Bull. Korean Math. Soc. **40** (2003), No. 4, pp. 613–631

SOME CURVATURE CONDITIONS OF n-DIMENSIONAL QR-SUBMANIFOLDS OF $(p-1)$ QR-DIMENSION IN A QUATERNIONIC PROJECTIVE SPACE $QP^{(n+p)/4}$

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Abstract. The purpose of this paper is to study n-dimensional QR-submanifolds of $(p-1)$ QR-dimension in a quaternionic projective space $QP^{(n+p)/4}$ and especially to determine such submanifolds under the curvature conditions appeared in (5.1) and (5.2).

1. Introduction

Let M be a connected real n-dimensional submanifold of real codimension p of a quaternionic Kähler manifold $\overline{M}$ with quaternionic Kähler structure $\{F, G, H\}$. If there exists an r-dimensional normal distribution ν of the normal bundle $TM^{\perp}$ such that

$$(1.1)\qquad \begin{cases} F\nu_x \subset \nu_x, \ G\nu_x \subset \nu_x, \ H\nu_x \subset \nu_x, \\ F\nu_x^{\perp} \subset T_xM, \ G\nu_x^{\perp} \subset T_xM, \ H\nu_x^{\perp} \subset T_xM \end{cases}$$

at each point x in M, then M is called a *QR-submanifold of r QR-dimension*, where $\nu^{\perp}$ denotes the complementary orthogonal distribution to ν in $TM^{\perp}$ (cf. [1, 8, 9]). Real hypersurfaces, which are typical examples of QR-submanifold with $r = 0$, have been investigated by many authors (cf. [10, 11, 14, 15, 17, 18]) in connection with the shape operator and the induced almost contact 3-structure (for definition, see [3, 5, 6, 7]). Recently, in their paper [8, 9], Kwon and Pak have studied QR-submanifolds of $(p-1)$ QR-dimension isometrically immersed

Received March 31, 2003.
2000 Mathematics Subject Classification: 53C40, 53C25.
Key words and phrases: quaternionic projective space, QR-submanifold, QR-dimension, shape operator, locally symmetric.
*This work was supported by ABRL grant Proj. No. R14-2002- 003-01000-0 from KOSEF.

in a quaternionic projective space $QP^{(n+p)/4}$ and proved the following theorem as quaternionic analogies to theorems given in [12], which are also natural extensions of theorems proved in [14] to the case of QR-submanifolds.

THEOREM K-P. *Let M be an n-dimensional QR-submanifold of $(p-1)$ QR-dimension isometrically immersed in a quaternionic projective space $QP^{(n+p)/4}$ and let the normal vector field N_1 be parallel with respect to the normal connection. If*

$$A_1\phi = \phi A_1, \quad A_1\psi = \psi A_1, \quad A_1\theta = \theta A_1$$

on M, then $\pi^{-1}(M)$ is locally a product of $M_1 \times M_2$ where M_1 and M_2 belong to some $(4n_1+3)$- and $(4n_2+3)$-dimensional spheres and A_1 denotes the shape operator corresponding to N_1 (π is the Hopf fibration $S^{n+p+3}(1) \to QP^{(n+p)/4}$).

On the other hand, when M is a real hypersurface of $QP^{(n+p)/4}$, if $\pi^{-1}(M)$ is (1) an Einstein space or (2) a locally symmetric space, M has parallel second fundamental form (cf. [10, 14, 17, 19]). Projecting the quantities on $\pi^{-1}(M)$ onto M in $QP^{(n+p)/4}$, we can consider QR-submanifolds of $(p-1)$ QR-dimension with the conditions corresponding to (1) or (2). In this paper we shall study such QR-submanifolds isometrically immersed in $QP^{(n+p)/4}$ and obtain the theorems stated in the last Section 6 as quaternionic analogies to theorems given in [16, 20] by using Theorem K-P.

2. Preliminaries

Let $\overline{M}$ be a real $(n+p)$-dimensional quaternionic Kähler manifold. Then, by definition, there is a 3-dimensional vector bundle V consisting with tensor fields of type (1,1) over $\overline{M}$ satisfying the following conditions (a), (b) and (c) :

(a) In any coordinate neighborhood $\overline{U}$, there is a local basis $\{F, G, H\}$ of V such that

$$\begin{cases} F^2 = -I, \ G^2 = -I, \ H^2 = -I, \\ FG = -GF = H, \ GH = -HG = F, \ HF = -FH = G. \end{cases} \tag{2.1}$$

(b) There is a Riemannian metric g which is hermite with respect to all of F, G and H.

(c) For the Riemannian connection $\overline{\nabla}$ with respect to g

$$\begin{pmatrix} \overline{\nabla}F \\ \overline{\nabla}G \\ \overline{\nabla}H \end{pmatrix} = \begin{pmatrix} 0 & r & -q \\ -r & 0 & p \\ q & -p & 0 \end{pmatrix} \begin{pmatrix} F \\ G \\ H \end{pmatrix} \tag{2.2}$$

where p, q and r are local 1-forms defined in $\overline{\mathcal{U}}$. Such a local basis $\{F, G, H\}$ is called a *canonical local basis* of the bundle V in $\overline{\mathcal{U}}$ (cf. [3, 4, 20]).

For canonical local bases $\{F, G, H\}$ and $\{'F, 'G, 'H\}$ of V in coordinate neighborhoods $\overline{\mathcal{U}}$ and $'\overline{\mathcal{U}}$, it follows that in $\overline{\mathcal{U}} \cap {}'\overline{\mathcal{U}}$

$$\begin{pmatrix} 'F \\ 'G \\ 'H \end{pmatrix} = (s_{xy}) \begin{pmatrix} F \\ G \\ H \end{pmatrix} \qquad (x,\ y = 1,\ 2,\ 3)$$

where s_{xy} are local differentiable functions with $(s_{xy}) \in SO(3)$ as a consequence of (2.1). As is well known (cf. [4]), every quaternionic Kähler manifold is orientable.

Now let M be an n-dimensional QR-submanifold of $(p-1)$ QR-dimension isometrically immersed in $\overline{M}$. Then by definition there is a unit normal vector field N such that $\nu_x^{\perp} = \mathrm{Span}\{N\}$ at each point x in M. We set

$$U = -FN, \quad V = -GN, \quad W = -HN. \tag{2.3}$$

Denoting by $\mathcal{D}_x$ the maximal quaternionic invariant subspace $T_xM \cap FT_xM \cap GT_xM \cap HT_xM$ of T_xM, we have $\mathcal{D}_x^{\perp} = \mathrm{Span}\{U, V, W\}$, where $\mathcal{D}_x^{\perp}$ means the complementary orthogonal subspace to $\mathcal{D}_x$ in T_xM (cf. [1, 8, 9]). Thus we have

$$T_xM = \mathcal{D}_x \oplus \mathrm{Span}\{U, V, W\}, \quad {}^{\forall}x \in M,$$

which together with (2.1) and (2.3) implies

$$FT_xM,\ GT_xM,\ HT_xM \subset T_xM \oplus \mathrm{Span}\{N\}.$$

Therefore, for any tangent vector field X and for a local orthonormal basis $\{N_\alpha\}_{\alpha=1,\ldots,p}$ $(N_1 := N)$ of normal vectors to M, we have

$$\begin{aligned} &FX = \phi X + u(X)N, \quad GX = \psi X + v(X)N, \\ &HX = \theta X + w(X)N, \end{aligned} \tag{2.4}$$

$$(2.5)\qquad \begin{aligned} &FN_\alpha = -U_\alpha + P_1 N_\alpha, \quad GN_\alpha = -V_\alpha + P_2 N_\alpha, \\ &HN_\alpha = -W_\alpha + P_3 N_\alpha \end{aligned}$$

($\alpha = 1, \ldots, p$). Then it is easily seen that $\{\phi, \psi, \theta\}$ and $\{P_1, P_2, P_3\}$ are skew-symmetric endomorphisms acting on T_xM and $T_xM^\perp$, respectively. Moreover, the hermitian property of $\{F, G, H\}$ implies

$$(2.6)\qquad \begin{aligned} &g(X, \phi U_\alpha) = -u(X) g(N_1, P_1 N_\alpha), \\ &g(X, \psi V_\alpha) = -v(X) g(N_1, P_2 N_\alpha), \ \alpha = 1, \ldots, p, \\ &g(X, \theta W_\alpha) = -w(X) g(N_1, P_3 N_\alpha), \end{aligned}$$

$$(2.7)\qquad \begin{aligned} &g(U_\alpha, U_\beta) = \delta_{\alpha\beta} - g(P_1 N_\alpha, P_1 N_\beta), \\ &g(V_\alpha, V_\beta) = \delta_{\alpha\beta} - g(P_2 N_\alpha, P_2 N_\beta), \ \alpha, \beta = 1, \ldots, p, \\ &g(W_\alpha, W_\beta) = \delta_{\alpha\beta} - g(P_3 N_\alpha, P_3 N_\beta). \end{aligned}$$

Also, from the hermitian properties $g(FX, N_\alpha) = -g(X, FN_\alpha)$, $g(GX, N_\alpha) = -g(X, GN_\alpha)$ and $g(HX, N_\alpha) = -g(X, HN_\alpha)$, it follows that

$$g(X, U_\alpha) = u(X)\delta_{1\alpha}, \quad g(X, V_\alpha) = v(X)\delta_{1\alpha}, \quad g(X, W_\alpha) = w(X)\delta_{1\alpha}$$

and hence

$$(2.8)\qquad \begin{aligned} &g(U_1, X) = u(X), \ g(V_1, X) = v(X), \ g(W_1, X) = w(X), \\ &U_\alpha = 0, \ V_\alpha = 0, \ W_\alpha = 0, \ \alpha = 2, \ldots, p. \end{aligned}$$

On the other hand, comparing (2.3) and (2.5) with $\alpha = 1$, we have $U_1 = U$, $V_1 = V$, $W_1 = W$, which together with (2.3) and (2.8) implies

$$(2.9)\qquad \begin{aligned} &g(U, X) = u(X), \quad g(V, X) = v(X) \quad g(W, X) = w(X), \\ &u(U) = 1, \quad v(V) = 1, \quad w(W) = 1. \end{aligned}$$

In the sequel we shall use the notations U, V, W instead of U_1, V_1, W_1.

Next, applying F to the first equation of (2.4) and using (2.5), (2.8) and (2.9), we have

$$\phi^2 X = -X + u(X)U, \quad u(X) P_1 N = -u(\phi X) N.$$

Similarly we have

$$\phi^2X = -X + u(X)U, \quad \psi^2X = -X + v(X)V, \quad \theta^2X = -X + w(X)W, \tag{2.10}$$

$$u(X)P_1N = -\,u(\phi X)N, \quad v(X)P_2N = -v(\psi X)N, \quad w(X)P_3N = -w(\theta X)N, \tag{2.11}$$

from which, taking account of the skew-symmetry of P_1, P_2 and P_3 and using (2.6), we also have

$$\begin{aligned} &u(\phi X) = 0, \quad v(\psi X) = 0, \quad w(\theta X) = 0,\\ &\phi U = 0, \quad \psi V = 0, \quad \theta W = 0,\\ &P_1N = 0, \quad P_2N = 0, \quad P_3N = 0. \end{aligned} \tag{2.12}$$

So (2.5) can be rewritten in the form

$$\begin{aligned} &FN = -U, \quad GN = -V, \quad HN = -W,\\ &FN_\alpha = P_1N_\alpha, \quad GN_\alpha = P_2N_\alpha, \quad HN_\alpha = P_3N_\alpha \end{aligned} \tag{2.13}$$

$(\alpha = 2, \ldots, p)$. Applying G and H to the first equation of (2.4) and using (2.1), (2.4) and (2.13), we have

$$\begin{aligned} \theta X + w(X)N &= -\psi(\phi X) - v(\phi X)N + u(X)V,\\ \psi X + v(X)N &= \theta(\phi X) + w(\phi X)N - u(X)W, \end{aligned}$$

and consequently

$$\begin{aligned} &\psi(\phi X) = -\theta X + u(X)V, \quad v(\phi X) = -w(X),\\ &\theta(\phi X) = \psi X + u(X)W, \quad w(\phi X) = v(X). \end{aligned} \tag{2.14}$$

Similarly the other equations of (2.4) yield

$$\begin{aligned} &\phi(\psi X) = \theta X + v(X)U, \quad u(\psi X) = w(X),\\ &\theta(\psi X) = -\phi X + v(X)W, \quad w(\psi X) = -u(X), \end{aligned} \tag{2.15}$$

$$\begin{aligned} &\phi(\theta X) = -\psi X + w(X)U, \quad u(\theta X) = -v(X),\\ &\psi(\theta X) = \phi X + w(X)V, \quad v(\theta X) = u(X). \end{aligned} \tag{2.16}$$

From the first three equations of (2.13), we also have

$$
\begin{aligned}
&\psi U = -W, \quad v(U) = 0, \quad \theta U = V, \quad w(U) = 0, \\
&\phi V = W, \quad u(V) = 0, \quad \theta V = -U, \quad w(V) = 0, \\
&\phi W = -V, \quad u(W) = 0, \quad \psi W = U, \quad v(W) = 0.
\end{aligned} \tag{2.17}
$$

The equations (2.8)-(2.10), (2.12) and (2.14)-(2.17) tell us that M admits the so-called almost contact 3-structure and consequently $n = 4m + 3$ for some integer m (cf. [6]).

Now let ∇ be the Levi-Civita connection on M and let $\nabla^{\perp}$ the normal connection induced from $\overline{\nabla}$ in the normal bundle of M. Then Gauss and Weingarten formulae are given by

$$
\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.18}
$$

$$
\overline{\nabla}_X N_\alpha = -A_\alpha X + \nabla^{\perp}_X N_\alpha, \quad \alpha = 1, \dots, p \tag{2.19}
$$

for X, Y tangent to M. Here h denotes the second fundamental form and A_α the shape operator corresponding to N_α. They are related by $h(X,Y) = \sum_{\alpha=1}^{p} g(A_\alpha X, Y) N_\alpha$. Furthermore, put

$$
\nabla^{\perp}_X N_\alpha = \sum_{\beta=1}^{p} s_{\alpha\beta}(X) N_\beta, \tag{2.20}
$$

where $(s_{\alpha\beta})$ is the skew-symmetric matrix of connection forms of $\nabla^{\perp}$.

Differentiating the first equation of (2.4) covariantly and using (2.2), (2.4), (2.5), (2.8), (2.18) and (2.19), we have

$$
\begin{aligned}
(\nabla_Y \phi)X &= r(Y)\psi X - q(Y)\theta X + u(X)A_1 Y - g(A_1 Y, X)U, \\
(\nabla_Y u)X &= r(Y)v(X) - q(Y)w(X) + g(\phi A_1 Y, X).
\end{aligned} \tag{2.21}
$$

From the other equations of (2.4) we also have

$$
\begin{aligned}
(\nabla_Y \psi)X &= -r(Y)\phi X + p(Y)\theta X + v(X)A_1 Y - g(A_1 Y, X)V, \\
(\nabla_Y v)X &= -r(Y)u(X) + p(Y)w(X) + g(\psi A_1 Y, X),
\end{aligned} \tag{2.22}
$$

$$
\begin{aligned}
(\nabla_Y \theta)X &= q(Y)\phi X - p(Y)\psi X + w(X)A_1 Y - g(A_1 Y, X)W, \\
(\nabla_Y w)X &= q(Y)u(X) - p(Y)v(X) + g(\theta A_1 Y, X).
\end{aligned} \tag{2.23}
$$

Next, differentiating the first equation of (2.13) covariantly and comparing the tangential and normal parts, we have

$$\begin{aligned} &\nabla_Y U = r(Y)V - q(Y)W + \phi A_1 Y, \\ (2.24)\qquad &g(A_\alpha U, Y) = -\sum_{\beta=2}^{p} s_{1\beta}(Y) P_{1\beta\alpha}, \quad \alpha = 2, \dots, p. \end{aligned}$$

From the other equations of (2.13), we have similarly

$$\begin{aligned} &\nabla_Y V = -r(Y)U + p(Y)W + \psi A_1 Y, \\ (2.25)\qquad &g(A_\alpha V, Y) = -\sum_{\beta=2}^{p} s_{1\beta}(Y) P_{2\beta\alpha}, \quad \alpha = 2, \dots, p, \end{aligned}$$

$$\begin{aligned} &\nabla_Y W = q(Y)U - p(Y)V + \theta A_1 Y, \\ (2.26)\qquad &g(A_\alpha W, Y) = -\sum_{\beta=2}^{p} s_{1\beta}(Y) P_{3\beta\alpha}, \quad \alpha = 2, \dots, p. \end{aligned}$$

Finally the equation of Gauss is given as follow (cf. [2]) :
(2.27)

$$\begin{aligned} g(\overline{R}(X,Y)Z,W) = g(R(X,Y)Z,W) + \sum_{\alpha} \{ & g(A_\alpha X, Z) g(A_\alpha Y, W) \\ & - g(A_\alpha Y, Z) g(A_\alpha X, W) \}, \end{aligned}$$

for X, Y, Z tangent to M, where $\overline{R}$ and R denote the Riemannian curvature tensor of $\overline{M}$ and M, respectively.

In the rest of this paper *we assume that the distinguished normal vector field $N_1 := N$ is parallel with respect to the normal connection $\nabla^\perp$*. Then it follows from (2.20) that $s_{1\beta} = 0$ and consequently (2.24)-(2.26) imply

$$(2.28)\qquad A_\alpha U = 0, \ A_\alpha V = 0, \ A_\alpha W = 0, \ \alpha = 2, \dots, p.$$

On the other hand, since the curvature tensor $\overline{R}$ of $QP^{(n+p)/4}$ is of the form

$$\begin{aligned} \overline{R}(\overline{X},\overline{Y})\overline{Z} = {} & g(\overline{Y},\overline{Z})\overline{X} - g(\overline{X},\overline{Z})\overline{Y} \\ & + g(F\overline{Y},\overline{Z})F\overline{X} - g(F\overline{X},\overline{Z})F\overline{Y} - 2g(F\overline{X},\overline{Y})F\overline{Z} \\ & + g(G\overline{Y},\overline{Z})G\overline{X} - g(G\overline{X},\overline{Z})G\overline{Y} - 2g(G\overline{X},\overline{Y})G\overline{Z} \\ & + g(H\overline{Y},\overline{Z})H\overline{X} - g(H\overline{X},\overline{Z})H\overline{Y} - 2g(H\overline{X},\overline{Y})H\overline{Z} \end{aligned}$$

for $\overline{X}, \overline{Y}, \overline{Z}$ tangent to $QP^{(n+p)/4}$. (2.27) reduces to

$$
\begin{aligned}
R(X,Y)Z = &\, g(Y,Z)X - g(X,Z)Y \\
&+ g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z \\
&+ g(\psi Y,Z)\psi X - g(\psi X,Z)\psi Y - 2g(\psi X,Y)\psi Z \\
&+ g(\theta Y,Z)\theta X - g(\theta X,Z)\theta Y - 2g(\theta X,Y)\theta Z \\
&+ \sum_\alpha \{g(A_\alpha Y,Z)A_\alpha X - g(A_\alpha X,Z)A_\alpha Y\}.
\end{aligned} \tag{2.29}
$$

3. Fibrations and immersions

From now on n-dimensional QR-submanifolds of $(p-1)$ QR- dimension isometrically immersed in $QP^{(n+p)/4}$ only will be considered. Moreover we shall use the assumption and the notations as in Section 2.

Let $S^{n+p+3}(a)$ be the hypersphere of radius $a(>0)$ in $Q^{(n+p+4)/4}$ the quaternionic space of quaternionic dimension $(n+p+4)/4$, which is identified with the Euclidean $(n+p+4)$-space $\mathbb{R}^{n+p+4}$. The unit sphere $S^{n+p+3}(1)$ will be briefly denoted by S^{n+p+3}. Let $\widetilde{\pi} : S^{n+p+3} \to QP^{(n+p)/4}$ be the natural projection of S^{n+p+3} onto $QP^{(n+p)/4}$ defined by the Hopf-fibration $S^3 \to S^{n+p+3} \to QP^{(n+p)/4}$. As is well known (cf. [3, 5, 22]), S^{n+p+3} admits a Sasakian 3-structure $\{\widetilde{\xi}, \widetilde{\eta}, \widetilde{\zeta}\}$ which are mutually orthogonal unit Killing vector fields. Thus it follows that

$$
\widetilde{\nabla}_{\widetilde{\xi}}\widetilde{\xi} = 0, \quad \widetilde{\nabla}_{\widetilde{\eta}}\widetilde{\eta} = 0, \quad \widetilde{\nabla}_{\widetilde{\zeta}}\widetilde{\zeta} = 0, \tag{3.1}
$$

$$
\widetilde{\nabla}_{\widetilde{\zeta}}\widetilde{\eta} = -\widetilde{\nabla}_{\widetilde{\eta}}\widetilde{\zeta} = \widetilde{\xi}, \quad \widetilde{\nabla}_{\widetilde{\xi}}\widetilde{\zeta} = -\widetilde{\nabla}_{\widetilde{\zeta}}\widetilde{\xi} = \widetilde{\eta}, \quad \widetilde{\nabla}_{\widetilde{\eta}}\widetilde{\xi} = -\widetilde{\nabla}_{\widetilde{\xi}}\widetilde{\eta} = \widetilde{\zeta}, \tag{3.2}
$$

where $\widetilde{\nabla}$ denotes the Riemannian connection with respect to the canonical metric $\widetilde{g}$ on S^{n+p+3} (cf. [3, 5, 6, 7, 14, 16, 19]). Moreover each fibre $\widetilde{\pi}^{-1}(x)$ of x in $QP^{(n+p)/4}$ is a maximal integral submanifold of the distribution spanned by $\widetilde{\xi}, \widetilde{\eta}$ and $\widetilde{\zeta}$. Thus the base space $QP^{(n+p)/4}$ admits the induced quaternionic Kähler structure of constant Q-sectional curvature 4 (cf. [3, 5]). Especially we have a fibration $\pi : \pi^{-1}(M) \to M$ which is compatible with the Hopf-fibration $\widetilde{\pi}$. More precisely speaking $\pi : \pi^{-1}(M) \to M$ is a fibration with totally geodesic fibers such that the following diagram is commutative :

$$\begin{array}{ccc} \pi^{-1}(M) & \xrightarrow{i'} & S^{n+p+3} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\tilde{\pi}} \\ M & \xrightarrow{i} & QP^{(n+p)/4} \end{array}$$

where $i' : \pi^{-1}(M) \to S^{n+p+3}$ and $i : M \to QP^{(n+p)/4}$ are isometric immersions.

Now, let ξ, η and ζ be the unit vector fields tangent to the fibers of $\pi^{-1}(M)$ such that $i'_*\xi = \tilde{\xi}$, $i'_*\eta = \tilde{\eta}$ and $i'_*\zeta = \tilde{\zeta}$. (In what follows we shall again delete the i' and i'_* in our notation.) Furthermore we denote by X^* the horizontal lift of a vector field X tangent to M. Then the horizontal lifts N^*_α $(\alpha = 1, \ldots, p)$ of the normal vectors N_α to M form an orthonormal basis of normal vectors to $\pi^{-1}(M)$ in S^{n+p+3}. Let A'_α and $s'_{\alpha\beta}$ be the corresponding shape operators and normal connection forms, respectively. Then, as shown in [8, 9, 13, 15, 19, 21], the fundamental equations for the submersion π are given by

$$\begin{aligned} {}'\nabla_{X^*}Y^* = (\nabla_X Y)^* &+ g'((\phi X)^*, Y^*)\xi \\ &+ g'((\psi X)^*, Y^*)\eta + g'((\theta X)^*, Y^*)\zeta, \end{aligned} \tag{3.3}$$

$$\begin{aligned} [X^*, Y^*] = [X, Y]^* &+ 2g'((\phi X)^*, Y^*)\xi \\ &+ 2g'((\psi X)^*, Y^*)\eta + 2g'((\theta X)^*, Y^*)\zeta, \end{aligned} \tag{3.4}$$

$$\begin{aligned} {}'\nabla_{X^*}\xi = {}'\nabla_\xi X^* = -(\phi X)^*, \quad {}'\nabla_{X^*}\eta = {}'\nabla_\eta X^* = -(\psi X)^*, \\ {}'\nabla_{X^*}\zeta = {}'\nabla_\zeta X^* = -(\theta X)^*, \end{aligned} \tag{3.5}$$

$$[X^*, \xi] = 0, \quad [X^*, \eta] = 0, \quad [X^*, \zeta] = 0 \tag{3.6}$$

where g' denotes the Riemannian metric of $\pi^{-1}(M)$ induced from $\tilde{g}$ that of S^{n+p+3} and ${}'\nabla$ the Levi-Civita connection with respect to g'. The similar equations are valid for the submersion $\tilde{\pi}$ by replacing ϕ, ψ, θ (resp. ξ, η, ζ) with F, G, H (resp. $\tilde{\xi}$, $\tilde{\eta}$, $\tilde{\zeta}$) respectively. We denote by $'\nabla^{\perp}$ the normal connection of $\pi^{-1}(M)$ induced from $\widetilde{\nabla}$. Since the diagram is commutative, $\widetilde{\nabla}_{X^*}N^*_\alpha$ implies

$$\begin{aligned}&'\nabla^{\perp}_{X^*}N^*_\alpha - A'_\alpha X^*\\ &= (\overline{\nabla}_X N_\alpha)^* + \widetilde{g}((FX)^*, N^*_\alpha)\widetilde{\xi} + \widetilde{g}((GX)^*, N^*_\alpha)\widetilde{\eta} + \widetilde{g}((HX)^*, N^*_\alpha)\widetilde{\zeta}\\ &= -(A_\alpha X)^* + g(U_\alpha, X)^*\xi + g(V_\alpha, X)^*\eta + g(W_\alpha, X)^*\zeta + (\nabla^{\perp}_X N_\alpha)^*\end{aligned}$$

because of (2.5), (2.19) and (3.3), from which, comparing the tangential part, we have

$$A'_\alpha X^* = (A_\alpha X)^* - g(U_\alpha, X)^*\xi - g(V_\alpha, X)^*\eta - g(W_\alpha, X)^*\zeta. \tag{3.7}$$

Next, calculating $\widetilde{\nabla}_\xi N^*_\alpha$ and using (2.5), (2.19) and (3.5), we have

$$'\nabla^{\perp}_\xi N^*_\alpha - A'_\alpha\xi = -(FN_\alpha)^* = U^*_\alpha - (P_1 N_\alpha)^*,$$

which yields

$$A'_\alpha\xi = -U^*_\alpha$$

and similarly

$$A'_\alpha\xi = -U^*_\alpha,\ A'_\alpha\eta = -V^*_\alpha,\ A'_\alpha\zeta = -W^*_\alpha. \tag{3.8}$$

Hence (3.7) and (3.8) with $\alpha = 1$ imply

$$A'_1 X^* = (A_1 X)^* - g(U, X)^*\xi - g(V, X)^*\eta - g(W, X)^*\zeta, \tag{3.9}$$

$$A'_1\xi = -U^*,\ A'_1\eta = -V^*,\ A'_1\zeta = -W^*. \tag{3.10}$$

4. Co-Gauss equations for the submersion $\pi : \pi^{-1}(M) \to M$

In this section we derive the co-Gauss and co-Codazzi equations of the submersion $\pi : \pi^{-1}(M) \to M$ for later use.

Differentiating (3.3) with $Y = U$ covariantly along $\pi^{-1}(M)$ and using (2.17) and (3.3)-(3.4), we have

$$\begin{aligned}&'\nabla_{Y^*}{}'\nabla_{X^*}U^*\\ &= (\nabla_Y\nabla_X U)^* + \{v(X)\theta Y - w(X)\psi Y\}^* + g(\phi Y, \nabla_X U)^*\xi\\ &\quad + \{g(\psi Y, \nabla_X U) + g(\nabla_Y X, W) + g(X, \nabla_Y W)\}^*\eta\\ &\quad + \{g(\theta Y, \nabla_X U) - g(\nabla_Y X, V) - g(X, \nabla_Y V)\}^*\zeta.\end{aligned} \tag{4.1}$$

Similarly (3.3) with $Y = V$ and (3.3) with $Y = W$ give

$$
\begin{aligned}
&'\nabla_{Y^*}{}'\nabla_{X^*}V^* \\
&= (\nabla_Y\nabla_X V)^* + \{w(X)\phi Y - u(X)\theta Y\}^* + \{g(\phi Y, \nabla_X V) \\
&\quad - g(\nabla_Y X, W) - g(X, \nabla_Y W)\}^*\xi + g(\psi Y, \nabla_X V)^*\eta \\
&\quad + \{g(\theta Y, \nabla_X V) + g(\nabla_Y X, U) + g(X, \nabla_Y U)\}^*\zeta,
\end{aligned} \tag{4.2}
$$

$$
\begin{aligned}
&'\nabla_{Y^*}{}'\nabla_{X^*}W^* \\
&= (\nabla_Y\nabla_X W)^* - \{v(X)\phi Y - u(X)\psi Y\}^* \\
&\quad + \{g(\phi Y, \nabla_X W) + g(\nabla_Y X, V) + g(X, \nabla_Y V)\}^*\xi \\
&\quad + \{g(\psi Y, \nabla_X W) - g(\nabla_Y X, U) - g(X, \nabla_Y U)\}^*\eta \\
&\quad + g(\theta Y, \nabla_X W)^*\zeta,
\end{aligned} \tag{4.3}
$$

respectively. On the other hand it follows from (2.12), (2.17), (3.3) and (3.4) that

$$
\begin{aligned}
'\nabla_{[Y^*,X^*]}U^* &= (\nabla_{[Y,X]}U)^* + 2g(\psi Y, X)^*W^* - 2g(\theta Y, X)^*V^* \\
&\quad + g([Y,X], W)^*\eta - g([Y,X], V)^*\zeta,
\end{aligned} \tag{4.4}
$$

$$
\begin{aligned}
'\nabla_{[Y^*,X^*]}V^* &= (\nabla_{[Y,X]}V)^* - 2g(\phi Y, X)^*W^* + 2g(\theta Y, X)^*U^* \\
&\quad - g([Y,X], W)^*\xi + g([Y,X], U)^*\zeta,
\end{aligned} \tag{4.5}
$$

$$
\begin{aligned}
'\nabla_{[Y^*,X^*]}W^* &= (\nabla_{[Y,X]}W)^* + 2g(\phi Y, X)^*V^* - 2g(\psi Y, X)^*U^* \\
&\quad + g([Y,X], V)^*\xi - g([Y,X], U)^*\eta.
\end{aligned} \tag{4.6}
$$

By means of (4.1) and (4.4), we have

$$
\begin{aligned}
&'R(Y^*, X^*)U^* \\
&= \{R(Y,X)U\}^* + \{w(Y)\psi X - w(X)\psi Y - v(Y)\theta X \\
&\quad + v(X)\theta Y + 2g(\theta Y, X)V - 2g(\psi Y, X)W\}^* \\
&\quad + \{g(\phi Y, \nabla_X U) - g(\phi X, \nabla_Y U)\}^*\xi \\
&\quad + \{g(\psi Y, \nabla_X U) - g(\psi X, \nabla_Y U) + g(X, \nabla_Y W) - g(Y, \nabla_X W)\}^*\eta \\
&\quad + \{g(\theta Y, \nabla_X U) - g(\theta X, \nabla_Y U) - g(X, \nabla_Y V) + g(Y, \nabla_X V)\}^*\zeta,
\end{aligned}
$$

where $'R$ denotes the curvature tensor of $\pi^{-1}M$ with respect to the connection $'\nabla$. Using (2.24)-(2.26), (2.28) and (2.29), we can easily see that

(4.7)
$$
\begin{aligned}
&'R(Y^*,X^*)U^* \\
&= \{u(X)Y - u(Y)X + u(A_1X)A_1Y - u(A_1Y)A_1X\}^* \\
&\quad + \{r(Y)w(X) - r(X)w(Y) + q(Y)v(X) - q(X)v(Y) \\
&\quad + u(X)u(A_1Y) - u(Y)u(A_1X)\}^*\xi \\
&\quad + \{p(X)v(Y) - p(Y)v(X) + v(X)u(A_1Y) - v(Y)u(A_1X)\}^*\eta \\
&\quad + \{p(X)w(Y) - p(Y)w(X) + w(X)u(A_1Y) - w(Y)u(A_1X)\}^*\zeta.
\end{aligned}
$$

By the same method we can easily verify that (4.2), (4.3), (4.5) and (4.6) yield

(4.8)
$$
\begin{aligned}
&'R(Y^*,X^*)V^* \\
&= \{v(X)Y - v(Y)X + v(A_1X)A_1Y - v(A_1Y)A_1X\}^* \\
&\quad + \{q(X)u(Y) - q(Y)u(X) - u(Y)v(A_1X) + u(X)v(A_1Y)\}^*\xi \\
&\quad + \{r(Y)w(X) - r(X)w(Y) + p(Y)u(X) - p(X)u(Y) \\
&\quad + v(X)v(A_1Y) - v(Y)v(A_1X)\}^*\eta \\
&\quad + \{q(X)w(Y) - q(Y)w(X) - w(Y)v(A_1X) + w(X)v(A_1Y)\}^*\zeta,
\end{aligned}
$$

(4.9)
$$
\begin{aligned}
&'R(Y^*,X^*)W^* \\
&= \{w(X)Y - w(Y)X + w(A_1X)A_1Y - w(A_1Y)A_1X\}^* \\
&\quad + \{r(X)u(Y) - r(Y)u(X) - u(Y)w(A_1X) + u(X)w(A_1Y)\}^*\xi \\
&\quad + \{r(X)v(Y) - r(Y)v(X) - v(Y)w(A_1X) + v(X)w(A_1Y)\}^*\eta \\
&\quad + \{q(Y)v(X) - q(X)v(Y) + p(Y)u(X) - p(X)u(Y) \\
&\quad + w(X)w(A_1Y) - w(Y)w(A_1X)\}^*\zeta.
\end{aligned}
$$

Differentiating (3.5) covariantly in the direction of Y^* and using (3.3), we have

$$
\begin{aligned}
'\nabla_{Y^*}{'\nabla_{X^*}}\xi = &- \{(\nabla_Y\phi)X + \phi(\nabla_YX)\}^* - g(\phi Y,\phi X)^*\xi \\
&- g(\psi Y,\phi X)^*\eta - g(\theta Y,\phi X)^*\zeta,
\end{aligned}
\tag{4.10}
$$

$$
\begin{aligned}
'\nabla_{Y^*}{'\nabla_{X^*}}\eta = &- \{(\nabla_Y\psi)X + \psi(\nabla_YX)\}^* - g(\phi Y,\psi X)^*\xi \\
&- g(\psi Y,\psi X)^*\eta - g(\theta Y,\psi X)^*\zeta,
\end{aligned}
\tag{4.11}
$$

$$\begin{aligned}{}'\nabla_{Y^*}{}'\nabla_{X^*}\zeta = &-\{(\nabla_Y\theta)X+\theta(\nabla_YX)\}^* - g(\phi Y,\theta X)^*\xi \\ &- g(\psi Y,\theta X)^*\eta - g(\theta Y,\theta X)^*\zeta.\end{aligned} \tag{4.12}$$

On the other hand (3.2), (3.4) and (3.5) imply

$$'\nabla_{[Y^*,X^*]}\xi = -(\phi[Y,X])^* - 2g(\psi Y,X)^*\zeta + 2g(\theta Y,X)^*\eta, \tag{4.13}$$

$$'\nabla_{[Y^*,X^*]}\eta = -(\psi[Y,X])^* + 2g(\phi Y,X)^*\zeta - 2g(\theta Y,X)^*\xi, \tag{4.14}$$

$$'\nabla_{[Y^*,X^*]}\zeta = -(\theta[Y,X])^* - 2g(\phi Y,X)^*\eta + 2g(\psi Y,X)^*\xi. \tag{4.15}$$

Using (4.10)-(4.12) and (4.13)-(4.15), we have

$$\begin{aligned}&'R(Y^*,X^*)\xi \\ &= -\{(\nabla_Y\phi)X-(\nabla_X\phi)Y\}^* + \{v(Y)u(X)-v(X)u(Y)\}^*\eta \\ &+ \{w(Y)u(X)-w(X)u(Y)\}^*\zeta,\end{aligned} \tag{4.16}$$

$$\begin{aligned}&'R(Y^*,X^*)\eta \\ &= -\{(\nabla_Y\psi)X-(\nabla_X\psi)Y\}^* + \{u(Y)v(X)-u(X)v(Y)\}^*\xi \\ &+ \{w(Y)v(X)-w(X)v(Y)\}^*\zeta,\end{aligned} \tag{4.17}$$

$$\begin{aligned}&'R(Y^*,X^*)\zeta \\ &= -\{(\nabla_Y\theta)X-(\nabla_X\theta)Y\}^* + \{u(Y)w(X)-u(X)w(Y)\}^*\xi \\ &+ \{v(Y)w(X)-v(X)w(Y)\}^*\eta.\end{aligned} \tag{4.18}$$

5. Some lemmas under the additional assumptions

In this section we investigate n-dimensional QR-submanifolds of $(p-1)$ QR- dimension in $QP^{(n+p)/4}$ under the additional assumptions

$$\begin{aligned}&('\nabla_\xi{}'R)(Y^*,X^*)U^*=0, \quad ('\nabla_\eta{}'R)(Y^*,X^*)V^*=0, \\ &('\nabla_\zeta{}'R)(Y^*,X^*)W^*=0,\end{aligned} \tag{5.1}$$

$$\begin{aligned}&('\nabla_\xi{}'R)(Y^*,X^*)\xi=0, \quad ('\nabla_\eta{}'R)(Y^*,X^*)\eta=0, \\ &('\nabla_\zeta{}'R)(Y^*,X^*)\zeta=0.\end{aligned} \tag{5.2}$$

We first consider the assumption

$$('\nabla_{\xi}{}'R)(Y^*, X^*)U^* = 0.$$

Differentiating (4.7) covariantly in the direction of ξ and using (2.12), (3.1), (3.2), (3.5) and the assumption $('\nabla'_{\xi}R)(Y^*, X^*)U^* = 0$, we have

$$\begin{aligned}
&- {}'R((\phi Y)^*, X^*)U^* - {}'R(Y^*, (\phi X)^*)U^* \\
&= \{-u(X)\phi Y + u(Y)\phi X - u(A_1X)\phi A_1 Y + u(A_1Y)\phi A_1 X\}^* \\
&\quad + \{p(X)w(Y) - p(Y)w(X) + w(X)u(A_1Y) - w(Y)u(A_1X)\}^*\eta \\
&\quad - \{p(X)v(Y) - p(Y)v(X) + v(X)u(A_1Y) - v(Y)u(A_1X)\}^*\zeta,
\end{aligned}$$

from which, taking the vertical component and using (2.14)-(2.17) and (4.7) itself, we can get

$$\begin{aligned}
&r(X)v(Y) - r(Y)v(X) - r(\phi Y)w(X) + r(\phi X)w(Y) \\
&- q(X)w(Y) + q(Y)w(X) - q(\phi Y)v(X) + q(\phi X)v(Y) \\
&- u(X)u(A_1\phi Y) + u(Y)u(A_1\phi X) = 0,
\end{aligned} \tag{5.3}$$

$$-u(A_1\phi Y)v(X) + u(A_1\phi X)v(Y) + p(\phi Y)v(X) - p(\phi X)v(Y) = 0, \tag{5.4}$$

$$-u(A_1\phi Y)w(X) + u(A_1\phi X)w(Y) + p(\phi Y)w(X) - p(\phi X)w(Y) = 0. \tag{5.5}$$

Putting $Y = U$ in (5.3) and using (2.12) and (2.17), we have

$$\phi A_1 U + r(U)V - q(U)W = 0. \tag{5.6}$$

and consequently

$$r(U) = w(A_1U) = u(A_1W), \quad q(U) = v(A_1U) = u(A_1V). \tag{5.7}$$

Putting $Y = W$ and $X = V$ in (5.4) and using (2.9) and (2.17) yield

$$p(V) = v(A_1U) = u(A_1V). \tag{5.8}$$

Also, putting $Y = V$ and $X = W$ in (5.5) and using (2.9) and (2.17), we have

$$p(W) = w(A_1U) = u(A_1W). \tag{5.9}$$

Summing up we have

$$\begin{aligned}
&A_1U = u(A_1U)U + p(V)V + p(W)W, \\
&p(V) = v(A_1U) = u(A_1V) = q(U), \\
&p(W) = w(A_1U) = u(A_1W) = r(U).
\end{aligned}$$

Thus we have

LEMMA 5.1. *Let M be an n-dimensional QR-submanifold of $(p-1)$ QR-dimension in a quaternionic projective space $QP^{(n+p)/4}$ and let the normal vector field N_1 be parallel with respect to the normal connection. If the equalities in* (5.1) *are established, then*

$$\begin{aligned}
A_1U &= u(A_1U)U + p(V)V + p(W)W,\\
A_1V &= q(U)U + v(A_1V)V + q(W)W,\\
A_1W &= r(U)U + r(V)V + w(A_1W)W,\\
p(V) &= v(A_1U) = u(A_1V) = q(U),\\
p(W) &= w(A_1U) = u(A_1W) = r(U),\\
q(W) &= w(A_1V) = v(A_1W) = r(V).
\end{aligned}$$

Next we assume the additional condition

$$({}'\nabla_\xi {}'R)(Y^*, X^*)\xi = 0.$$

Differentiating (4.16) covariantly in the direction of ξ and using (3.1), (3.2), (3.5) and the assumption $({}'\nabla'_\xi R)(Y^*, X^*)\xi = 0$, we have

$$\begin{aligned}
-{}'R((\phi Y)^*, X^*)\xi - {}'R(Y^*, (\phi X)^*)\xi &= (\phi(\nabla_Y\phi)X - \phi(\nabla_X\phi)Y)^*\\
&\quad + \{w(Y)u(X) - w(X)u(Y)\}^*\eta - \{v(Y)u(X) - v(X)u(Y)\}^*\zeta,
\end{aligned}$$

from which, using (2.14)-(2.16), (2.21) and (4.16), we can easily obtain

$$\begin{aligned}
&2r(Y)\theta X - 2r(X)\theta Y + 2q(Y)\psi X - 2q(X)\psi Y\\
&- u(Y)(\phi A_1X - A_1\phi X) + u(X)(\phi A_1Y - A_1\phi Y)\\
&+ r(Y)(v(X)U - u(X)V) - r(X)(v(Y)U - u(Y)V)\\
&- q(Y)(w(X)U - u(X)W) + q(X)(w(Y)U - u(Y)W)\\
&- r(\phi Y)\psi X + r(\phi X)\psi Y + q(\phi Y)\theta X - q(\phi X)\theta Y = 0.
\end{aligned}\tag{5.10}$$

Putting $X = U$ in (5.10) and using (2.9), (2.12) and (2.17), we have

$$\begin{aligned}
&\phi A_1Y - A_1\phi Y - 2r(U)\theta Y - 2q(U)\psi Y - u(Y)\phi A_1U\\
&- r(U)\{v(Y)U - u(Y)V\} + q(U)\{w(Y)U - u(Y)W\}\\
&+ r(Y)V - q(Y)W + r(\phi Y)W + q(\phi Y)V = 0,
\end{aligned}\tag{5.11}$$

or equivalently

$$\begin{aligned}&g(\phi A_1Y - A_1\phi Y, X) - 2r(U)g(\theta Y, X)\\ &- 2q(U)g(\psi Y, X) + u(Y)u(A_1\phi X)\\ &- r(U)\{v(Y)u(X) - v(X)u(Y)\} + q(U)\{w(Y)u(X) - w(X)u(Y)\}\\ &+ r(Y)v(X) - q(Y)w(X) + r(\phi Y)w(X) + q(\phi Y)v(X) = 0,\end{aligned}$$

from which, taking the skew-symmetric part, we find

$$\begin{aligned}&- 4r(U)g(\theta Y, X) - 4q(U)g(\psi Y, X) + u(Y)u(A_1\phi X)\\ &- u(X)u(A_1\phi Y) - 2r(U)\{v(Y)u(X) - v(X)u(Y)\}\\ &+ 2q(U)\{w(Y)u(X) - w(X)u(Y)\} + r(Y)v(X)\\ &- r(X)v(Y) - q(Y)w(X) + q(X)w(Y) + r(\phi Y)w(X)\\ &- r(\phi X)w(Y) + q(\phi Y)v(X) - q(\phi X)v(Y) = 0.\end{aligned} \tag{5.12}$$

Now we replace Y with θY in (5.12). Then we have with the aid of (2.14)-(2.16)

$$\begin{aligned}&4r(U)g(Y, X) - 4q(U)g(\phi Y, X) - 3q(U)w(Y)v(X) + 2q(U)v(Y)w(X)\\ &+ u(X)u(A_1\psi Y) - u(A_1U)u(X)w(Y) - v(Y)u(A_1\phi X) + r(\theta Y)v(X)\\ &- r(X)u(Y) - r(\psi Y)w(X) - q(\theta Y)w(X) - q(\psi Y)v(X) - q(\phi X)u(Y)\\ &- r(U)\{2u(Y)u(X) + 2v(Y)v(X) + 3w(Y)w(X)\} = 0.\end{aligned}$$

Now we consider the following orthonormal basis

$$\{U, V, W, e_1, \cdots, e_m, \phi(e_1), \cdots, \phi(e_m), \psi(e_1), \cdots, \psi(e_m), \theta(e_1), \cdots, \theta(e_m)\},$$

which is the so-called Q-basis, where $4m + 3 = dimM$. Taking the trace of the above equation with respect to the Q-basis and using Lemma 5.1, we can easily see $8(2m + 1)r(U) = 0$, that is,

$$r(U) = u(A_1W) = w(A_1U) = p(W) = 0.$$

Similarly, replacing Y with ψY in (5.12) and using (2.9), (2.12) and (2.14)-(2.17), we have

$$q(U) = u(A_1V) = v(A_1U) = p(V) = 0,$$

and consequently (5.6) reduces to

$$A_1U = u(A_1U)U.$$

Thus we have

LEMMA 5.2. *Let M be as in Lemma 5.1 and let the normal vector field N_1 be parallel with respect to the normal connection. If the equalities in (5.1) and (5.2) are established, then*

$$(5.13)\quad \begin{aligned} &A_1U = u(A_1U)U, \quad A_1V = v(A_1V)V, \quad A_1W = w(A_1W)W, \\ &p(V) = p(W) = q(U) = q(W) = r(U) = r(V) = 0. \end{aligned}$$

6. Main results

In this section we shall investigate n-dimensional QR-submanifolds of $(p-1)$ QR- dimension in $QP^{(n+p)/4}$ under the additional assumptions (5.1) and (5.2), and prove the following theorem.

THEOREM 6.1. *Let M be an n-dimensional QR-submanifold of $(p-1)$ QR-dimension in a quaternionic projective space $QP^{(n+p)/4}$ and let the normal vector field N_1 be parallel with respect to the normal connection. If the equalities (5.1) and (5.2) are established and if*

$$u(A_1U) = v(A_1V) = w(A_1W)$$

on M, then $\pi^{-1}(M)$ is locally a product of $M_1 \times M_2$ where M_1 and M_2 belong to some $(4n_1+3)$- and $(4n_2+3)$-dimensional spheres (π is the Hopf fibration $S^{n+p+3}(1) \to QP^{(n+p)/4}$).

Proof. By means of Lemma 5.2, (5.11) and (5.12) yield

$$(6.1)\quad \phi A_1Y - A_1\phi Y + r(Y)V - q(Y)W + r(\phi Y)W + q(\phi Y)V = 0,$$

$$(6.2)\quad \begin{aligned} &r(Y)v(X) - r(X)v(Y) - q(Y)w(X) + q(X)w(Y) + r(\phi Y)w(X) \\ &- r(\phi X)w(Y) + q(\phi Y)v(X) - q(\phi X)v(Y) = 0, \end{aligned}$$

respectively. Putting $X = V$ in (6.2) and using (5.13), we have

$$(6.3)\quad r(Y) + q(\phi Y) + \{q(V) - r(W)\}w(Y) = 0.$$

On the other hand, putting $Y = W$ in (6.1) and using (5.13), we have

$$q(V) - r(W) = v(A_1V) - w(A_1W).$$

If $v(A_1V) = w(A_1W)$, then $q(V) = r(W)$, which together with (6.3) yields

$$r(Y) + q(\phi Y) = 0. \tag{6.4}$$

Putting $Y = \phi X$ in (6.4) and using (2.10) and (5.13), we also have

$$-q(Y) + r(\phi Y) = 0. \tag{6.5}$$

Hence it follows from (6.1), (6.4) and (6.5) that

$$A_1\phi = \phi A_1.$$

By the same way we can obtain

$$A_1\psi = \psi A_1, \quad A_1\theta = \theta A_1.$$

By means of Theorem K-P we complete the proof. □

In the previous paper ([15]) we have already proved the following result without the assumption

$$u(A_1U) = v(A_1V) = w(A_1W).$$

REMARK. Let M be as in Theorem 6.1 and let the normal vector field N_1 be parallel with respect to the normal connection. If the following equalities

$$'\nabla_\xi{}'R = 0, \quad '\nabla_\eta{}'R = 0, \quad '\nabla_\zeta{}'R = 0,$$

are established, then $\pi^{-1}(M)$ is locally a product of $M_1 \times M_2$ where M_1 and M_2 belong to some $(4n_1 + 3)$- and $(4n_2 + 3)$-dimensional spheres.

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