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AN EXTENSION OF THE FUGLEDGE-PUTNAM THEOREM TO w-HYPONORMAL OPERATORS

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ABSTRACT. The Fuglede-Putnam Theorem is that if A and B are normal operators and X is an operator such that $AX = XB$, then $A^*X = X^*B^*$. In this paper, we show that if A is w-hyponormal and B^* is invertible w-hyponormal such that $AX = XB$ for a Hilbert-Schmidt operator X, then $A^*X = XB^*$.

1. INTRODUCTION

Let $\mathcal{H}$ be a separable complex Hilbert space with inner product $(\, , \,)$ and $\mathcal{L}(\mathcal{H})$ denote the $*$-algebra of all bounded linear operators acting on $\mathcal{H}$. An operators T in $\mathcal{L}(\mathcal{H})$ is called normal $T^*T = TT^*$ and p-hyponormal if $(T^*T)^p \geq (TT^*)^p$, where $0 < p \leq 1$. In particular, 1-hyponormal is called hyponormal and $\frac{1}{2}$-hyponormal is called semi-hyponormal. The Löwner-Heinz inequality implies that if T is p-hyponormal, then it is q-hyponormal for any $0 < q \leq p$. Let $T = U|T|$ be the polar decomposition of T, where U is partial isometry, then $|T|$ is a positive square root of T^*T and $\ker T = \ker |T| = \ker U$.

Aluthge [1] introduced the operator $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$, which is called the Aluthge transform. An operator T in $\mathcal{L}(\mathcal{H})$ is said to be w-hyponormal if $|\widetilde{T}| \geq |T| \geq |\widetilde{T}^*|$. Evidently, if T is w-hyponormal, then $\widetilde{T}$ is semi-hyponormal. Aluthge & Wang [2, 3] proved that if an operator T in $\mathcal{L}(\mathcal{H})$ is p-hyponormal, then it is w-hyponormal and also show the following results:

Theorem 1.1 (Aluthge & Wang [2]). *If T is an invertible w-hyponormal operator, then so is T^{-1}.*

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Theorem 1.2 (Aluthge & Wang [2]). *Let T be a w-hyponormal operator. If $Tx = \lambda x, \lambda \neq 0$, then $T^*x = \bar{\lambda}x$.*

Theorem 1.3 (Aluthge & Wang [3]). *An operator T is w-hyponormal if and ony if*

$$|T| \geq (|T|^{\frac{1}{2}}|T^*||T|^{\frac{1}{2}})^{\frac{1}{2}} \quad and \quad |T^*| \leq (|T^*|^{\frac{1}{2}}|T||T^*|^{\frac{1}{2}})^{\frac{1}{2}}.$$

An operator T in $\mathcal{L}(\mathcal{H})$ is said to be M-hyponormal for $M \geq 0$ if $\|(T-zI)^*x\| \leq M\|(T-zI)x\|$ for all complex numbers z and for all x in $\mathcal{H}$, and dominant if $R(T-zI) \subset R(T^* - \bar{z}I)$ for all complex numbers z, where $R(T)$ is the range of an operator T.

The following inclusions are well-known and proper (*cf.* Aluthge & Wang [2], Patel [12], Stampfli & Wadahwa [13], Wadahwa [14] and Xia [15]).

$$\begin{aligned} \text{Normal} &\subset \text{Hyponormal} \subset p-\text{Hyponormal for } \frac{1}{2} < p < 1 \\ &\subset \text{semihyponormal} \subset p-\text{Hyponormal for } 0 < p < \frac{1}{2} \\ &\subset w-\text{Hyponormal}, \\ \text{Hyponormal} &\subset M-\text{hyponormal} \subset \text{Dominant}. \end{aligned}$$

The well known Fugledge-Putnam theorem asserts that if A and B are normal operators and $AX = XB$ for some operator X, then $A^*X = XB^*$(Conway [6], Halmos [8]). In past years several authors have extended this theorem for non-normal operators (*cf.* Berberian [4], Chō & Huruya [5], Duggal [7], Moore, Rogers & Trent [9], Radjabalipour [11], Patel [12] and Stampfli & Wadahwa [13]).

Duggal [7] has extended the result by assuming A and B^* to be dominant and M-hyponormal respectively. However the result fails to hold in case when A is M-hyponomal and B^* is dominant.

Berberian [4] has extended the result by assuming A and B^* are hyponormal and X is a Hilbert-Schmidt operator. Recently, Chō & Huruya [5] have extended the result by assuming A, B^* and X to be p-hyponormal, invertible p-hyponormal and Hilbert-Schmidt respectively and also Patel [12] has extended the result by assuming A and B^* are p-hyponormals which have reducing normal part.

In this paper, we show that the p-hyponormality of A and B^* in the result of Chō & Huruya can be replaced by the w-hyponormality.

2. The Main Theorem

Let T be an operator in $\mathcal{L}(\mathcal{H})$ and suppose that $\{e_n\}$ is an orthonormal basis for $\mathcal{H}$.

We define the Hilbert -Schmidt norm of T to be

$$\|T\|_2 = (\sum_{n=1}^{\infty} \|Te_n\|^2)^{\frac{1}{2}}.$$

This definition is independent of the choice of basis (*cf.* Conway [6], Murphy [10]). If $\|T\|_2 < \infty$, T is said to be a Hilbert-Schmidt operator and we denote the set of all Hilbert-Schmidt operators in $\mathcal{L}(\mathcal{H})$ by $B_2(\mathcal{H})$.

Let $B_1(\mathcal{H})$ be the set $\{C = AB \mid A, B \in B_2(\mathcal{H})\}$. Then operators belonging to be $B_1(\mathcal{H})$ are called trace class operators.

We define a linear functional

$$\mathrm{tr} : B_1(\mathcal{H}) \to \mathbb{C}$$

by $\mathrm{tr}(C) = \sum_{n=1}^{\infty}(Ce_n, e_n)$ for an orthonormal basis $\{e_n\}$ for $\mathcal{H}$.

In this case, the definition of $tr(C)$ does not depend on the choice of an orthonormal basis and $tr(C)$ is called the trace of C. Then we know the followings:

Theorem 2.1 (Conway [6], Murphy [10]). *We have the following properties.*

a) *The set $B_2(\mathcal{H})$ is self-adjoint ideal of $\mathcal{L}(\mathcal{H})$.*

b) *If $(A, B) = \sum_{n=1}^{\infty}(Ae_n, Be_n) = tr(B^*A) = \mathrm{tr}(AB^*)$ for A and B in $B_2(\mathcal{H})$ and for any orthonormal basis $\{e_n\}$ for $\mathcal{H}$, then $(\cdot,\cdot)$ is an inner product on $B_2(\mathcal{H})$ and $B_2(\mathcal{H})$ ia a Hilbert space with respect to this inner product.*

Theorem 2.2 (Conway [6], Murphy [10]). *If $T \in \mathcal{L}(\mathcal{H})$ and $A \in B_2(\mathcal{H})$, then $\|A\| \le \|A\|_2 = \|A^*\|_2$, $\|TA\|_2 \le \|T\|\|A\|_2$ and $\|AT\|_2 \le \|A\|_2\|T\|$.*

For each pair operators A, B in $B_2(\mathcal{H})$, there is an operator $\mathcal{J}$ defined on $B_2(\mathcal{H})$ via the formula $\mathcal{J}X = AXB$, which is due to Berberian [4]. Evidently, by the above Theorem 2.1 and Theorem 2.2, $\|\mathcal{J}\| \le \|A\|\|B\|$ and the adjoint of $\mathcal{J}$ is given by the formula $\mathcal{J}^*X = A^*XB^*$, as one sees from the calculation $(\mathcal{J}^*X, Y) = (X, \mathcal{J}Y) = (X, AYB) = \mathrm{tr}((AYB)^*X) = \mathrm{tr}(XB^*Y^*A^*) = \mathrm{tr}(A^*XB^*Y^*) = (A^*XB^*, Y)$. If $A \ge 0$ and $B \ge 0$, then also $\mathcal{J} \ge 0$ and $\mathcal{J}^{\frac{1}{2}}X = A^{\frac{1}{2}}XB^{\frac{1}{2}}$ because of

$$\begin{aligned}(\mathcal{J}X, X) &= \mathrm{tr}(AXBX^*) = \mathrm{tr}(A^{\frac{1}{2}}XBX^*A^{\frac{1}{2}}) \\ &= \mathrm{tr}((A^{\frac{1}{2}}XB^{\frac{1}{2}})(A^{\frac{1}{2}}XB^{\frac{1}{2}})^*) \ge 0.\end{aligned}$$

Lemma 2.3. *If A and B^* are w-hyponormal operators, then the operator $\mathcal{J}$ in $\mathcal{L}(B_2(\mathcal{H}))$ defined by $\mathcal{J}X = AXB$ is also w-hyponormal.*

Proof. Since $\mathcal{J}^*\mathcal{J}X = A^*AXBB^*$ and $\mathcal{J}\mathcal{J}^*X = AA^*XB^*B$ for any operator X in $B_2(\mathcal{H})$, we get $|\mathcal{J}|X = |A|X|B^*|$ and $|\mathcal{J}^*|X = |A^*|X|B|$ and so, $|\mathcal{J}|^{\frac{1}{2}}X = |A|^{\frac{1}{2}}X|B^*|^{\frac{1}{2}}$ and $|\mathcal{J}^*|^{\frac{1}{2}}X = |A^*|^{\frac{1}{2}}X|B|^{\frac{1}{2}}$. Thus, we have

$$|\mathcal{J}|^{\frac{1}{2}}|\mathcal{J}^*|\,|\mathcal{J}|^{\frac{1}{2}}X = |A|^{\frac{1}{2}}|A^*|\,|A|^{\frac{1}{2}}X|B^*|^{\frac{1}{2}}|B|\,|B^*|^{\frac{1}{2}}$$

and

$$|\mathcal{J}^*|^{\frac{1}{2}}|\mathcal{J}|\,|\mathcal{J}^*|^{\frac{1}{2}}X = |A^*|^{\frac{1}{2}}|A|\,|A^*|^{\frac{1}{2}}X|B|^{\frac{1}{2}}|B^*|\,|B|^{\frac{1}{2}}$$

and hence,

$$(|\mathcal{J}|^{\frac{1}{2}}|\mathcal{J}^*|\,|\mathcal{J}|^{\frac{1}{2}})^{\frac{1}{2}}X = (|A|^{\frac{1}{2}}|A^*|\,|A|^{\frac{1}{2}})^{\frac{1}{2}}X(|B^*|^{\frac{1}{2}}|B|\,|B^*|^{\frac{1}{2}})^{\frac{1}{2}}$$

and

$$(|\mathcal{J}^*|^{\frac{1}{2}}|\mathcal{J}|\,|\mathcal{J}^*|^{\frac{1}{2}})^{\frac{1}{2}}X = (|A^*|^{\frac{1}{2}}|A|\,|A^*|^{\frac{1}{2}})^{\frac{1}{2}}X(|B|^{\frac{1}{2}}|B^*|\,|B|^{\frac{1}{2}})^{\frac{1}{2}}.$$

Since operator A and B^* are w-hyponormal, by Theorem 1.3, we obtain

$$(|\mathcal{J}|^{\frac{1}{2}}|\mathcal{J}^*|\,|\mathcal{J}|^{\frac{1}{2}})^{\frac{1}{2}}X \leq |A|X|B^*| = |J|X$$

and

$$(|\mathcal{J}^*|^{\frac{1}{2}}|\mathcal{J}|\,|\mathcal{J}^*|^{\frac{1}{2}})^{\frac{1}{2}}X \geq |A^*|X|B| = |J^*|X,$$

which completes the proof. □

Theorem 2.4. *If A is w-hyponormal and B^* is invertible w-hyponormal such that $AX = XB$ for any operator X in $B_2(\mathcal{H})$, then $AX^* = XB^*$*

Proof. Let $\mathcal{J}$ be the operator on $B_2(\mathcal{H})$ defined by $\mathcal{J}X = AYB^{-1}$. Since $(B^*)^{-1} = (B^{-1})^*$ is w-hyponormal by Theorem 1.1, by Lemma 2.3, $\mathcal{J}$ is also w-hyponormal. The hypothesis $AX = XB$ implies $\mathcal{J}X = AXB^{-1} = X$ and so, by Theorem 1.2, $J^*X = X$. Hence we have $A^*X(B^{-1})^* = J^*X = X$. Therefore, $A^*X = XB^*$ which is the desired relation. □

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