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ESSENTIAL NORMS AND STABILITY CONSTANTS OF WEIGHTED COMPOSITION OPERATORS ON $C(X)$

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Abstract. For a weighted composition operator uC_φ on $C(X)$, we determine its essential norm and the constant for its Hyers-Ulam stability, in terms of the set $\varphi(\{x \in X : |u(x)| \geq r\})$ $(r > 0)$.

0. Introduction

Let X be a compact Hausdorff space and let $C(X)$ denote the Banach space of all continuous functions on X with the supremum norm. For any $u \in C(X)$, we put $S(u) = \{x \in X : u(x) \neq 0\}$. Fix a function $u \in C(X)$ and a selfmap φ of X which is continuous on $S(u)$. Then u and φ induce an operator uC_φ defined by

$$(uC_\varphi f)(x) = u(x)\, f(\varphi(x)) \qquad (x \in X)$$

for all $f \in C(X)$. Clearly, uC_φ is a bounded linear operator on $C(X)$. We call uC_φ a *weighted composition operator* on $C(X)$. The properties of this operator are studied by Kamowitz [5], Singh and Summers [10], Feldman [2] and many other mathematicians (see also [4, 11]). The book [9] is a nice reference on this type of operator.

In this paper, we determine two kinds of constants of uC_φ. One is the essential norm of uC_φ, which is computed in Section 1 (Theorem 1). The other is the constant for the Hyers-Ulam stability of uC_φ. In Section 2, we determine it by comparing with the norm of the inverse of the one-to-one operator induced by uC_φ. Indeed, we remark that a bounded linear operator between Banach spaces has the Hyers-Ulam stability if and only if it has closed range (Theorem 2). Using this fact, we also give a necessary and sufficient condition for uC_φ to have the

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Hyers-Ulam stability (Theorem 3). The results on uC_φ in this paper are related to the set $\varphi(\{x \in X : |u(x)| \geq r\})$, where r is a positive number.

1. Essential norm

Let A be a Banach space and $\mathcal{K}$ be the set of all compact operators on A. For any bounded linear operator T on A, the *essential norm* of T means the distance from T to $\mathcal{K}$ in the operator norm, namely

$$\|T\|_e = \inf\{\|T - S\| : S \in \mathcal{K}\}.$$

Clearly, T is compact if and only if $\|T\|_e = 0$. As is seen in [8], the essential norm plays an interesting role in the compact problem of concrete operators.

We are concerned with the case that T is a weighted composition operator uC_φ on $C(X)$. In [5], Kamowitz has showed that uC_φ is compact if and only if

> for each connected component C of $S(u)$, there exists an open set $U \supset C$ such that φ is constant on U.

As is mentioned in [4, 10], this condition is equivalent to the following:

(1) For any $r > 0$, $\varphi(\{x \in X : |u(x)| \geq r\})$ is finite.

From this point of view, we compute the essential norm of uC_φ.

THEOREM 1. *Let uC_φ be a weighted composition operator on $C(X)$. The essential norm of uC_φ is given by*

(2) $\|uC_\varphi\|_e = \inf\{\, r > 0 : \varphi(\{x \in X : |u(x)| \geq r\})$ *is finite*$\,\}$.

Considering the case $\|uC_\varphi\|_e = 0$ in (2), we know that (1) is necessary and sufficient for uC_φ to be compact.

Proof. Denote the right side of (2) by ρ. We first show that $\|uC_\varphi\|_e \geq \rho$. If $\rho = 0$, there is nothing to prove, and so we assume $\rho > 0$. Take $\varepsilon > 0$ arbitrarily. Since $\varphi(\{x \in X : |u(x)| \geq \rho - \varepsilon\})$ is infinite, we find a sequence $\{x_k\}$ in X such that $|u(x_k)| \geq \rho - \varepsilon$ and $\varphi(x_k) \neq \varphi(x_l)$ $(k \neq l)$. By using the fact that X is a compact Hausdorff space, we can select a subsequence $\{x_{k_n}\}$ of $\{x_k\}$ such that each $\varphi(x_{k_n})$ has an open neighborhood U_n and $\{U_n\}$ is pairwise disjoint. For simplicity, we write $x_n = x_{k_n}$ $(n = 1, 2, \ldots)$. For each n, we use Urysohn's lemma to get a function $f_n \in C(X)$ such that $0 \leq f_n \leq 1$, $f_n(\varphi(x_n)) = 1$ and $f_n(X \backslash U_n) = \{0\}$. Then $\|f_n\| = 1$ for $n = 1, 2, \ldots$, and our choice of $\{U_n\}$ and $\{f_n\}$ shows that $f_n(x) \to 0$ for each $x \in X$. These facts imply that $\{f_n\}$ converges weakly to zero in $C(X)$ (see [1, Corollary IV.6.4]). Now, take a compact operator S on $C(X)$ so that $\|uC_\varphi - S\| < \|uC_\varphi\|_e + \varepsilon$. Then we have

$$\begin{aligned}\|uC_\varphi\|_e &> \|uC_\varphi - S\| - \varepsilon \geq \|uC_\varphi f_n - Sf_n\| - \varepsilon \\ &\geq \|uC_\varphi f_n\| - \|Sf_n\| - \varepsilon \geq |u(x_n)f_n(\varphi(x_n))| - \|Sf_n\| - \varepsilon \\ &> \rho - \varepsilon - \|Sf_n\| - \varepsilon\end{aligned}$$

for all $n = 1, 2, \ldots$. Since a compact operator maps weakly convergent sequences into norm convergent ones, it follows $\|Sf_n\| \to 0$. Hence $\|uC_\varphi\|_e \geq \rho - 2\varepsilon$. Since ε was arbitrary, we obtain $\|uC_\varphi\|_e \geq \rho$.

For the opposite inequality, take $\varepsilon > 0$ arbitrarily. Put $F = \{x \in X : |u(x)| \leq \rho + \varepsilon\}$ and $G = \{x \in X : |u(x)| \geq \rho + 2\varepsilon\}$. Since F and G are disjoint closed sets, Urysohn's lemma gives a function $g \in C(X)$ such that $0 \leq g \leq 1$, $g(F) = \{0\}$ and $g(G) = \{1\}$ (we understand $g \equiv 1$ if $F = \emptyset$, or $g \equiv 0$ if $G = \emptyset$). Put $v = ug$. Then $v \in C(X)$ and φ is continuous on $S(v)$, because $S(v) \subset S(u)$. Thus we can define a weighted composition operator vC_φ on $C(X)$. Here we observe that vC_φ has finite rank: Since $X\backslash F \subset \{x \in X : |u(x)| \geq \rho+\varepsilon/2\}$, the definition of ρ implies that $\varphi(X \setminus F)$ is a finite set. If $\varphi(X \setminus F)$ is empty, then $G \subset X \setminus F = \emptyset$ and so $v \equiv 0$, which says that vC_φ is a zero operator and has finite rank. Otherwise we can write $\varphi(X \setminus F) = \{y_1, \ldots, y_m\}$, where $y_1, \ldots, y_m$ are distinct. For $i = 1, \ldots, m$, put $F_i = \{x \in X \setminus F : \varphi(x) = y_i\}$ and define a function v_i on X by

$$v_i(x) = \begin{cases} v(x) & \text{if } x \in F_i \\ 0 & \text{if } x \in X \setminus F_i. \end{cases}$$

Since F_j is open in $X \setminus F$ and so in X for $j = 1, \ldots m$, it follows that v_i is continuous at each point in $X \setminus F$. On the other hand, from the fact that v vanishes on F, it is shown that v_i is continuous at each point in F. Hence $v_i \in C(X)$. In addition, the equation $v = \sum_{i=1}^m v_i$ shows that

$$vC_\varphi f = \sum_{i=1}^m f(y_i)\, v_i$$

for all $f \in C(X)$. This says that $\{v_1, \ldots, v_m\}$ spans the range of vC_φ. Hence vC_φ has finite rank. Noting that vC_φ is compact, we have

$$\begin{aligned}\|uC_\varphi\|_e &\leq \|uC_\varphi - vC_\varphi\| = \sup_{\|f\|\leq 1} \|uC_\varphi f - vC_\varphi f\| \\ &= \sup_{\|f\|\leq 1} \sup_{x \in X} |u(x)f(\varphi(x)) - v(x)f(\varphi(x))|\end{aligned}$$

$$\begin{aligned} &= \sup_{\|f\|\le 1}\sup_{x\in X}|u(x)-v(x)|\,|f(\varphi(x))| \\ &\le \sup_{x\in X}|u(x)-v(x)| = \sup_{x\in X}|u(x)|\,|1-g(x)| \\ &\le \sup_{x\in X\setminus G}|u(x)| \le \rho + 2\varepsilon. \end{aligned}$$

Since ε was arbitrary, we get $\|uC_\varphi\|_e \le \rho$. This completes the proof. □

2. Hyers-Ulam stability

Let A and B be Banach spaces and T be a mapping from A into B. We say that T has the *Hyers-Ulam stability*, if there exists a constant K with the following property:

For any $g \in T(A), \varepsilon > 0$ and $f \in A$ satisfying $\|Tf - g\| \le \varepsilon$, we can find an $f_0 \in A$ such that $Tf_0 = g$ and $\|f - f_0\| \le K\varepsilon$.

We call K an *HUS constant* for T, and denote the infimum of all *HUS* constants for T by K_T. These concepts are based on the research by Hyers [3] or Ulam [14], and are introduced in the paper [7]. One of their concrete examples may be found in the papers [6, 12].

In this paper, we focus on the case that T is a bounded linear operator. In the sequel, we use the symbol $\mathcal{N}(T)$ to denote the kernel of T, and consider the induced one-to-one operator $\tilde{T}$ from the quotient space $A/\mathcal{N}(T)$ into B :

$$\tilde{T}(f + \mathcal{N}(T)) = Tf \qquad (f \in A).$$

The inverse operator $\tilde{T}^{-1}$ from $T(A)$ into $A/\mathcal{N}(T)$ is closely related to the Hyers-Ulam stability of T.

THEOREM 2. *Let A and B be Banach spaces and T be a bounded linear operator from A into B. Then the following statements are equivalent:*

(a) *T has the Hyers-Ulam stability.*
(b) *T has closed range.*
(c) *$\tilde{T}^{-1}$ is bounded.*

Moreover, if one of (hence all of) the conditions (a), (b) *and* (c) *is true, then we have $K_T = \|\tilde{T}^{-1}\|$.*

Proof. The equivalence of (b) and (c) is well known as an application of the open mapping theorem (see [13, Theorem IV.5.9]). We here show the equivalence of (a) and (c).

By the linearity of T, T has the Hyers-Ulam stability if and only if there exists a constant K with the following property:

For any $\varepsilon > 0$ and $f \in A$ satisfying $\|Tf\| \leq \varepsilon$,

we can find an $f_0 \in \mathcal{N}(T)$ such that $\|f - f_0\| \leq K\varepsilon$.

Another way of stating this property is to say:

(3)

For any $f \in A$, we can find an $f_0 \in \mathcal{N}(T)$ such that $\|f - f_0\| \leq K\|Tf\|$.

If (3) holds, then

$$\|f + \mathcal{N}(T)\| \leq K\|Tf\|$$

for all $f \in A$, and hence $\tilde{T}^{-1}$ is bounded and $\|\tilde{T}^{-1}\| \leq K$. This shows (a) $\Rightarrow$ (c). If we note that K is an arbitrary HUS constant for T, we reach

$$\|\tilde{T}^{-1}\| \leq K_T. \tag{4}$$

Conversely, assume that $\tilde{T}^{-1}$ is bounded and $\|\tilde{T}^{-1}\| < L$. For any $f \in A$, we have

$$\|f + \mathcal{N}(T)\| = \|\tilde{T}^{-1}(Tf)\| \leq \|\tilde{T}^{-1}\| \, \|Tf\| < L\|Tf\|,$$

and so, we can find an $f_0 \in \mathcal{N}(T)$ such that

$$\|f - f_0\| < L\|Tf\|.$$

Thus (3) holds, and so T has the Hyers-Ulam stability. Hence (c) implies (a). More precisely, we have now proved that L is an HUS constant for T whenever $\|\tilde{T}^{-1}\| < L$. Hence $K_T \leq \|\tilde{T}^{-1}\|$. Together with (4), we obtain $K_T = \|\tilde{T}^{-1}\|$. The proof is completed. □

Now we characterize the weighted composition operators on $C(X)$ which have the Hyers-Ulam stability.

THEOREM 3. *Let uC_φ be a weighted composition operator on $C(X)$. Then uC_φ has the Hyers-Ulam stability if and only if there exists a positive constant r such that*

$$\varphi(\{x \in X : |u(x)| \geq r\}) = \varphi(S(u)). \tag{5}$$

Moreover, if R is the supremum of all r such that (5) holds, then $K_{uC_\varphi} = 1/R$.

For the proof, we use the following lemma.

LEMMA. *Let uC_φ be a weighted composition operator on $C(X)$. Then we have*

$$\|f + \mathcal{N}(uC_\varphi)\| = \sup\{\, |f(y)| : y \in \varphi(S(u)) \,\},$$

for any $f \in C(X)$.

Proof. Pick $f \in C(X)$ and put $\alpha = \sup\{\,|f(y)| : y \in \varphi(S(u))\,\}$. For any $h \in \mathcal{N}(uC_\varphi)$, we have $h(y) = 0$ for all $y \in \varphi(S(u))$, and so

$$\alpha = \sup\{\,|f(y) + h(y)| : y \in \varphi(S(u))\,\} \le \|f + h\|.$$

Hence $\alpha \le \|f + \mathcal{N}(uC_\varphi)\|$.

To verify $\|f + \mathcal{N}(uC_\varphi)\| \le \alpha$, take $\varepsilon > 0$ arbitrarily. Let F be the closure of $\varphi(S(u))$, and put $G = \{y \in X : |f(y)| \ge \alpha + \varepsilon\}$. It is easy to see that F and G are disjoint closed sets in X. It follows from Urysohn's lemma that there exists a $g \in C(X)$ such that $0 \le g \le 1$, $g(F) = \{0\}$ and $g(G) = \{1\}$. Put $h = fg$. Clearly, $h \in C(X)$ and $h(y) = 0$ for $y \in \varphi(S(u))$. This implies $h \in \mathcal{N}(uC_\varphi)$. On the other hand, we have

$$|f(x) - h(x)| = |f(x)|\,|1 - g(x)| \le \begin{cases} 0 & \text{if } x \in G \\ \alpha + \varepsilon & \text{if } x \in X \setminus G. \end{cases}$$

Hence

$$\|f + \mathcal{N}(uC_\varphi)\| \le \|f - h\| \le \alpha + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we get $\|f + \mathcal{N}(uC_\varphi)\| \le \alpha$, concluding the proof. □

Proof of Theorem 3. Suppose that there exists an $r > 0$ that satisfies (5). We use Lemma and compute as follows:

$$\begin{aligned}
\|f + \mathcal{N}(uC_\varphi)\| &= \sup\{\,|f(y)| : y \in \varphi(S(u))\,\} \\
&= \sup\{\,|f(y)| : y \in \varphi(\{x \in X : |u(x)| \ge r\})\,\} \\
&= \sup\{\,|f(\varphi(x))| : |u(x)| \ge r\,\} \\
&= \sup\left\{\frac{1}{|u(x)|}|(uC_\varphi f)(x)| : |u(x)| \ge r\right\} \\
&\le \frac{1}{r}\sup\{\,|(uC_\varphi f)(x)| : |u(x)| \ge r\} \\
&\le \frac{1}{r}\,\|uC_\varphi f\|,
\end{aligned}$$

for all $f \in C(X)$. Hence $u\widetilde{C}_\varphi^{\,-1}$ is bounded and $\|u\widetilde{C}_\varphi^{\,-1}\| \le 1/r$. According to Theorem 2, uC_φ has the Hyers-Ulam stability. If r is taken all over the numbers satisfying (5), we obtain

$$\|u\widetilde{C}_\varphi^{\,-1}\| \le \frac{1}{R}. \tag{6}$$

Conversely, suppose that uC_φ has the Hyers-Ulam stability. By Theorem 2, $u\widetilde{C}_\varphi^{-1}$ is bounded. Assume $\|u\widetilde{C}_\varphi^{-1}\| < 1/r$ and

$$\varphi(\{x \in X : |u(x)| \geq r\}) \neq \varphi(S(u))$$

for some $r > 0$. Then there is a point $y_0 \in \varphi(S(u))$ with $y_0 \notin \varphi(\{x \in X : |u(x)| \geq r\})$. Here $\varphi(\{x \in X : |u(x)| \geq r\})$ is a closed set, because $\{x \in X : |u(x)| \geq r\}$ is a compact subset of $S(u)$ and φ is continuous on $S(u)$. By Urysohn's lemma, we find an $f_0 \in C(X)$ such that $0 \leq f_0 \leq 1$, $f_0(y_0) = 1$ and $f_0(y) = 0$ for all $y \in \varphi(\{x \in X : |u(x)| \geq r\})$. Then we have

$$|(uC_\varphi f_0)(x)| = |u(x)|\,|f_0(\varphi(x))| \leq \begin{cases} |u(x)| \cdot 0 = 0 & \text{if } |u(x)| \geq r \\ r|f_0(\varphi(x))| \leq r & \text{if } |u(x)| < r, \end{cases}$$

and so $\|uC_\varphi f_0\| \leq r$. Hence we use Lemma to see that

$$\begin{aligned} 1 &= |f_0(y_0)| \leq \sup\{|f_0(y)| : y \in \varphi(S(u))\} = \|f_0 + \mathcal{N}(uC_\varphi)\| \\ &= \|u\widetilde{C}_\varphi^{-1}(uC_\varphi f_0)\| \leq \|u\widetilde{C}_\varphi^{-1}\|\,\|uC_\varphi f_0\| < \frac{1}{r} \cdot r = 1, \end{aligned}$$

which is a contradiction. Thus we conclude that if $\|u\widetilde{C}_\varphi^{-1}\| < 1/r$, then (5) holds. This implies $1/R \leq \|u\widetilde{C}_\varphi^{-1}\|$. Together with (6), we get $\|u\widetilde{C}_\varphi^{-1}\| = 1/R$. The equality $K_{uC_\varphi} = 1/R$ follows from Theorem 2. □

The next corollaries are the immediate consequences of Theorem 3.

COROLLARY 1. *Let uC_φ be a weighted composition operator on $C(X)$. If u has an inverse $u^{-1} \in C(X)$, then uC_φ has the Hyers-Ulam stability and $K_{uC_\varphi} \leq \|u^{-1}\|$. In addition, if φ is one-to-one, then $K_{uC_\varphi} = \|u^{-1}\|$.*

Proof. Put $r = 1/\|u^{-1}\|$. Then we have $|u(x)| \geq r$ for all $x \in X$. It follows that $\{x \in X : |u(x)| \geq r\} = X = S(u)$, and so (5) holds. Hence uC_φ has the Hyers-Ulam stability and $K_{uC_\varphi} \leq 1/r = \|u^{-1}\|$. Here we also note that $\|u^{-1}\|$ is equal to the supremum of all r satisfying $\{x \in X : |u(x)| \geq r\} = S(u)$. Hence if φ is one-to-one, then $\|u^{-1}\|$ is the supremum of all r such that (5) holds, and so $K_{uC_\varphi} = \|u^{-1}\|$. □

COROLLARY 2. *Let $u \in C(X)$ and $M_u : f \to u \cdot f$ be a multiplication operator on $C(X)$. Then M_u has the Hyers-Ulam stability if and only if $S(u)$ is a compact set. Moreover, we have $K_{M_u} = 1/\inf\{|u(x)| : x \in S(u)\}$.*

Proof. Consider the case that φ is the identity map of X. Then $uC_\varphi = M_u$. Hence Theorem 3 says that M_u has the Hyers-Ulam stability if and only if there exists an $r > 0$ such that $\{x \in X : |u(x)| \geq r\} = S(u)$. If there exists such an $r > 0$, then $S(u)$ is clearly a compact set. Conversely, if $S(u)$ is compact, then $|u|$ attains its minimum r (> 0) on $S(u)$ and we get $\{x \in X : |u(x)| \geq r\} = S(u)$. Thus we proved the first assertion. Noting that the supremum of all r such that $\{x \in X : |u(x)| \geq r\} = S(u)$ is equal to $\inf\{|u(x)| : x \in S(u)\}$, we obtain $K_{M_u} = 1/\inf\{|u(x)| : x \in S(u)\}$. □

REMARK. In Corollary 1, if φ is not one-to-one, the equality $K_{uC_\varphi} = \|u^{-1}\|$ does not necessarily hold. Indeed, let $X = [0,1]$ and $u(x) = (x - 1/2)^2 + 1$ for all $x \in X$. Put

$$\varphi_1(x) = \begin{cases} 0 & \text{if } x \in [0, 3/4] \\ 4x - 3 & \text{if } x \in (3/4, 1]. \end{cases}$$

Then it is easy to see that $[0,1] = \varphi_1(S(u)) = \varphi_1(\{x \in X : |u(x)| \geq r\})$ for all $r \leq 17/16$. We also note that if $r > 17/16$ then $\varphi_1(\{x \in X : |u(x)| \geq r\}) \subsetneq [0,1]$. Hence $K_{uC_{\varphi_1}} = 16/17$ by Theorem 3. Therefore we get $K_{uC_{\varphi_1}} < 1 = \|u^{-1}\|$.

Note that if we consider the function

$$\varphi_2(x) = \begin{cases} 0 & \text{if } x \in [0, 1/2] \\ 2x - 1 & \text{if } x \in (1/2, 1], \end{cases}$$

then we see that φ_2 is not one-to-one but $K_{uC_{\varphi_2}} = 1 = \|u^{-1}\|$ in a way similar to the above.

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