\}$ and $\{\theta_k, k \geq 1\}$ respectively. We assume that both Z and θ are not degenerate at 0. Throughout, the relative safety loading condition holds:

$$\rho = \frac{c\mathbb{E}\theta - \mu}{\mu} > 0. \tag{1.3}$$

Let $u \geq 0$ be the initial surplus of the insurance company. Then, the ruin time of the risk process (1.2) is $T_u = \inf\{t : S(t) > u\}$ and the deficit at ruin is $A_u = S(T_u) - u$. Here we define $\inf \phi = \infty$ as usual.

In risk theory one particularly interesting problem is to determine the distribution of A_u. Related work can be found in Gerber et al. [6], Picard [8], Schmidli [10], Willmot and Lin [13], among others. The cited references, however, only dealt with the problem in the compound Poisson model, i.e. the counting process $N(t)$ is a homogenous Poisson process.

In this short communication we first establish an asymptotic relationship for the tail probability of the deficit A_u. Since the quantity A_u can not be defined provided that the ruin does not occur (i.e. $T_u = \infty$), we have to consider the tail probability of A_u accompanied by the event $(T_u < \infty)$. In order to state our result, we need the notation below. Let F be a d.f. supported on $[0, \infty)$. We say that F has a dominated variation, denoted by $F \in \mathcal{D}$, if and only if

$$\limsup_{x \to \infty} \frac{\overline{F}(xy)}{\overline{F}(x)} < \infty$$

for any $0 < y < 1$ (or equivalently, for $y = 1/2$).

THEOREM 1.1. *In the renewal model with the relative safety loading condition* (1.3), *if the claimsize distribution $F \in \mathcal{D}$, then we have, uniformly for $x > 0$,*

$$\mathbb{P}\left(A_u > x, T_u < \infty\right) \sim \rho^{-1}\overline{F_e}(u + x) \qquad \text{as } u \to \infty. \tag{1.4}$$

REMARK. The uniformity of the asymptotic relation (1.4) is understood as

$$\lim_{u\to\infty}\sup_{x>0}\left|\frac{\mathbb{P}\left(A_u > x, T_u < \infty\right)}{\rho^{-1}\overline{F_e}(u+x)} - 1\right| = 0,$$

which is crucial for our purpose. For example, it allows us to use (1.4) in deriving asymptotic relationships for the moments of the deficit $A_u \mathbb{I}_{(T_u<\infty)}$ as $u \to \infty$. Further related studies can be found in Cheng *et al.* [2].

2. Preliminaries

Heavy-tailedness properties are often considered when one aims to establish some tail asymptotic relationships in extremal value theory; see Embrechts et al. [3] for a thorough review. Besides the class $\mathcal{D}$, another very important class of heavy-tailed distributions is the subexponential class, denoted by $\mathcal{S}$. By definition, a d.f. F supported on $[0,\infty)$ belongs to $\mathcal{S}$ if and only if

$$\lim_{x\to\infty}\frac{\overline{F^{*n}}(x)}{\overline{F}(x)} = n$$

for any $n \geq 2$ (or equivalently, for some $n \geq 2$). It is well-known that if $F \in \mathcal{D}$ with a finite mean then $F_e \in \mathcal{S}$; see Embrechts and Omey [5].

In the sequel we will need the following closure property of the class $\mathcal{S}$:

LEMMA 2.1 (Lemma A3.28 in Embrechts et al. [3]). *Let F_1 and F_2 be two d.f.'s supported on $[0,\infty)$. If there is some d.f. $F \in \mathcal{S}$ and some numbers $c_i \geq 0$ such that $\lim_{x\to\infty}\overline{F_i}(x)/\overline{F}(x) = c_i$ for $i = 1, 2$, then $\lim_{x\to\infty}\overline{F_1 * F_2}(x)/\overline{F}(x) = c_1 + c_2$.*

Best to our knowledge, the following result about the class $\mathcal{D}$ is new in the literature; closely related discussions can be found in Tang [11].

LEMMA 2.2. *Let F be a d.f. with support $[0,\infty)$ and finite mean, if $F \in \mathcal{D}$ then*

$$\lim_{y\nearrow 1}\limsup_{x\to\infty}\frac{\overline{F_e}(xy)}{\overline{F_e}(x)} = 1. \tag{2.1}$$

Proof. For $0 < y < 1$ arbitrarily fixed, we have that

$$\frac{\overline{F_e}(xy)}{\overline{F_e}(x)} = 1 + \frac{\int_{xy}^{x}\overline{F}(t)\mathrm{d}t}{\int_x^\infty \overline{F}(t)\mathrm{d}t},$$

and that

$$0 \leq \frac{\int_{xy}^{x} \overline{F}(t)\mathrm{d}t}{\int_{x}^{\infty} \overline{F}(t)\mathrm{d}t} \leq \frac{\overline{F}(xy)(1-y)x}{\int_{x}^{2x} \overline{F}(t)\mathrm{d}t} \leq \frac{\overline{F}(xy)(1-y)}{\overline{F}(2x)}.$$

Since $F \in \mathcal{D}$ implies that $\overline{F}(xy)/\overline{F}(2x)$ is uniformly bounded, it follows that

$$\lim_{y \nearrow 1} \limsup_{x \to \infty} \frac{\int_{xy}^{x} \overline{F}(t)\mathrm{d}t}{\int_{x}^{\infty} \overline{F}(t)\mathrm{d}t} = 0.$$

Hence (2.1) holds. □

Following Kalashnikov [7], now we introduce some important characteristics of the risk process (1.2):

(i) the ladder epochs: $\pi_1 = \inf\{t : S(t) > 0\}$, $\pi_n = \inf\{t : S(t) > S(\pi_{n-1})\}$, $n \geq 2$;

(ii) the ladder heights: $L = L_1 = S(\pi_1)$, $L_n = S(\pi_n) - S(\pi_{n-1})$, $n \geq 2$.

For notational convenience we define $\pi_0 = L_0 = 0$. As an important complementary situation, we define $\pi_n = L_n = \infty$ in case the ladder epoch π_{n-1} is well-defined but the set $\{t : S(t) > S(\pi_{n-1})\}$ is empty. Clearly, the conditional random variables

$$L_n \,|(\pi_{n-1} < \infty), \qquad n \geq 1,$$

comprise an i.i.d. sequence. Write H as the d.f. of the ladder L. Of course, H is supported on $(0, \infty)$. We remark that, under the safety loading condition (1.3), H is a defective d.f. with a deficit

$$q = 1 - \mathbb{P}(L < \infty) > 0; \tag{2.2}$$

see Veraverbeke [12] (pp. 28–31). In addition, Veraverbeke [12] (Theorem 1(C)) indicates that:

LEMMA 2.3. *In the renewal model with the relative safety loading condition* (1.3), *if the claimsize distribution* $F \in \mathcal{D}$, *then*

$$\overline{H}(x) = 1 - q - H(x) \sim \rho^{-1} q \overline{F_e}(x) \qquad \text{as } x \to \infty. \tag{2.3}$$

It is well-known that the ruin probability of the risk process (1.2), defined by $\psi(u) = \mathbb{P}(T_u < \infty)$, can be reduced to the following series:

$$\psi(u) = q \sum_{n=1}^{\infty} \overline{H^{*n}}(u); \tag{2.4}$$

see Kalashnikov [7]. We introduce an auxiliary function that

$$R(u) = 1 - \psi(u) = q \sum_{n=0}^{\infty} H^{*n}(u). \tag{2.5}$$

From (2.2) we see that R is a standard distribution on $[0, \infty)$ with $R\{0\} = q$. Embrechts and Veraverbeke [4] obtained that:

LEMMA 2.4. *In the renewal model with the safety loading condition* (1.3), *if the claimsize distribution* $F \in \mathcal{D}$, *then*

$$\psi(u) \sim \rho^{-1} \overline{F_e}(u) \qquad \text{as } u \to \infty. \tag{2.6}$$

3. Proof of the main result

First of all, we establish a general expression for the tail probability of the deficit A_u.

LEMMA 3.1. *In the renewal model with the relative safety loading condition* (1.3), *we have, for any* $x > 0$ *and any* $u > 0$,

$$\mathbb{P}(A_u > x, T_u < \infty) = \frac{1}{q} \int_{[0,u]} \overline{H}(u + x - t) R(\mathrm{d}t). \tag{3.1}$$

Proof. We write $\tau_u = \inf\{n : \pi_n = T_u\}$, which denotes the number of the ladders before ruin. Note that the ruin, if occurs, should be at the point of some ladder epoch. Therefore $\{T_u < \infty\} = \{\tau_u < \infty\}$ and

$$A_u = \sum_{k=1}^{\tau_u} L_k - u.$$

From this and (2.5) we have

$$\begin{aligned} & \mathbb{P}(A_u > x, T_u < \infty) \\ = \; & \sum_{n=1}^{\infty} \mathbb{P}\left(\sum_{k=1}^{n} L_k - u > x, \tau_u = n \right) \\ = \; & \sum_{n=1}^{\infty} \mathbb{P}\left(\infty > \sum_{k=1}^{n} L_k > u + x, \sum_{k=1}^{n-1} L_k \le u \right) \\ = \; & \sum_{n=1}^{\infty} \int_0^u \mathbb{P}(\infty > L_n > u + x - t) \, \mathbb{P}\left(\sum_{k=1}^{n-1} L_k \in \mathrm{d}t \right) \end{aligned}$$

$$= \overline{H}(u+x) + \sum_{n=2}^{\infty} \int_0^u \overline{H}(u+x-t)\, H^{*(n-1)}(\mathrm{d}t)$$
$$= \frac{1}{q} \int_{[0,u]} \overline{H}(u+x-t) R(\mathrm{d}t).$$

This ends the proof of Lemma 3.1. □

Now we are ready to prove Theorem 1.1.

Proof. From (3.1) we have

$$\begin{aligned} &\mathbb{P}(A_u > x, T_u < \infty) \\ &= \frac{1}{q} \left(\int_{[0,u+x]} - \int_u^{u+x} \right) \overline{H}(u+x-t) R(\mathrm{d}t) \\ &= \frac{1}{q} \Big(\overline{H * R}(u+x) - (1-q)\psi(u+x) \\ &\quad - \int_u^{u+x} \overline{H}(u+x-t) R(\mathrm{d}t) \Big) \\ &= \frac{1}{q} (I_1 - I_2 - I_3). \end{aligned} \tag{3.2}$$

By virtue of Lemma 2.1, simple combination of (2.3) and (2.6) yields that

$$I_1 \sim \overline{H}(u+x) + (1-q)\psi(u+x) \sim \rho^{-1} q \overline{F_e}(u+x) + I_2.$$

Now we deal with I_3. For any $0 < y < 1$, we subdivide I_3 into two parts as

$$\begin{aligned} I_3 &= \int_u^{y(u+x)} \overline{H}(u+x-t) R(\mathrm{d}t) + \int_{y(u+x)}^{u+x} \overline{H}(u+x-t) R(\mathrm{d}t) \\ &\le \overline{H}((1-y)(u+x))\, \psi(u) + (1-q)\left(\psi(y(u+x)) - \psi(u+x)\right) \\ &= I_{31} + (1-q) I_{32}. \end{aligned}$$

Recalling (2.3) and the condition $F \in \mathcal{D}$, we see that, for arbitrarily fixed $0 < y < 1$,

$$1 \le \inf_{u \ge 0} \frac{\overline{H}((1-y)(u))}{\overline{F_e}(u)} \le \sup_{u \ge 0} \frac{\overline{H}((1-y)(u))}{\overline{F_e}(u)} < \infty.$$

It follows that, for arbitrarily fixed $0 < y < 1$,

$$\limsup_{u \to \infty} \sup_{x > 0} \frac{I_{31}}{\overline{F_e}(u+x)} = 0.$$

As for I_{32}, from (2.6) and by virtue of Lemma 2.2, we obtain

$$\begin{aligned}\lim_{y\nearrow 1}\limsup_{u\to\infty}\sup_{x>0}\frac{I_{32}}{\psi(u+x)} &= \lim_{y\nearrow 1}\limsup_{u\to\infty}\sup_{x>0}\frac{\psi(y(u+x))}{\psi(u+x)}-1\\ &= \lim_{y\nearrow 1}\limsup_{u\to\infty}\frac{\overline{F_e}(yu)}{\overline{F_e}(u)}-1\\ &= 0.\end{aligned}$$

Hence $I_3 = o\left(\overline{F_e}(u+x)\right)$ uniformly for $x > 0$. Finally, substituting I_1 and I_3 into (3.2) we complete the proof of (1.4). □

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