Bull. Korean Math. Soc. **40** (2003), No. 4, pp. 577–581

RANDOM FIXED POINTS OF WEAKLY INWARD MULTIVALUED RANDOM MAPS IN BANACH SPACES

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Abstract. Some random fixed point theorems for weakly inward multivalued random maps are obtained in Banach spaces.

1. Introduction and preliminaries

Throughout this note, (Ω, Σ) denotes a measurable space. Let S be a subset of a Banach space $X = (X, ||.||)$. Let $C(S)$ be the family of all nonempty closed subsets of S, $CC(S)$ all nonempty closed convex subsets of S, and $CK(S)$ all nonempty compact convex subsets of S, respectively. A mapping $F : \Omega \to C(S)$ is called measurable if for any open subset B of S, $F^{-1}(B) = \{\omega \in \Omega : F(\omega) \cap B \neq \emptyset\} \in \Sigma$. A mapping $F : \Omega \times S \to C(X)$ is called a random operator if for each $x \in S$, $F(., x)$ is measurable. A mapping $\xi : \Omega \to S$ is called a random fixed point of a random operator $F : \Omega \times S \to C(X)$ if ξ is measurable and for each $\omega \in \Omega$, $\xi(\omega) \in F(\omega, \xi(\omega))$.

For any $x \in X$ and any subsets A and B of X, we denote

$$d(x, B) = \inf\{||x - y|| : y \in B\}$$

and

$$d(A, B) = \inf\{||x - y|| : x \in A, y \in B\}.$$

A mapping $F : S \to C(X)$ is said to satisfy condition (B) [11] if for any sequence $\{x_n\} \subset S$ and $D \in C(S)$ such that $d(x_n, D) \to 0$ and $|d(x_n, F(x_n)) - d(F(x_n), S)| \to 0$ as $n \to \infty$, there exists an $x_0 \in D$ with $d(x_0, F(x_0)) = d(F(x_0), S)$. A mapping $F : S \to CK(X)$ is called k-Lipschitz $(k \geq 0)$ if for any $x, y \in S$, $H(F(x), F(y)) \leq k||x - y||$, where H is the Hausdorff metric induced by the norm $||.||$. If $k < 1$, then F is called a contraction and if $k = 1$, then F is called nonexpansive. A

Received January 6, 2003.
2000 Mathematics Subject Classification: 47H10, 60H25.
Key words and phrases: random fixed point, continuous map, multivalued random operator, Banach space.

random operator $F : \Omega \times S \to C(X)$ or $CK(X)$ is called continuous (k-Lipschitz, nonexpansive, etc.) if for each $\omega \in \Omega$, $F(\omega, .)$ is continuous (k-Lipschitz, nonexpansive, etc.). Let S be a subset of X and $x \in X$. The inward set $I_S(x)$ is defined by

$$I_S(x) = \{x + \lambda(y - x) : y \in S \text{ and } \lambda \geq 0\}.$$

If S is convex and $x \in S$, then

$$I_S(x) = x + \{\lambda(y - x) : y \in S \text{ and } \lambda \geq 1\}.$$

By $\overline{I_S(x)}$, we denote the closure of $I_S(x)$ in X.

Random fixed point theory is presently an active field of research lying at the intersection of nonlinear analysis and probability theory and is required for the study of random equations arising in various applied areas. Although its study was initiated about almost half a century ago, however, it received the attention it deserved after the appearance of the paper by Bharucha-Reid [3] in 1976. Since then there has been a lot of activity in this area and many interesting results have appeared, see, for instance, the work of Beg and Shahzad [1, 2], Itoh [4], Lin [5], Liu [6], O'Regan [7], Papageorgiou [9], Sehgal and Singh [10], Shahzad [12], Tan and Yuan [13], etc. Recently, Shahzad [11] established a general random approximation for continuous multivalued random operators and obtained a random fixed point theorem in the general setting. The aim of this note is to prove some random fixed point theorems for weakly inward multivalued random operators using a general approximation result of Shahzad [11]. Our results extend and include, as special cases, the work of Beg and Shahzad [2], Bharucha-Reid [3], and Sehgal and Singh [10].

2. Main results

We need the following approximation result, which is a special case of Shahzad [11, Theorem 3.3], in the sequel.

LEMMA 2.1. *Let S be a nonempty separable closed subset of a Banach space X, and let $F : \Omega \times S \to C(X)$ be a continuous random operator satisfying condition (B). If for any $\omega \in \Omega$, the set $\{x \in S : d(x, F(\omega, x)) = d(F(\omega, x), S)\}$ is nonempty, then there exists a measurable map $\xi : \Omega \to S$ such that*

$$d(\xi(\omega), F(\omega, \xi(\omega))) = d(F(\omega, \xi(\omega)), S)$$

for each $\omega \in \Omega$.

THEOREM 2.1. *Let S be a nonempty separable closed convex subset of a Banach space X, and let $F : \Omega \times S \to CC(X)$ be a continuous random operator satisfying condition (B). If for any $\omega \in \Omega$, the set $\{x \in S : d(x, F(\omega, x)) = d(F(\omega, x), S)\}$ is nonempty and F is weakly inward (i.e., for each $\omega \in \Omega$, $F(\omega, x) \cap \overline{I_S(x)} \neq \emptyset$ for $x \in S$), then F has a random fixed point.*

Proof. By Lemma 2.1, there exists a measurable map $\xi : \Omega \to S$ such that

$$d(\xi(\omega), F(\omega, \xi(\omega))) = d(F(\omega, \xi(\omega)), S) \tag{1}$$

for each $\omega \in \Omega$. This ξ is the desired random fixed point of F. Indeed, fix $\omega \in \Omega$. Since F is weakly inward, we can choose z (depending on ω) such that

$$z \in F(\omega, \xi(\omega)) \cap \overline{I_S(\xi(\omega))}.$$

This implies that there exists $\{z_n\} \subset I_S(\xi(\omega))$ (depending on ω) such that $z_n \to z$ as $n \to \infty$. Since $z_n \in I_S(\xi(\omega))$, there exist $y_n \in S$ and $\lambda_n \geq 1$ (both depending on ω) such that

$$z_n = \xi(\omega) + \lambda_n(y_n - \xi(\omega))$$

or

$$y_n = \frac{1}{\lambda_n} z_n + (1 - \frac{1}{\lambda_n})\xi(\omega) \in S.$$

Since F is convex-valued, it follows from (1) that

$$\begin{aligned} & d(\xi(\omega), F(\omega, \xi(\omega))) \\ \leq \; & d(y_n, F(\omega, \xi(\omega))) \\ \leq \; & \frac{1}{\lambda_n} d(z_n, F(\omega, \xi(\omega))) + (1 - \frac{1}{\lambda_n}) d(\xi(\omega), F(\omega, \xi(\omega))). \end{aligned}$$

Thus,

$$d(\xi(\omega), F(\omega, \xi(\omega))) \leq d(z_n, F(\omega, \xi(\omega))).$$

But $z \in F(\omega, \xi(\omega))$. Therefore,

$$d(\xi(\omega), F(\omega, \xi(\omega))) \leq d(z_n, z),$$

which on taking the limit as $n \to \infty$ gives

$$d(\xi(\omega), F(\omega, \xi(\omega))) = 0.$$

Since F is closed-valued, $\xi(\omega) \in F(\omega, \xi(\omega))$. □

THEOREM 2.2. *Let S be a nonempty compact convex subset of a Banach space X, and let $F : \Omega \times S \to CK(X)$ be a continuous random operator. If F is weakly inward, then F has a random fixed point.*

Proof. As in Shahzad [11], F satisfies condition (B). By Reich [8], the set $\{x \in S : d(x, F(\omega, x)) = d(F(\omega, x), S)\}$ is nonempty for each $\omega \in \Omega$. Theorem 2.1 further implies that F has a random fixed point. □

The following is an easy consequence of Theorem 2.2.

COROLLARY 2.1. *Let S be a nonempty compact convex subset of a Banach space X, and let $F : \Omega \times S \to CK(X)$ be a nonexpansive random operator. If F is weakly inward, then F has a random fixed point.*

REMARK 2.1.

1. Theorem 2.2 extends Beg and Shahzad [2, Theorem 2], Bharucha-Reid [3, Theorem 10], and Sehgal and Singh [10, Corollary 1].
2. Corollary 2.1 includes Beg and Shahzad [2, Theorem 5 and Corollary 6] as special cases.
3. Let S be a nonempty separable closed convex subset of a Banach space X, and let $F : \Omega \times S \to CC(X)$ be continuous and weakly inward. If the set $\{x \in S : d(x, F(\omega, x)) = d(F(\omega, x), S)\}$ is nonempty for each $\omega \in \Omega$, then, following the proof of Theorem 2.1, we can find an $x_0 \in S$ such that $x_0 \in F(\omega, x_0)$ for each $\omega \in \Omega$. In view of this, we observe that Theorem 2.2 also follows from Shahzad [12, Theorem 2.1].

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