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ON TWO-DIMENSIONAL LANDSBERG SPACE WITH A SPECIAL (α, β)-METRIC

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ABSTRACT. In the present paper, we treat a Finsler space with a special (α, β)-metric $L(\alpha, \beta) = c_1\alpha + c_2\beta + \alpha^2/\beta$ satisfying some conditions. We find a condition that a Finsler space with a special (α, β)-metric be a Berwald space. Then it is shown that if a two-dimensional Finsler space with a special (α, β)-metric is a Landsberg space, then it is a Berwald space.

1. INTRODUCTION

In the Cartan connection $C\Gamma$, a Finsler space is called *Landsberg space*, if the covariant derivative $C_{hij|k}$ of the C-torsion tensor $C_{hij} = \dot{\partial}_h\dot{\partial}_i\dot{\partial}_j(L^2/4)$ satisfies $C_{hij|k}(x, y)y^k = 0$. A Berwald space is characterized by $C_{hij|k} = 0$. Berwald spaces are specially interesting and important, because the connection is linear, and many examples of a Berwald space have been known. But any concrete example of a Landsberg space which is not a Berwald space is not known yet. If a Finsler space is a Landsberg space and satisfies some additional conditions, then it is merely a Berwald space (*cf.* Bácsó & Matsumoto [3]). On the other hand, in the two-dimensional case, a general Finsler space is a Landsberg space, if and only if its main scalar $I(x, y)$ satisfies $I_{|i}y^i = 0$ (*cf.* Matsumoto [6]).

The purpose of the present paper is to find a two-dimensional Landsberg space with a special (α, β)-metric $L(\alpha, \beta) = c_1\alpha + c_2\beta + \alpha^2/\beta$ satisfying some conditions, where c_1, c_2 are constants and $c_1 \neq 0$. First we find the condition that a Finsler space with a special (α, β)-metric be a Berwald space (see Theorem 3.1). Next we determine the difference vector and the main scalar of F^2 with the metric above.

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Finally we derive the condition that a two-dimensional Finsler space F^2 with a special (α,β)-metric be a Landsberg space, and we show that if F^2 with the metric above is a Landsberg space, then it is a Berwald space (see Theorem 4.1).

2. PRELIMINARIES

Let $F^n = (M^n, L(\alpha,\beta))$ be an n-dimensional Finsler space with an (α,β)-metric and $R^n = (M^n, \alpha)$ the associated Riemannian space, where $\alpha^2 = a_{ij}(x)y^iy^j$, $\beta = b_i(x)y^i$. We put $(a_{ij}) = (a_{ij})^{-1}$.

The Riemannian metric α is *not supposed to be positive-definite* and we shall restrict our discussions to a domain of (x,y) where β does not vanish. The covariant differentiation in the Levi-Civita connection $(\gamma_j{}^i{}_k(x))$ of R^n is denoted by the semicolon. Let us list the symbols here for the late use:

$$\begin{array}{llll} 2r_{ij} = b_{i;j} + b_{j;i} & 2s_{ij} = b_{i;j} - b_{j;i} & r^i{}_j = a^{ir}r_{rj} & s^i{}_j = a^{ir}s_{rj}, \\ r_i = b_r r^r{}_i, & s_i = b_r s^r{}_i, & b^i = a^{ir}b_r, & b^2 = a^{rs}b_r b_s, \\ L_\alpha = \partial L/\partial\alpha, & L_\beta = \partial L/\partial\beta, & L_{\alpha\alpha} = \partial L_\alpha/\partial & \text{and} \quad y_k = a_{kr}y^r. \end{array}$$

The Berwald connection $B\Gamma = (G_j{}^i{}_k, G^i{}_j, 0)$ of F^n plays one of the leading roles in the present paper. Denote by $B_j{}^i{}_k$ the difference tensor Matsumoto [7] of $G_j{}^i{}_k$ from $\gamma_j{}^i{}_k$:

$$G_j{}^i{}_k(x,y) = \gamma_j{}^i{}_k(x) + B_j{}^i{}_k(x,y). \tag{2.1}$$

With the subscript 0, the transvection by y^i, we have

$$G^i{}_j = \gamma_0{}^i{}_j + B^i{}_j, \quad 2G^i = \gamma_0{}^i{}_0 + 2B^i, \tag{2.2}$$

and then $B^i{}_j = \dot\partial_j B^i$ and $B_j{}^i{}_k = \dot\partial_k B^i{}_j$. On account of Matsumoto [7], the Berwald connection $B\Gamma$ of a Finsler space with (α,β)-metric $L(\alpha,\beta)$ is given by (2.1) and (2.2), where $B_j{}^i{}_k$ are the components of a Finsler tensor of $(1,2)$-type which is determined by

$$L_\alpha B_j{}^k{}_i y^i y_k = \alpha L_\beta (b_{j;i} - B_j{}^k{}_i b_k) y^j. \tag{2.3}$$

According to Matsumoto [7], $B^i(x,y)$ is called the *difference vector.* If

$$\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha} \neq 0,$$

where $\gamma^2 = b^2\alpha^2 - \beta^2$, then B^i is written as follows:

$$B^i = \frac{E}{\alpha}y^i + \frac{\alpha L_\beta}{L_\alpha}s^i{}_0 - \frac{\alpha L_{\alpha\alpha}}{L_\alpha}C^*\left(\frac{1}{\alpha}y^i - \frac{\alpha}{\beta}b^i\right), \tag{2.4}$$

where

$$E = \frac{\beta L_\beta}{L} C^* \quad \text{and} \quad C^* = \frac{\alpha\beta(r_{00}L_\alpha - 2\alpha s_0 L_\beta)}{2(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha})}.$$

Furthermore, by means of Hashiguchi, Hōjō & Matsumoto [4] we have

$$\alpha_{|i} = -\frac{L_\beta}{L_\alpha}\beta_{|i}. \tag{2.5}$$

$$\beta_{|i}y^i = r_{00} - 2b_r B^r. \tag{2.6}$$

$$b^2_{|i}y^i = 2(r_0 + s_0). \tag{2.7}$$

$$\gamma^2_{|i}y^i = 2(r_0+s_0)\alpha^2 - 2\left(\frac{L_\beta}{L_\alpha}b^2\alpha + \beta\right)(r_{00} - 2b_r B^r). \tag{2.8}$$

The following Lemmas have been shown:

Lemma 2.1 (Bácsó & Matsumoto [2]). *If $\alpha^2 \equiv 0 \ (mod\, \beta)$, that is, $a_{ij}(x)y^iy^j$ contains $b_i(x)y^i$ as a factor, then the dimension n is equal to two and b^2 vanishes. In this case we have $\delta = d_i(x)y^i$ satisfying $\alpha^2 = \beta\delta$ and $d_ib^i = 2$.*

Lemma 2.2 (Hashiguchi, Hōjō & Matsumoto [4]). *We consider the two-dimensional case.*

(1) *If $b^2 \neq 0$, then there exist a sign $\varepsilon = \pm 1$ and $\delta = d_i(x)y^i$ such that $\alpha^2 = \beta^2/b^2 + \varepsilon\delta^2$ and $d_ib^i = 0$.*
(2) *If $b^2 = 0$, then there exists $\delta = d_i(x)y^i$ such that $\alpha^2 = \beta\delta$ and $d_ib^i = 2$.*

If there are two functions $f(x)$ and $g(x)$ satisfying $f\alpha^2 + g\beta^2 = 0$, then $f = g = 0$ is obvious, because $f \neq 0$ implies a contradition $\alpha^2 = (-g/f)\beta^2$.

Throughout the paper, we shall say "homogeneous polynomial(s) in (y^i) of degree r"as $hp(r)$ for brevity. Thus $\gamma_0{}^i{}_0$ are $hp(2)$.

3. Berwald space

In the present section, we find the condition that a Finsler space F^n with a special (α,β)-metric be a Berwald space.

Let $F^n = (M^n, L(\alpha,\beta))$ be an n-dimensional Finsler space with a special (α,β)-metric given by

$$L(\alpha,\beta) = c_1\alpha + c_2\beta + \alpha^2/\beta, \tag{3.1}$$

where c_1, c_2 are constants, and $c_1 \neq 0$.

We shall assume $b^2 \neq 0$. If $b^2 = 0$, then from Lemma 2.2 we have $\alpha^2 = \beta\delta$, so $L = c_1\alpha + (c_2\beta + \delta)$, which is a Randers metric. So the assumption $b^2 \neq 0$ is reasonable.

Then from the above we have

$$L_\alpha = c_1 + 2\alpha/\beta, \quad L_\beta = c_2 - \alpha^2/\beta^2, \quad L_{\alpha\alpha} = 2/\beta. \tag{3.2}$$

Substituting (3.2) into (2.3), we obtain

$$c_1\beta^2 B_j{}^k{}_i y^i y_k + \alpha\{2\beta B_j{}^k{}_i y^j y_k + (\alpha^2 - c_2\beta^2)(b_{j;i} - B_j{}^k{}_i b_k)y^j\} = 0. \tag{3.3}$$

Assume that the Finsler space with (3.1) be a Berwald space, that is, $G_j{}^i{}_k = G_j{}^i{}_k(x)$. Then we have $B_j{}^k{}_i = B_j{}^k{}_i(x)$, so the left-hand side of (3.3) has a form

$$P(x,y) + \alpha Q(x,s) = 0, \tag{3.4}$$

where P, Q are polynomials in (y^i) while α is irrational in (y^i). Hence the above (3.3) shows $P = Q = 0$. By Lemma 2.1, the assumption $b^2 \neq 0$ implies $\alpha^2 - c_2\beta^2 \neq 0$. Thus we have

$$B_j{}^k{}_i a_{kh} y^j y^h = 0 \quad \text{and} \quad (b_{j;i} - B_j{}^k{}_i b_k)y^j = 0. \tag{3.5}$$

The former yields $B_j{}^k{}_i a_{kh} + B_h{}^k{}_i a_{kj} = 0$, so we have $B_j{}^k{}_i = 0$. Then the latter leads to $b_{j;i} = 0$ directly.

Conversely, if $b_{i;j} = 0$, then $(\gamma_j{}^i{}_k, \gamma_0{}^i{}_j, 0)$ becomes the Berwald connection of F^n due to the well-known Okada's axioms. Thus F^n is a Berwlad space. Therefore we have.

Theorem 3.1. *The Finsler space F^n with a special (α,β)-metric (3.1) satisfying $b^2 \neq 0$ is a Berwald space if and only if $b_{j;i} = 0$, and then the Berwald connection is essentially Riemannian $(\gamma_j{}^i{}_k, \gamma_0{}^i{}_j, 0)$.*

4. Two-dimensional Landsberg space

In the present section, we find the necessary and sufficient conditions that a two-dimensional Finsler space with a special (α,β)-metric (3.1) be a Landsberg space.

The difference vector B^i of the Finsler space has been first given in Shibata, Shimada, Azuma & Yasuda [11]. Here, by means of (2.4) and (3.2), we have

$$2B^i = \frac{AB}{\beta(c_1\beta + 2\alpha)L\Omega}\left(y^i + \frac{2\alpha^3 L}{B}b^i\right) + \frac{2\alpha(c_2\beta^2 - \alpha^2)}{\beta(c_1\beta + 2\alpha)}s^i{}_0, \tag{4.1}$$

where

$$A = \beta(2\alpha + c_1\beta)r_{00} + 2\alpha(\alpha^2 - c_2\beta^2)s_0,$$
$$B = c_1c_2\beta^3 - 3c_1\alpha^2\beta - 4\alpha^3,$$
$$\Omega = c_1\beta^3 + 2b^2\alpha^3.$$

It is trivial that $\beta \neq 0$, $c_1\beta + 2\alpha \neq 0$ and $\Omega \neq 0$, because α is irrational in (y^i). It follows from (4.1) that

$$r_{00} - 2b_rB^r = \frac{\alpha(c_1\beta + 2\alpha)A}{L\Omega}. \tag{4.2}$$

Now we deal with the necessary and sufficient conditions that a two-dimensional Finsler space F^2 with (3.1) be a Landsberg space. It is well known that in the two-dimensional case, a general Finsler space is a Landsberg space, if and only if its main scalar $I_{|i}y^i = 0$. Owing to Antonelli, Ingarden & Matsumoto [1], Kitayama, Azuma & Matsumoto [5], the main scalar I of a two-dimensional Finsler space F^2 with (3.1) is obtained as follows:

$$\varepsilon I^2 = \frac{9\gamma^2M^2}{4\alpha\beta L\Omega^3}, \quad \text{where} \quad M = c_1c_2\beta^5 - c_1\alpha^2\beta^3 - 2c_1b^2\alpha^4\beta - 4b^2\alpha^5. \tag{4.3}$$

The covariant differentiation of (4.3) leads to

$$\begin{aligned}(4.4)\quad & 4\alpha^2\beta^2L\Omega^4\varepsilon I^2_{|i} \\ &= 9M(\alpha\beta\Omega M\gamma^2_{|i} + 2\alpha\beta\Omega\gamma^2M_{|i} - \beta\Omega\gamma^2M\alpha_{|i} - \alpha\Omega\gamma^2M\beta_{|i} - 3\alpha\beta\gamma^2M\Omega_{|i}).\end{aligned}$$

Trasvecting (4.4) by y^i, we have

$$4\alpha^2\beta^2L\Omega^4\varepsilon I^2_{|i}y^i = 9M(U\gamma^2_{|i}y^i + QM_{|i}y^i - R\alpha_{|i}y^i - S\beta_{|i}y^i - T\Omega_{|i}y^i), \tag{4.5}$$

where

$$\begin{aligned}U =\ & c_1^2c_2\alpha\beta^9 - c_1^2\alpha^3\beta^7 + 2c_1c_2b^2\alpha^4\beta^6 - 2c_1^2b^2\alpha^5\beta^5 - 6c_1b^2\alpha^6\beta^4 - 4c_1b^4\alpha^8\beta^2 \\ & - 8b^4\alpha^9\beta, \\ Q =\ & - 2c_1\alpha\beta^6 + 2c_1b^2\alpha^3\beta^4 - 4b^2\alpha^4\beta^3 + 4b^4\alpha^6\beta, \\ R =\ & - c_1^2c_2\beta^{11} + c_1^2(c_2b^2 + 1)\alpha^2\beta^9 - 2c_1c_2b^2\alpha^3\beta^8 + c_1^2b^2\alpha^4\beta^7 + 2c_1b^2(c_2b^2 + 3)\alpha^5\beta^6 \\ & - 2c_1^2b^4\alpha^6\beta^5 - 2c_1b^4\alpha^7\beta^4 + 8b^4\alpha^8\beta^3 - 4c_1b^6\alpha^9\beta^2 - 8b^6\alpha^{10}\beta, \\ S =\ & - c_1^2c_2\alpha\beta^{10} + c_1^2(c_2b^2 + 1)\alpha^3\beta^8 - 2c_1c_2b^2\alpha^4\beta^7 + c_1^2b^2\alpha^5\beta^6 \\ & + 2c_1b^2(c_2b^2 + 3)\alpha^6\beta^5 - 2c_1^2b^4\alpha^7\beta^4 - 2c_1b^4\alpha^8\beta^3 + 8b^4\alpha^9\beta^2 - 4c_1b^6\alpha^{10}\beta \\ & - 8b^6\alpha^{11},\end{aligned}$$

$$T = -3c_1c_2\alpha\beta^8 + 3c_1(c_2b^2+1)\alpha^3\beta^6 + 3c_1b^2\alpha^5\beta^4 + 12b^2\alpha^6\beta^3 - 6c_1b^4\alpha^7\beta^2 - 12b^4\alpha^8\beta.$$

Thus the equation (4.5) is rewritten in the form

$$4\alpha^2\beta^2 L\Omega^4\varepsilon I^2_{|i}y^i = 9M(U\gamma^2_{|i}y^i + V\alpha_{|i}y^i + W\beta_{|i}y^i + Xb^2_{|i}y^i), \tag{4.6}$$

where

$$\begin{aligned} V =\ & c_1^2c_2\beta^{11} - c_1^2(c_2b^2-3)\alpha^2\beta^9 + 20c_1c_2b^2\alpha^3\beta^8 - 13c_1^2b^2\alpha^4\beta^7 \\ & - 4c_1b^2(5c_2b^2-18)\alpha^5\beta^6 + 14c_1^2b^4\alpha^6\beta^5 - 32c_1b^2\alpha^7\beta^4 - 72c_1b^6\alpha^9\beta^2, \\ W =\ & -4c_1^2\alpha^3\beta^8 - 18c_1c_2b^2\alpha^4\beta^7 - 12c_1^2b^2\alpha^5\beta^6 + 6c_1b^2(3c_1c_2b^2-5)\alpha^6\beta^5 \\ & - 2c_1b^4(c_1-9)\alpha^7\beta^4 + 34c_1b^4\alpha^8\beta^3 - 8b^4\alpha^9\beta^2 - 4c_1b^6\alpha^{10}\beta + 8b^6\alpha^{11}, \\ X =\ & 6c_1c_2\alpha^4\beta^8 + 4c_1^2\alpha^5\beta^7 - 2c_1(3c_2b^2-1)\alpha^6\beta^6 - 4c_1^2b^2\alpha^7\beta^5 - 6c_1b^2\alpha^8\beta^4 \\ & - 8b^2\alpha^9\beta^3 + 4c_1b^4\alpha^{10}\beta^2 + 8b^4\alpha^{11}\beta. \end{aligned}$$

Consequently, the two-dimensional Finsler space F^2 with (3.1) is a Landsberg space, if and only if

$$U\gamma^2_{|i}y^i + V\alpha_{|i}y^i + W\beta_{|i}y^i + Xb^2_{|i}y^i = 0, \tag{4.7}$$

where $M \neq 0$. If $M = 0$, then $b^2 = 0$, namely, it is a contradiction.

By means of (2.5), (2.6), (2.7) and (2.8), the equation above is written as

$$\begin{aligned} & 2\beta(c_1\beta+2\alpha)(\alpha^2U+X)(r_0+s_0) + [(\alpha^2-c_2\beta^2)V + \beta(c_1\beta+2\alpha)W \\ & \qquad - 2\{c_1\beta^3 + (c_2b^2+2)\alpha\beta^2 - b^2\alpha^3\}U](r_{00} - 2b_rB^r) = 0. \end{aligned} \tag{4.8}$$

Substituting (4.2), U, V, W and X into (4.8), we obtain

$$\begin{aligned} & [2c_1^3c_2^2\alpha^3\beta^{14} + 2c_1^2c_2(c_1^2+6c_2)\alpha^4\beta^{13} + 20c_1^3c_2\alpha^5\beta^{12} \\ & + 2c_1^2(3c_1^2 - 2c_2^2b^2 + 8c_2)\alpha^6\beta^{11} + 2c_1(12c_2^2b^2 - 8c_1^2c_2b^2 + 5c_1^2)\alpha^7\beta^{10} \\ & + 4c_1^2(2c_2b^2 - 3c_1^2b^2 + 1)\alpha^8\beta^9 + 8c_1b^2(2c_2 - 3c_1^2 - 2c_2^2b^2)\alpha^9\beta^8 \\ & - 20c_1^2b^2(2c_2b^2+1)\alpha^{10}\beta^7 - 8c_1b^2(3c_1^2b^2 + 8c_2b^2 + 1)\alpha^{11}\beta^6 \\ & \qquad - 8b^4(9c_1^2 + 4c_2)\alpha^{12}\beta^5 - 80c_1b^4\alpha^{13}\beta^4 - 32b^4\alpha^{14}\beta^3](r_0+s_0) \\ & + [c_1^3c_2^2\alpha\beta^{15} - c_1^2c_2(5c_1^2 - 2c_2)\alpha^2\beta^{14} - c_1^3c_2(2c_2b^2 + c_2 - 6)\alpha^3\beta^{13} \\ & + 4c_1^2(4c_2^2b^2 - c_1^2 + 8c_2)\alpha^4\beta^{12} + c_1(-31c_1^2c_2b^2 + 40c_2^2b^2 + c_1^2c_2 - 3c_1^2)\alpha^5\beta^{11} \\ & - 2c_1^2(12c_2^2b^4 + 4c_1^2b^2 + 99c_2b^2 - c_1 - 1)\alpha^6\beta^{10} \\ & + c_1b^2(40c_1c_2b^2 - 48c_2^2b^2 - 16c_1^2 + 13c_1 - 272c_2 - 36)\alpha^7\beta^9 \end{aligned} \tag{4.9}$$

$$
\begin{aligned}
&+ 2c_1b^2\{c_1b^2(72c_2 - c_1^2 + 9c_1) + 67c_1 - 72\}\alpha^8\beta^8\\
&+ 4b^2\{c_1b^2(3c_1^2 + 18c_1 - 4c_2) + 54c_1 - 36\}\alpha^9\beta^7\\
&+ 8b^4(10c_1^2c_2b^2 + 14c_1^2 + 9c_1 - 40c_2)\alpha^{10}\beta^6\\
&+ 4c_1b^2\{(84c_2 + c_1^2)b^4 + 84b^2 - 2c_1\}\alpha^{11}\beta^5\\
&- 8b^2\{(7c_1^2 - 44c_2)b^4 - 44b^2 + 2c_1^2 + 2c_1\}\alpha^{12}\beta^4\\
&\qquad\qquad - 32c_1b^2(9b^4 + 1)\alpha^{13}\beta^3 - 320b^6\alpha^{14}\beta^2\big]r_{00}\\
+ \big[&- 2c_1^2c_2^3\alpha^2\beta^{15} + 10c_1^3c_2^2\alpha^3\beta^{14} + 2c_1^2c_2^2(2c_2b^2 + c_2 - 15)\alpha^4\beta^{13}\\
&- 2c_1c_2(32c_2^2b^2 + 3c_1^2)\alpha^5\beta^{12} + 2c_1^2c_2(29c_2b^2 - 3c_2 + 15)\alpha^6\beta^{11}\\
&+ 4c_1\{2c_2b^2(6c_2^2b^2 + 2c_1^2 + 34c_2 + 5) - c_1^2\}\alpha^7\beta^{10}\\
&- 2c_1\{c_2b^2(40c_1c_2b^2 + 13c_1 - 36) - c_1(-31c_2b^2 + c_2 + 1)\}\alpha^8\beta^9\\
&- 2b^2\{c_1c_2b^2(60c_2 - c_1^2 + 9c_1) + 8c_1^3 + 190c_1c_2 - 36c_2\}\alpha^9\beta^8\\
&+ 2b^2\{-4c_2b^2(-6c_1^2 + 9c_1 - 40c_2) + c_1(13c_1 - 16)\}\alpha^{10}\beta^7\\
&- 2b^2\{c_1b^2(40c_2^2b^2 + 4c_2 + c_1^2 - 9c_1) + 54c_1 - 36\}\alpha^{11}\beta^6\\
&- 8b^2\{c_2(44c_2 + c_1^2)b^4 + (84c_2 - 4c_1^2 - 9c_1)b^2 - 8b^2 - 2c_1c_2\}\alpha^{12}\beta^5\\
&+ 32c_1b^2(9c_2b^4 + 5b^2 + c_2)\alpha^{13}\beta^4 + 4b^2\{(84c_2 + c_1^2)b^4 + 44b^2 - 2c_1\}\alpha^{14}\beta^3\\
&\qquad\qquad - 32c_1b^2(4b^4 + 1)\alpha^{15}\beta^2 - 320b^6\alpha^{16}\beta\big]s_0\\
= 0.&
\end{aligned}
$$

Separating (4.9) in the rational and irrational terms with respect to (y^i), we have

$$
\begin{aligned}
(4.10)\quad &\{\alpha^4\beta^2D_1(r_0 + s_0) + \alpha^2\beta E_1r_{00} + 2\alpha^2F_1s_0\}\\
&\qquad + \alpha\{\alpha^2\beta^3D_2(r_0 + s_0) + \beta^2E_2r_{00} + 2\alpha^2\beta F_2s_0\} = 0,
\end{aligned}
$$

where

$$
\begin{aligned}
D_1 =&2c_1^2c_2(c_1^2 + 6c_2)\beta^{10} + 2c_1^2(3c_1^2 - 2c_2^2b^2 + 8c_2)\alpha^2\beta^8\\
&+ 4c_1^2(2c_2b^2 - 3c_1^2b^2 + 1)\alpha^4\beta^6 - 20c_1^2b^2(2c_2b^2 + 1)\alpha^6\beta^4\\
&- 8b^4(9c_1^2 + 4c_2)\alpha^8\beta^2 - 32b^4\alpha^{10},\\
D_2 =&2c_1^3c_2^2\beta^{10} + 20c_1^3c_2\alpha^2\beta^8 + 2c_1(12c_2^2b^2 - 8c_1^2c_2b^2 + 5c_1^2)\alpha^4\beta^6\\
&+ 8c_1b^2(2c_2 - 3c_1^2 - 2c_2^2b^2)\alpha^6\beta^4 - 8c_1b^2(3c_1^2b^2 + 8c_2b^2 + 1)\alpha^8\beta^2 - 80c_1b^4\alpha^{10},\\
E_1 =& - c_1^2c_2(5c_1^2 - 2c_2)\beta^{12} + 4c_1^2(4c_2^2b^2 - c_1^2 + 8c_2)\alpha^2\beta^{10}\\
&- 2c_1^2(12c_2^2b^4 + 4c_1^2b^2 + 99c_2b^2 - c_1 - 1)\alpha^4\beta^8
\end{aligned}
$$

$$
\begin{aligned}
&+ 2c_1b^2\{c_1b^2(72c_2 - c_1^2 + 9c_1) + 67c_1 - 72\}\alpha^6\beta^6 \\
&+ 8b^4(10c_1^2c_2b^2 + 14c_1^2 + 9c_1 - 40c_2)\alpha^8\beta^4 \\
&- 8b^2\{(7c_1^2 - 44c_2)b^4 - 44b^2 + 2c_1^2 + 2c_1\}\alpha^{10}\beta^2 - 320b^6\alpha^{12}, \\
E_2 =& c_1^3c_2^2\beta^{12} - c_1^3c_2(2c_2b^2 + c_2 - 6)\alpha^2\beta^{10} \\
&+ c_1(-31c_1^2c_2b^2 + 40c_2^2b^2 + c_1^2c_2 - 3c_1^2)\alpha^4\beta^8 \\
&+ c_1b^2(40c_1c_2b^2 - 48c_2^2b^2 - 16c_1^2 + 13c_1 - 272c_2 - 36)\alpha^6\beta^6 \\
&+ 4b^2\{c_1b^2(3c_1^2 + 18c_1 - 4c_2) + 54c_1 - 36\}\alpha^8\beta^4 \\
&+ 4c_1b^2\{(84c_2 + c_1^2)b^4 + 84b^2 - 2c_1\}\alpha^{10}\beta^2 - 32c_1b^2(9b^4 + 1)\alpha^{12}, \\
F_1 =& - c_1^2c_2^3\beta^{14} + c_1^2c_2^2(2c_2b^2 + c_2 - 15)\alpha^2\beta^{12} + c_1^2c_2(29c_2b^2 - 2c_2 + 15)\alpha^4\beta^{10} \\
&- c_1\{c_2b^2(40c_1c_2b^2 + 13c_1 - 36) - c_1(-31c_2b^2 + c_2 + 1)\}\alpha^6\beta^8 \\
&+ b^2\{-4c_2b^2(-6c_1^2 + 9c_1 + 40c_2) + c_1(13c_1 - 36)\}\alpha^8\beta^6 \\
&- 4b^2\{c_2(44c_2 + c_1^2)b^4 + (84c_2 - c_1^2 - 9c_1)b^2 - 2c_1c_2\}\alpha^{10}\beta^4 \\
&+ 4b^2\{(84c_2 + c_1^2)b^4 + 44b^2 - 2c_1\}\alpha^{12}\beta^2 - 160b^6\alpha^{14}, \\
F_2 =& 5c_1^3c_2^2\beta^{12} - c_1c_2(32c_2^2b^2 + 3c_1^2)\alpha^2\beta^{10} \\
&+ 2c_1\{2c_2b^2(6c_2^2b^2 + 2c_1^2 + 34c_2 + 5) - c_1^2\}\alpha^4\beta^8 \\
&- b^2\{c_1c_2b^2(60c_2 - c_1^2 + 9c_1) + 8c_1^3 + 190c_1c_2 - 36c_2\}\alpha^6\beta^6 \\
&- 2b^2\{c_1b^2(40c_2^2b^2 + 4c_2 + c_1^2 - 9c_1) + 54c_1 - 36\}\alpha^8\beta^4 \\
&+ 16c_1b^2(9c_2b^4 + 5b^2 + c_2)\alpha^{10}\beta^2 - 16c_1b^2(4b^4 + 1)\alpha^{12}.
\end{aligned}
$$

which yield two equations as follows:

$$\alpha^2\beta^2D_1(r_0 + s_0) + \beta E_1r_{00} + 2F_1s_0 = 0, \tag{4.11}$$

$$\alpha^2\beta^2D_2(r_0 + s_0) + \beta E_2r_{00} + 2\alpha^2F_2s_0 = 0. \tag{4.12}$$

From (4.12) we obtain

$$c_1^3c_2^2\beta^{13}r_{00} \equiv 0 \pmod{\alpha^2}. \tag{4.13}$$

If $c_2 \neq 0$, then there exists a function $f(x)$ such that $r_{00} = \alpha^2f(x)$. Thus we have

$$r_{ij} = a_{ij}f(x). \tag{4.13$'$}$$

Transvection by b^iy^j leads to

$$r_0 = \beta f(x); \quad r_j = b_jf(x). \tag{4.13$''$}$$

Elimination $(r_0 + s_0)$ from (4.11) and (4.12), from (4.13′) we have

$$f(x)\beta\alpha^2(D_2E_1 - D_1E_2) + 2(D_2F_1 - \alpha^2 D_1F_2)s_0 = 0. \tag{4.14}$$

From $\alpha^2 \not\equiv 0 \pmod{\beta}$ it follows that there exists a function $g(x)$ satisfying $s_0 = g\beta$. Hence (4.14) is reduced to

$$\alpha^2\{f(x)(D_2E_1 - D_1E_2) - 2g(x)D_1F_2\} + 2g(x)D_2F_1 = 0. \tag{4.14′}$$

Since only the term $-4c_1^5c_2^5g(x)\beta^{24}$ of $2g(x)D_2F_1$ seemingly does not contain α^2, we must have $hp(22)$ V_{22} such that $\beta^{24} = \alpha^2V_{22}$. Thus it is a contradiction because of $\alpha^2 \not\equiv 0 \pmod{\beta}$, that is, D_2F_1 does not contain α^2 as a factor. Hence from (4.14′) we have $g(x) = 0$, which leads to $s_0 = 0$ and $s_i = 0$. Further, substituting $g(x) = 0$ into (4.14′), we obtain

$$f(x)(D_2E_1 - D_1E_2) = 0. \tag{4.14″}$$

If $(D_2E_1 - D_1E_2) = 0$, then the term of $D_2E_1 - D_1E_2$ which does not contain α^2 as a factor is $-4c_1^5c_2^3(3c_1^2 + 2c_2)\beta^{22}$. If $3c_1^2 + 2c_2 \neq 0$, then there exists $hp(20)$ V_{20} such that $\beta^{22} = \alpha^2V_{20}$. From $\alpha^2 \not\equiv 0 \pmod{\beta}$ and $b^2 \neq 0$ we have $V_{22} = 0$. It is a contradiction, which leads to $D_2E_1 - D_1E_2 \neq 0$. Thus from (4.14″) we have $f(x) = 0$. From (4.13′) we get $r_{ij} = 0$.

In each exceptional case where $c_2 = 0$ or $3c_1^2 + 2c_2 = 0$, we have the same conclusion similarly.

Summarizing up, we obtain $r_{ij} = 0$ and $s_i = 0$, that is,

$$b_{i;j} + b_{j;i} = 0, \quad b^r b_{r;i} = 0. \tag{4.15}$$

Therefore $b_i(x)$ is the so-called Killing vector field with a constant length.

According to Hashiguchi, Hōjō & Matsumoto [4], the condition (4.15) is equivalent to $b_{i;j} = 0$. So we have

Theorem 4.1. *Let F^2 be a two-dimensional Finsler space with a special (α,β)-metric (3.1) satisfying $b^2 \neq 0$. If F^2 is a Landsberg space, then F^2 is a Berwald space.*

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