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SOME RESULTS CONCERNING (θ, φ)-DERIVATIONS ON PRIME RINGS

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ABSTRACT. Let R be a prime ring with characteristic different from two and let $\theta, \varphi, \sigma, \tau$ be the automorphisms of R. Let $d : R \to R$ be a nonzero (θ, φ)-derivation. We prove the following results: (i) if $a \in R$ and $[d(R), a]_{\theta o \sigma, \varphi o \tau} = 0$, then $\sigma(a) + \tau(a) \in Z$, the center of R, (ii) if $d([R, a]_{\sigma,\tau}) = 0$, then $\sigma(a) + \tau(a) \in Z$, (iii) if $[ad(x), x]_{\sigma,\tau} = 0$ for all $x \in R$, then $a = 0$ or R is commutative.

1. INTRODUCTION

Throughout, R will represent an associative ring, and Z will be its center. Let $x, y \in R$. As usual, the commutator $xy - yx$ will be denoted by $[x, y]$. Let $\theta, \varphi, \sigma, \tau : R \to R$ be automorphisms. We write $[x, y]_{\sigma,\tau}$ for $x\sigma(y) - \tau(y)x$, and will make extensive use of the following basic commutator identities: $[xy, z] = x[y, z] + [x, z]y$, $[xy, z]_{\sigma,\tau} = x[y, z]_{\sigma,\tau} + [x, \tau(z)]y = x[y, \sigma(z)] + [x, z]_{\sigma,\tau}y$, $[x, yz]_{\sigma,\tau} = [x, y]_{\sigma,\tau}\sigma(z) + \tau(y)[x, z]_{\sigma,\tau}$. An additive mapping $d : R \to R$ is called a (θ, φ)-derivation if

$$d(xy) = d(x)\theta(y) + \varphi(x)d(y) \quad \text{for all} \quad x, y \in R.$$

A $(1, 1)$-derivation is called simply a derivation, where $1 : R \to R$ is the identity map on R. A derivation d is inner if there exists an $a \in R$ such that $d(x) = [x, a]$ for all $x \in R$. For subsets A and B of R, let $[A, B]$ (*resp.* $[A, B]_{\sigma,\tau}$) be the additive subgroup generated by $[a, b]$ (*resp.* $[a, b]_{\sigma,\tau}$) for all $a \in A$ and $b \in B$. We recall that a Lie ideal L is an additive subgroup of R such that $[R, L] \subset L$.

In Kaya, Gölbaşi & Adyın [2], firsty introduced the generalized Lie ideal as following: Let U be an additive subgroup of R. Then

(i) U is a (σ, τ)-right Lie ideal of R if $[U, R]_{\sigma,\tau} \subset U$.
(ii) U is a (σ, τ)-left Lie ideal of R if $[R, U]_{\sigma,\tau} \subset U$.

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(iii) U is a (σ,τ)-Lie ideal of R if U is both a (σ,τ)-left Lie ideal and a (σ,τ)-right Lie ideal of R. Now every Lie ideal of R is a $(1,1)$-Lie ideal of R.

Let R be a prime ring with characteristic different from two and let $a \in R$. Let $d: R \to R$ be a nonzero derivation. Herstein [1], proved that if $[d(R),a]=0$, then $a \in Z$. Lee & Lee [3], proved that if $[d(R),d(R)] \subset Z$, then R is commutative. It is our main goal in this note to improve their results to (θ,φ)-derivations.

2. Results

We shall need the following lemma which is similar to Posner [4, Lemma 1].

Lemma 1. *Let R be a prime ring and let d be a (θ,φ)-derivation of R. Suppose that either (i) $ad(x)=0$, $x \in R$ or (ii) $d(x)a=0$, $x \in R$ holds. Then we have either $a=0$ or $d=0$.*

Proof. Suppose that

$$ad(x)=0 \text{ for all } x \in R.$$

Replacing x by xy in this relation, we obtain

$$ad(x)\theta(y)+a\varphi(x)d(y)=0 \text{ for all } x,y \in R,$$

which is reduced to

$$a\varphi(x)d(y)=0 \text{ for all } x,y \in R.$$

Since φ is an automorphism of R, we see that

$$azd(y)=0 \text{ for all } y,z \in R.$$

By primeness of R, we have either $a=0$ or $d=0$. The proof of (ii) is similar to the one of (i).

Let us start our investigation with the following result. □

Theorem 2. *Let R be a prime ring with characteristic different from two and let d be a nonzero (θ,φ)-derivation of R. If $a \in R$ and $[d(R),a]_{\theta\circ\sigma,\varphi\circ\tau}=0$, then $\sigma(a)+\tau(a) \in Z$.*

Proof. If $a \in Z$, then the proof of the theorem is obvious. So we assume that $a \notin Z$. By hypothesis, for all $x \in R$ we have,

$$\begin{aligned} 0 &= [d(x\sigma(a)), a]_{\theta o\sigma,\varphi o\tau} \\ &= [d(x)\theta(\sigma(a)) + \varphi(x)d(\sigma(a)), a]_{\theta o\sigma,\varphi o\tau} \\ &= d(x)[\theta(\sigma(a)),\ \theta(\sigma(a))] + [d(x), a]_{\theta o\sigma,\varphi o\tau}\theta(\sigma(a)) \\ &\quad + \varphi(x)[d(\sigma(a)), a]_{\theta o\sigma,\varphi o\tau} + [\varphi(x), \varphi(\tau(a))]d(\sigma(a)). \end{aligned}$$

Hence we obtain

(1) $$[\varphi(x), \varphi(\tau(a))]d(\sigma(a)) = 0 \text{ for all } x \in R.$$

Since φ is an automorphism of R, relation (1) can be written as

(2) $$[z, \varphi(\tau(a))]d(\sigma(a)) = 0 \text{ for all } z \in R.$$

Replacing z by zy in (2) and using (2), we get

$$[z, \varphi(\tau(a))]yd(\sigma(a)) = 0 \text{ for all } y, z \in R.$$

Since R is prime and $a \notin Z$, we obtain $d(\sigma(a)) = 0$.

Let us substitute $x\tau(a)$ for x in the hypothesis. Then for all $x \in R$ we have,

$$\begin{aligned} 0 &= [d(x\tau(a)), a]_{\theta o\sigma,\varphi o\tau} \\ &= [d(x)\theta(\tau(a)) + \varphi(x)d(\tau(a)), a]_{\theta o\sigma,\varphi o\tau} \\ &= d(x)[\theta(\tau(a)), \theta(\sigma(a))] + [d(x), a]_{\theta o\sigma,\varphi o\tau}\theta(\tau(a)) \\ &\quad + \varphi(x)[d(\tau(a)), a]_{\theta o\sigma,\varphi o\tau} + [\varphi(x), \varphi(\tau(a))]d(\tau(a)), \end{aligned}$$

whence

(3) $$d(x)[\theta(\tau(a)), \theta(\sigma(a))] + [\varphi(x), \varphi(\tau(a))]d(\tau(a)) = 0 \text{ for all } x \in R.$$

Since τ is an automorphism of R, relation (3) can be restated as

(4) $$d(x)[\theta(\tau(a)), \theta(\sigma(a))] + [z, \varphi(\tau(a))]d(\tau(a)) = 0 \text{ for all } x, z \in R.$$

Putting $x = \sigma(a)$ in (4), we get

(5) $$[z, \varphi(\tau(a))]d(\tau(a)) = 0 \text{ for all } z \in R,$$

and replacing z by zy in (5) and using (5) yield

$$[z, \varphi(\tau(a))]yd(\tau(a)) = 0 \text{ for all } y, z \in R.$$

The primeness of R and $a \notin Z$ force $d(\tau(a)) = 0$.

Now we see that for all $x \in R$,

$$(6) \qquad \begin{aligned} d([x,a]_{\sigma,\tau}) &= d(x\sigma(a) - \tau(a)x) \\ &= d(x)\theta(\sigma(a)) + \varphi(x)d(\sigma(a)) - d(\tau(a))\theta(x) - \varphi(\tau(a))d(x) \\ &= [d(x),a]_{\theta\circ\sigma,\varphi\circ\tau} = 0. \end{aligned}$$

We can use the hypothesis and (6) to obtain, for all $x, y \in R$,

$$\begin{aligned} 0 &= [d(x[y,a]_{\sigma,\tau}),a]_{\theta\circ\sigma,\varphi\circ\tau} \\ &= [d(x)\theta([y,a]_{\sigma,\tau}) + \varphi(x)d([y,a]_{\sigma,\tau}),a]_{\theta\circ\sigma,\varphi\circ\tau} \\ &= [d(x)\theta([y,a]_{\sigma,\tau}),a]_{\theta\circ\sigma,\varphi\circ\tau} \\ &= d(x)[\theta([y,a]_{\sigma,\tau}),\ \theta(\sigma(a))] + [d(x),a]_{\theta\circ\sigma,\varphi\circ\tau}\theta([y,a]_{\sigma,\tau}) \\ &= d(x)[\theta([y,a]_{\sigma,\tau}),\ \theta(\sigma(a))], \end{aligned}$$

and by invoking Lemma 1 and considering that θ is an automorphism of R, we arrive at

$$[\theta([y,a]_{\sigma,\tau}),\theta(\sigma(a))] = 0 \ \text{ for all } \ y \in R,$$

and so

$$(7) \qquad [[y,a]_{\sigma,\tau},\sigma(a)] = 0 \ \text{ for all } \ y \in R,$$

which on substitution of $\tau(a)y$ for y in (7) yields

$$\begin{aligned} 0 &= [[\tau(a)y,a]_{\sigma,\tau},\sigma(a)] = [\tau(a)[y,a]_{\sigma,\tau} + [\tau(a),\tau(a)]y,\sigma(a)] \\ &= [\tau(a)[y,a]_{\sigma,\tau},\sigma(a)] = \tau(a)[[y,a]_{\sigma,\tau},\sigma(a)] + [\tau(a),\sigma(a)][y,a]_{\sigma,\tau}. \end{aligned}$$

Thus relation (7) gives

$$(8) \qquad [\tau(a),\sigma(a)][y,a]_{\sigma,\tau} = 0 \ \text{ for all } \ y \in R.$$

Replacing y by yz in (8) and using (8), we have

$$[\tau(a),\sigma(a)]y[z,\sigma(a)] = 0 \ \text{ for all } \ y,\ z \in R,$$

and so the primeness of R and $a \notin Z$ guarantee

$$(9) \qquad [\tau(a),\sigma(a)] = 0.$$

Now, expanding (7) and utilizing (9) yield

$$(10) \qquad [[y,\sigma(a)],a]_{\sigma,\tau} = 0 \ \text{ for all } \ y \in R.$$

Indeed,

$$\begin{aligned}0 &= [[y,a]_{\sigma,\tau},\sigma(a)] = [y\sigma(a)-\tau(a)y,\sigma(a)]\\ &= [y,\sigma(a)]\sigma(a)-\tau(a)[y,\sigma(a)] = [[y,\sigma(a)],a]_{\sigma,\tau}.\end{aligned}$$

Let us write in (10) yz instead of y, thereby obtaining, for all $y,z\in R$,

$$\begin{aligned}0 &= [[yz,\sigma(a)],a]_{\sigma,\tau}\\ &= [[y,\sigma(a)]z+y[z,\sigma(a)],a]_{\sigma,\tau}\\ &= [y,\sigma(a)][z,\sigma(a)]+[[y,\sigma(a)],a]_{\sigma,\tau}z\\ &\quad + y[[z,\sigma(a)],a]_{\sigma,\tau}+[y,\tau(a)][z,\sigma(a)]\\ &= [y,\sigma(a)][z,\sigma(a)]+[y,\tau(a)][z,\sigma(a)],\end{aligned}$$

that is,

$$[y,\sigma(a)+\tau(a)][z,\sigma(a)] = 0 \quad \text{for all} \quad y,z\in R.$$

Since the mapping $z\mapsto [z,\ \sigma(a)]$ is a nonzero derivation of the prime ring R, we can use Posner [4, Lemma 1] to conclude that $\sigma(a)+\tau(a)\in Z$. □

Corollary 3. *Let R be a prime ring with characteristic different from two, let d be a nonzero (θ,φ)-derivation of R and let U be a (σ,τ)-left Lie ideal. If $[d(R),\ U]_{\theta\circ\sigma,\varphi\circ\tau} = 0$, then $\sigma(u)+\tau(u)\in Z$ for all $u\in U$.*

We here establish an example to support Theorem 2.

Example. Consider the prime ring

$$R = \left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix} :\ a,\ b,\ c,\ d\in I\right\},$$

where I is the set of integers.

$$\sigma\begin{pmatrix} a & b\\ c & d\end{pmatrix} = \begin{pmatrix} d & -c\\ -b & a\end{pmatrix},\ \tau\begin{pmatrix} a & b\\ c & d\end{pmatrix} = \begin{pmatrix} a & -b\\ -c & d\end{pmatrix}$$

be two automorphisms of R and $a = \begin{pmatrix} 1 & 0\\ 0 & 2\end{pmatrix}\notin Z$. If we define $d: R\to R$ by

$$d\begin{pmatrix} a & b\\ c & d\end{pmatrix} = \begin{pmatrix} 0 & s-b\\ c & 0\end{pmatrix},$$

then d is a $(1,1)$-derivation of R such that $d([R,a]_{\sigma,\tau}) = 0$, but $\sigma(a)+\tau(a)\in Z$.

Theorem 4. *Let R be a prime ring with characteristic different from two and let d be a nonzero (θ,φ)-derivation of R. If $a\in R$ and $d([R,a]_{\sigma,\tau}) = 0$, then $\sigma(a)+\tau(a)\in Z$.*

Proof. If $a \in Z$, then it is clear that $\sigma(a)+\tau(a) \in Z$. We now let $a \notin Z$. From the hypothesis, for all $x \in R$we have,

$$0 = d([\tau(a)x, a]_{\sigma,\tau}) = d(\tau(a)[x,a]_{\sigma,\tau}) = d(\tau(a))\theta([x,a]_{\sigma,\tau}) + \varphi(\tau(a))d([x,a]_{\sigma,\tau}),$$

and so,

$$d(\tau(a))\theta([x,a]_{\sigma,\tau}) = 0 \text{ for all } x \in R. \tag{11}$$

Substituting xy for x in (11), we obtain for any $x, y \in R$,

$$\begin{aligned} 0 &= d(\tau(a))\theta([xy,a]_{\sigma,\tau}) \\ &= d(\tau(a))\theta(x[y,\sigma(a)] + [x,a]_{\sigma,\tau}y) \\ &= d(\tau(a))\theta(x)\theta([y,\sigma(a)]) + d(\tau(a))\theta([x,a]_{\sigma,\tau})\theta(y), \end{aligned}$$

and use (11) in order to get

$$d(\tau(a))\theta(x)\theta([y,\sigma(a)]) = 0 \text{ for all } x, y \in R,$$

from which it follows that

$$d(\tau(a))z\theta([y,\sigma(a)]) = 0 \text{ for all } y, z \in R$$

because θ is an automorphism of R. Since R is prime and $a \notin Z$, we know that $d(\tau(a)) = 0$. We claim that $d(\sigma(a)) = 0$. Replacing x by $x\sigma(a)$ in the hypothesis shows that for all $x \in R$,

$$\begin{aligned} 0 &= d([x\sigma(a),a]_{\sigma,\tau}) = d([x,a]_{\sigma,\tau}\sigma(a)) \\ &= d([x,a]_{\sigma,\tau})\theta(\sigma(a)) + \varphi([x,a]_{\sigma,\tau})d(\sigma(a)), \end{aligned}$$

and hence,

$$\varphi([x,a]_{\sigma,\tau})d(\sigma(a)) = 0 \text{ for all } x \in R. \tag{12}$$

If we substitute xy for x in (12), for all $x, y \in R$ we obtain,

$$\begin{aligned} 0 &= \varphi\big([xy,a]_{\sigma,\tau}d(\sigma(a))\big) \\ &= \varphi\big(x[y,a]_{\sigma,\tau} + [x,\tau(a)]y\big)d(\sigma(a)) \\ &= \varphi(x)\varphi([y,a]_{\sigma,\tau})d(\sigma(a)) + \varphi([x,\tau(a)])\varphi(y)d(\sigma(a)), \end{aligned}$$

thus, the relation (12) gives

$$\varphi([x,\tau(a)])\varphi(y)d(\sigma(a)) = 0 \text{ for all } x, y \in R.$$

We can use the fact that φ is an automorphism of R to get

$$\varphi([x,\tau(a)])zd(\sigma(a)) = 0 \text{ for all } x, z \in R,$$

and the primeness of R and $a \notin Z$ yield $d(\sigma(a)) = 0$, as claimed.

According to the hypothesis, we now have, for any $x \in R$,

$$\begin{aligned} 0 &= d([x,a]_{\sigma,\tau}) = d(x\sigma(a) - \tau(a)x) \\ &= d(x)\theta(\sigma(a)) + \varphi(x)d(\sigma(a)) - d(\tau(a))\theta(x) - \varphi(\tau(a))d(x), \end{aligned}$$

which, in conjunction with $d(\sigma(a)) = 0$ and $d(\tau(a)) = 0$, gives

$$[d(x), a]_{\theta\circ\sigma,\varphi\circ\tau} = 0 \quad \text{for all} \quad x \in R.$$

Hence we conclude from Theorem 2 that $\sigma(a) + \tau(a) \in Z$. □

Corollary 5. *Let R be a prime ring with characteristic different from two, let d be a nonzero (θ,φ)-derivation of R and let U be a (σ,τ)-left Lie ideal. If $d([R,U])_{\sigma,\tau} = 0$, then $\sigma(u) + \tau(u) \in Z$ for all $u \in U$.*

Theorem 6. *Let R be a prime ring and let d be a nonzero (θ,φ)-derivation of R. If $a \in R$ and $[ad(x), x]_{\sigma,\tau} = 0$ for all $x \in R$, then $a = 0$ or R is commutative.*

Proof. Linearizing our hypothesis, we get

$$[ad(x), y]_{\sigma,\tau} + [ad(y), x]_{\sigma,\tau} = 0 \quad \text{for all} \quad x, y \in R. \tag{13}$$

Replacing y by yx in (13), we have for all $x, y \in R$,

$$\begin{aligned} 0 &= [ad(x), yx]_{\sigma,\tau} + [ad(y)\theta(x) + a\varphi(y)d(x), x]_{\sigma,\tau} \\ &= [ad(x), y]_{\sigma,\tau}\sigma(x) + \tau(y)[ad(x), x]_{\sigma,\tau} + ad(y)[\theta(x), \sigma(x)] \\ &\qquad + [ad(y), x]_{\sigma,\tau}\theta(x) + a\varphi(y)[d(x), x]_{\sigma,\tau} + [a\varphi(y), \tau(x)]d(x). \end{aligned}$$

Since σ and θ are the automorphisms of R, the last relation implies that for all $w,\ x,\ y,\ z \in R$

$$\begin{aligned} 0 &= [ad(x), y]_{\sigma,\tau}z + \tau(y)[ad(x), x]_{\sigma,\tau} + ad(y)[w, z] \\ &\qquad + [ad(y), x]_{\sigma,\tau}w + a\varphi(y)[d(x), x]_{\sigma,\tau} + [a\varphi(y), \tau(x)]d(x). \end{aligned}$$

In particular, for all $x, y, z \in R$ we have,

$$\begin{aligned} 0 &= [ad(x), y]_{\sigma,\tau}z + \tau(y)[ad(x), x]_{\sigma,\tau} + ad(y)[z, z] \\ &\qquad + [ad(y), x]_{\sigma,\tau}z + a\varphi(y)[d(x), x]_{\sigma,\tau} + [a\varphi(y), \tau(x)]d(x). \end{aligned}$$

By hypothesis and (13), this relation is reduced to

$$a\varphi(y)[d(x), x]_{\sigma,\tau} + [a\varphi(y), \tau(x)]d(x) = 0 \quad \text{for all} \quad x, y \in R,$$

which gives, for all $x, y \in R$,

$$a\varphi(y)[d(x), x]_{\sigma,\tau} + a[\varphi(y), \tau(x)]d(x) + [a, \tau(x)]\varphi(y)d(x) = 0. \tag{14}$$

Taking $\varphi^{-1}(a)y$ for y in (14), for all $x, y \in R$ we get

$$a^2\varphi(y)[d(x), x]_{\sigma,\tau} + a^2[\varphi(y), \tau(x)]d(x) \\ + a[a, \tau(x)]\varphi(y)d(x) + [a, \tau(x)]a\varphi(y)d(x) = 0,$$

which leads to, in view of (14),

$$[a, \tau(x)]a\varphi(y)d(x) = 0 \quad \text{for all} \quad x, y \in R.$$

Since R is prime and φ is an automorphism of R, we see that for all $x \in R$, either $d(x) = 0$ or $[a, \tau(x)]a = 0$. That is, R is the union of its additive subgroups $\{x \in R : d(x) = 0\}$ and $\{x \in R : [a, \tau(x)]a = 0\}$.

Since a group cannot be the union of two proper subgroups and d is nonzero, it follows that $[a, \tau(x)]a = 0$ for all $x \in R$. But then Posner [4, Lemma 1] implies that $a \in Z$, which gives $a[d(x), x]_{\sigma,\tau} = 0$ since $a[d(x), x]_{\sigma,\tau} + [a, \tau(x)]d(x) = [ad(x), x]_{\sigma,\tau} = 0$ for all $x \in R$. Hence relation (14) is reduced to $a[\varphi(y), \tau(x)]d(x) = 0$ for all $x, y \in R$, and so also $aw[\varphi(y), \tau(x)]d(x) = 0$ for all $w, x, y \in R$. Therefore we have either $[\varphi(y), \tau(x)]d(x) = 0$ or $a = 0$ for all $x, y \in R$.

In the first case, by putting yz instead of y, we get $[\varphi(y), \tau(x)]\varphi(z)d(x) = 0$ for all $x, y, z \in R$, and so, $[u, \tau(x)]rd(x) = 0$ for all $r, u, x, y \in R$ because φ is an automorphism of R. Again using the fact that a group cannot be the union of two proper subgroups, we see that either $d = 0$ or R is commutative. But d is nonzero, and so it follows that R is commutative. □

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