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ON THE FEKETE-SZEGÖ PROBLEM FOR CERTAIN ANALYTIC FUNCTIONS

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ABSTRACT. Let $\mathcal{CS}_\alpha(\beta)$ denote the class of normalized strongly α-close-to-convex functions of order β, defined in the open unit disk $\mathcal{U}$ of $\mathbb{C}$ by

$$\left|\arg\left\{(1-\alpha)\frac{f(z)}{g(z)}+\alpha\frac{zf'(z)}{g(z)}\right\}\right| \le \frac{\pi}{2}\beta \ \ (\alpha,\beta\ge 0),$$

such that $g \in \mathcal{S}^*$, the class of normalized starlike unctions. In this paper, we obtain the sharp Fekete-Szegö inequalities for functions belonging to $\mathcal{CS}_\alpha(\beta)$.

1. INTRODUCTION

Let $\mathcal{S}$ denote the class of analytic functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are univalent in the open unit disk $\mathcal{U} = \{z \subset \mathbb{C} : |z| < 1\}$ & let $\mathcal{S}^*$ be the subclass of $\mathcal{S}$ consisting of all starlike functions. A classical theorem of Fekete and Szegö [4] states that, for $f \in \mathcal{S}$ given by (1.1),

$$|a_3 - \mu a_2^2| \le \begin{cases} 3-4\mu & \text{if } \ \mu \le 0, \\ 1+2e^{-2\mu/(1-\mu)} & \text{if } \ 0 \le \mu < 1, \\ 4\mu - 3 & \text{if } \ \mu \ge 1, \end{cases}$$

The inequality is sharp in the sense that for each μ, there exists a function in $\mathcal{S}$ such that equality holds. There are also several results of this type in the literature. Various interesting developments involving the Fekete-Szegö problem can be found in Abdel-Gawad & Thomas [1], Keogh & Merkes [7] and London [8].

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We denote by $\mathcal{K}(\beta)$ the class of strongly close-to-convex functions of order β. Thus $f \in \mathcal{K}(\beta)$ if and only if there exists $g \in \mathcal{S}^*$ such that

$$\left|\arg \frac{zf'(z)}{g(z)}\right| \leq \frac{\pi}{2}\beta \quad (\beta \geq 0;\ z \in \mathcal{U}).$$

For $0 \leq \beta \leq 1$, the class $\mathcal{K}(\beta)$ is a subclass of close-to-convex functions introduced by Kaplan [6] and hence contains only univalent functions. However, Goodman [5] showed that $\mathcal{K}(\beta)$ can contain functions with infinite valence for $\beta > 1$. The Fekete-Szegö problems for $\mathcal{K}(1)$ and $\mathcal{K}(\beta)$, respectively, have been solved by Keogh & Merkes [7] and London [8], respectively.

We now introduce a new class which covers the class $\mathcal{K}(\beta)$ as follows:

Definition. A function $f \in \mathcal{S}$, given by (1.1) is said to be strongly α-close-to-convex of order β if there exists a function $g \in \mathcal{S}^*$ such that

$$\left|\arg\left\{(1-\alpha)\frac{f(z)}{g(z)} + \alpha\frac{zf'(z)}{g(z)}\right\}\right| \leq \frac{\pi}{2}\beta \quad (\alpha, \beta \geq 0;\ z \in \mathcal{U}). \tag{1.2}$$

We denote by $\mathcal{CS}_\alpha(\beta)$ the class of strongly α-close-to-convex functions of order β. We note that $\mathcal{CS}_0(1) = \mathcal{CS}$, the class of close-to-star functions introduced by Reade [10] and $\mathcal{CS}_1(\beta) = \mathcal{K}(\beta)$.

The purpose of the present paper is to prove the sharp Fekete-Szegö inequalities for the functions belonging to the class $\mathcal{CS}_\alpha(\beta)$.

2. Main Results

Theorem. *Let $f \in \mathcal{CS}_\alpha(\beta)$ and be given by* (1.1). *Then for $\alpha, \beta \geq 0$, we have*

$$2(1+2\alpha)|a_3 - \mu a_2^2|$$
$$\leq \begin{cases} 1 + \dfrac{2(1+\beta)^2((1+\alpha)^2 - 2(1+2\alpha)\mu)}{(1+\alpha)^2} & \text{if } \mu \leq \frac{\beta(1+\alpha)^2}{2(1+\beta)(1+2\alpha)}, \\[2ex] 1 + 2\beta + \dfrac{2((1+\alpha)^2 - 2(1+2\alpha)\mu)}{(1+\alpha)^2 - \beta((1+\alpha)^2 - 2(1+2\alpha)\mu)} & \text{if } \frac{\beta(1+\alpha)^2}{2(1+\beta)(1+2\alpha)} \leq \mu \leq \frac{(1+\alpha)^2}{2(1+2\alpha)}, \\[2ex] 1 + 2\beta & \text{if } \frac{(1+\alpha)^2}{2(1+2\alpha)} \leq \mu \leq \frac{(\beta+2)(1+\alpha)^2}{2(\beta+1)(1+2\alpha)}, \\[2ex] -1 + \dfrac{2(1+\beta)^2(2(1+2\alpha)\mu - (1+\alpha)^2)}{(1+\alpha)^2} & \text{if } \mu \geq \frac{(\beta+2)(1+\alpha)^2}{2(\beta+1)(1+2\alpha)}. \end{cases}$$

For each μ, there is a function in $\mathcal{CS}_\alpha(\beta)$ such that equality holds in all cases.

To prove above Theorem, we need the following.

Lemma. *Let p be analytic in $\mathcal{U}$ and satisfying* $\operatorname{Re} p(z) > 0$ *for* $z \in \mathcal{U}$, *with* $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$. *Then*

$$| p_n | \leq 2 \tag{2.1}$$

and

$$\left| p_2 - \frac{p_1^2}{2} \right| \leq 2 - \frac{|p_1|^2}{2}. \tag{2.2}$$

The inequality (2.1) can be first proved by Carathéodory [2] (also, see Duren [3], p. 41) and the inequality (2.2) can be found in [Pommerenke [9], p. 166].

Proof of **Theorem**. Let $f \in \mathcal{CS}_\alpha(\beta)$. Then it follows from (1.2) that we may write

$$(1-\alpha)\frac{f(z)}{g(z)} + \alpha\frac{zf'(z)}{g(z)} = p^\beta(z), \tag{2.3}$$

where g is starlike and p has positive real part. Let $g(z) = z + b_2 z^2 + b_3 z^3 + \cdots$, and let $p(z)$ be given as in Lemma. Then by equating coefficients of both side of (2.3), we obtain

$$(1+\alpha)a_2 = b_2 + \beta p_1$$

and

$$(1+2\alpha)a_3 = b_3 + \beta p_1 b_2 + \frac{\beta(\beta-1)}{2}p_1^2 + \beta p_2.$$

So, with

$$x = \frac{(1+\alpha)^2 - 2(1+2\alpha)\mu}{(1+\alpha)^2},$$

we have

$$(1+2\alpha)(a_3 - \mu a_2^2) = b_3 + \frac{1}{2}(x-1)b_2^2 + \beta(p_2 + \frac{1}{2}(\beta x - 1)p_1^2) + \beta x p_1 b_2. \tag{2.4}$$

Since rotations of f also belong to $\mathcal{CS}_\alpha(\beta)$, without loss of generality, we may assume that $a_3 - \mu a_2^2$ is positive. Thus we now estimate $\operatorname{Re}(a_3 - \mu a_2^2)$.

Since $g \in \mathcal{S}^*$, there exists $h(z) = 1 + k_1 z + k_2 z^2 + \cdots$ ($|z| < 1$) with positive real part such that $zg'(z) = g(z)h(z)$, and so equating coefficients, we have $b_2 = k_1$ and $b_3 = (k_2 + k_1^2)/2$. Hence, by Lemma,

$$\begin{aligned}\operatorname{Re}\left(b_3 + \frac{1}{2}(x-1)b_2^2\right) &= \frac{1}{2}\operatorname{Re}\left(k_2 - \frac{1}{2}k_1^2\right) + \frac{1+2x}{4}\operatorname{Re}\ k_1^2 \\ &\leq 1 - \rho^2 + (1+2x)\rho^2\cos 2\phi,\end{aligned} \tag{2.5}$$

where $b_2 = k_1 = 2\rho e^{i\phi}$ for some ρ in [0,1]. We also have

$$\begin{aligned}\operatorname{Re}\left(p_2+\frac{1}{2}(\beta x-1)p_1^2\right) &= \operatorname{Re}\left(p_2-\frac{1}{2}p_1^2\right)+\frac{1}{2}\beta x \operatorname{Re}\, p_1^2 \\ &\le 2(1-r^2)+2\beta x r^2\cos 2\theta,\end{aligned} \tag{2.6}$$

where $p_1 = 2re^{i\theta}$ for some r in [0,1]. From (2.4), (2.5) and (2.6), we obtain

$$\begin{aligned}&\operatorname{Re}(1+2\alpha)(a_3-\mu a_2^2)\\ &\le 1-\rho^2+(1+2x)\rho^2\cos 2\phi+2\beta(1-r^2+r^2\beta x\cos 2\theta)+4\beta x r\rho\cos(\theta+\phi),\end{aligned} \tag{2.7}$$

and we now proceed to maximize the right-hand side of (2.7). This function will be denote $\psi(x)$ whenever all parameters except x are held constant.

At first, we assume that

$$\frac{\beta(1+\alpha)^2}{2(1+\beta)(1+2\alpha)} \le \mu \le \frac{(1+\alpha)^2}{2(1+2\alpha)},$$

so that $0 \le x \le 1/(1+\beta)$. Since the expression $-t^2+t^2\beta x\cos 2\theta+2xt$ is the largest when $t = x/1-\beta x\cos 2\theta$, we have

$$-t^2+t^2\beta x\cos 2\theta+2xt \le \frac{x^2}{1-\beta x\cos 2\theta} \le \frac{x^2}{1-\beta x}.$$

Thus

$$\begin{aligned}\psi(x) &\le 1+2x+2\beta\left(1+\frac{x^2}{1-\beta x}\right)\\ &= 1+2\beta+\frac{2((1+\alpha)^2-2(1+2\alpha)\mu)}{(1+\alpha)^2-\beta((1+\alpha)^2-2(1+2\alpha)\mu)}\end{aligned}$$

and with (2.7) this establishes the second inequality in the theorem. Equality occurs only if

$$p_1 = \frac{2((1+\alpha)^2-2(1+2\alpha)\mu)}{(1+\alpha)^2-\beta((1+\alpha)^2-2(1+2\alpha)\mu)},\ p_2 = 2,\ b_2 = 2,\ b_3 = 3,$$

and the corresponding function f is defined by

$$(1-\alpha)f(z)+\alpha z f'(z) = \frac{z}{(1-z)^2}\left(\lambda\frac{1+z}{1-z}+(1-\lambda)\frac{1-z}{1+z}\right)^{\beta},$$

where

$$\lambda = \frac{(1+\alpha)^2+(1-\beta)((1+\alpha)^2-2(1+2\alpha)\mu)}{2((1+\alpha)^2-\beta((1+\alpha)^2-2(1+2\alpha)\mu))}.$$

We now prove the first inequity. Let

$$\mu \le \frac{\beta(1+\alpha)^2}{2(\beta+1)(1+2\alpha)},$$

so that $x \geq 1/(1+\beta)$. With $x_0 = 1/(1+\beta)$, we have

$$\begin{aligned}\psi(x) &= \psi(x_0) + 2(x - x_0)(\rho^2 \cos 2\phi + \beta^2 r^2 \cos 2\theta + 2\rho\beta r \cos(\theta + \phi)) \\ &\leq \psi(x_0) + 2(x - x_0)(1+\beta)^2 \\ &\leq 1 + \frac{2(1+\beta)^2\big((1+\alpha)^2 - 2(1+2\alpha)\mu\big)}{(1+\alpha)^2},\end{aligned}$$

as required. Equality occurs only if $p_1 = p_2 = 2$, $b_2 = 2$, $b_3 = 3$, and the corresponding function f is defined by

$$(1-\alpha)f(z) + \alpha z f'(z) = \frac{z}{(1-z)^2}\left(\frac{1+z}{1-z}\right)^{\beta}.$$

Let $x_1 = -1/(1+\beta)$. We shall find that $\psi(x_1) \leq 1 + 2\beta$, and the remaining inequalities follow easily from this one. By an argument similar to the one above, we obtain

$$\begin{aligned}\psi(x) &\leq \psi(x_1) + 2|x - x_1|(1+\beta)^2 \\ &\leq -1 + \frac{2(1+\beta)^2(2(1+2\alpha)\mu - (1+\alpha)^2)}{(1+\alpha)^2},\end{aligned}$$

if $x \leq x_1$, that is,

$$\mu \geq \frac{(\beta+2)(1+\alpha)^2}{2(\beta+1)(1+2\alpha)}.$$

Equality occurs only if $p_1 = 2i$, $p_2 = -2$, $b_2 = 2i$, $b_3 = -3$, and the corresponding function f is defined by

$$(1-\alpha)f(z) + \alpha z f'(z) = \frac{z}{(1-iz)^2}\left(\frac{1+iz}{1-iz}\right)^{\beta}.$$

Also, for $0 \leq \lambda \leq 1$,

$$\begin{aligned}\psi(\lambda x_1) &= \lambda\psi(x_1) + (1-\lambda)\psi(0) \\ &\leq \lambda(1+2\beta) + (1-\lambda)(1+2\beta) = 1 + 2\beta,\end{aligned}$$

so $\psi(x) \leq 1 + 2\beta$ for $x_1 \leq x \leq 0$, *i. e.*,

$$\frac{(1+\alpha)^2}{2(1+2\alpha)} \leq \mu \leq \frac{(\beta+2)(1+\alpha)^2}{2(\beta+1)(1+2\alpha)}.$$

Equality occurs only if $p_1 = b_2 = 0$, $p_2 = 2$, $b_3 = 1$, and the corresponding function f is defined by

$$(1-\alpha)f(z) + \alpha z f'(z) = \frac{z(1+z^2)^{\beta}}{(1-z^2)^{1+\beta}}.$$

We now show that $\psi(x_1) \le 1+2\beta$. Since

$$
\begin{aligned}
&-(1-\beta x_1\cos 2\theta)t^2 + 2x_1\rho\cos(\theta+\phi)t \\
&\qquad = (1-\beta x_1\cos 2\theta)\left(t - \frac{x_1\rho\cos(\theta+\phi)}{1-\beta x_1\cos 2\theta}\right)^2 + \frac{x_1^2\rho^2\cos^2(\theta+\phi)}{1-\beta x_1\cos 2\theta}
\end{aligned}
$$

and

$$1-\beta x_1\cos 2\theta = 1+\frac{\beta}{1+\beta}\cos 2\theta \ge 1-\frac{\beta}{1+\beta} \ge 0,$$

we have

$$\psi(x_1)-(1+2\beta) \le \rho^2\left(-1+(1+2x_1)\cos 2\phi + \frac{\beta x_1^2(1+\cos 2(\theta+\phi))}{1-\beta x_1\cos 2\theta}\right).$$

Thus we consider the inequality

$$\beta x_1^2(1+\cos 2(\theta+\phi)) + (1-\beta x_1\cos 2\theta)(-1+(1+2x)\cos 2\phi) \le 0.$$

After some simplifications, this becomes

$$\beta^2(\cos 2\phi - 1)(\cos 2\theta+1) - \beta(1+\cos 2\theta + \sin 2\theta\sin 2\phi) - 1 - \cos 2\phi \le 0,$$

which is true if

$$2\beta^2\sin^2\phi\cos^2\phi + 2\beta\cos\theta\sin\theta\cos\phi\sin\phi + \cos^2\phi \ge 0. \tag{2.8}$$

Now, for all real t,

$$2t^2 + 2t\sin\theta\cos\phi + \cos^2\phi \ge 0,$$

so, by taking $t=\beta\sin\phi\cos\theta$, we obtain (2.8). This completes the proof of Theorem.

For the case $\alpha = 0$ in Theorem, we have the following. □

Corollary. *Let $f \in \mathcal{CS}_0(\beta)$ and be given by* (1.1). *Then for $\beta \ge 0$, we have*

$$|a_3-\mu a_2^2| \le \begin{cases} 1+2(1+\beta)^2(1-2\mu) & \text{if } \mu \le \dfrac{\beta}{2(1+\beta)}, \\ 1+2\beta+\dfrac{2(1-2\mu)}{1-\beta(1-2\mu)} & \text{if } \dfrac{\beta}{2(1+\beta)} \le \mu \le \dfrac{1}{2}, \\ 1+2\beta & \text{if } \frac{1}{2} \le \mu \le \dfrac{2+\beta}{2(1+\beta)}, \\ -1+\dfrac{2(1+\beta)^2(2(1+2\alpha)\mu-(1+3\alpha))}{(1+\alpha)^2} & \text{if } \mu \ge \dfrac{2+\beta}{2(1+\beta)}. \end{cases}$$

For each μ, there is a function in $\mathcal{CS}_0(\beta)$ such that equality holds in all cases.

Remark. (i) Putting $\alpha = \beta = 1$ in Theorem, we have the result by Keogh & Merkes [7].

(ii) Taking $\alpha = 1$ in Theorem, we obtain the corresponding results by Abdel-Gawad & Thomas [1] and London [8].

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