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TWO FORMS OF THE AXIOM OF CHOICE AND WELL-ORDERING IN AN ELEMENTARY TOPOS

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ABSTRACT. The purpose of this paper is to show the relationships between three forms of the axiom of choice and well-ordering in an elementary topos.

1. INTRODUCTION

In the topos *Set*, the *axiom of choice* can be expressed as the following three equivalent forms of (AC1), (AC2) and (AC3).

(AC1) Every epimorphism is a retraction (*cf.* Goldblatt [2]).

(AC2) For any noninitial object A and $f : A \to B$, there exists a morphism $g : B \to A$ such that $f \circ g \circ f = f$ (*cf.* Johnstone [3]).

(AC3) For any noninitial object A, there exists $\sigma : \Omega^A \to A$ such that for all $f : 1 \to \Omega^A$, we have $\sigma \circ f \in f'$ where $f' : A' \to A$ is a monomorphism, provided that $ev \circ (f \times i_A)$ is not the characteristic morphism of $0 \to A$ (*cf.* Penk [7]).

Penk [7] showed that (AC2) is equivalent to (AC3) in a topos with the terminal object as a generator. Also Mawanda [6] showed that (AC1) is equivalent to (WO) in a Boolean topos. In this paper, we show that (WO) implies (AC2) in a topos in which every object is an abstract retract and the reverse holds in a Boolean topos. Also we show that (WO) implies (AC3) in a bivalent topos and the reverse holds in a Boolean topos.

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2. PRELIMINARIES

In this section, we state some definitions and properties which will serve as the basic tools for the arguments used to prove our results.

Definition 2.1. An *elementary topos* is a category $\mathcal{E}$ that satisfies the following;

(T1) $\mathcal{E}$ is finitely complete,
(T2) $\mathcal{E}$ has exponentiation,
(T3) $\mathcal{E}$ has a subobject classifier.

Definition 2.2. A topos $\mathcal{E}$ is called *bivalent* if $\Omega = \{\top, \bot\}$.

Definition 2.3. A topos $\mathcal{E}$ is called *Boolean* if for every object D in $\mathcal{E}$, $(\mathrm{Sub}(D), \in)$ is a Boolean algebra where $\mathrm{Sub}(D)$ is the class of monomorphisms with common codomain D, and we say $g \in f$ if there exists a morphism $h : B \to A$ such that $f \circ h = g$ where $f : A \to D$ and $g : B \to D$ are monomorphisms.

Proposition 2.4. *For any topos $\mathcal{E}$, the following statements are equivalent;*

(1) *$\mathcal{E}$ is Boolean,*
(2) $\mathrm{Sub}(\Omega)$ *is a Boolean algebra,*
(3) $\top : 1 \to \Omega$ *has a complement in* $\mathrm{Sub}(\Omega)$,
(4) $\bot : 1 \to \Omega$ *is the complement of* $\top$ *in* $\mathrm{Sub}(\Omega)$,
(5) $\top \cup \bot \simeq 1_\Omega$ *in* $\mathrm{Sub}(\Omega)$,
(6) *$\mathcal{E}$ is classical,*
(7) $i_1 : 1 \to 1 + 1$ *is a subobject classifier.*

For the proof see Goldblatt [2].

Example 2.5. The category M-Set is a non-Boolean topos. For the proof see Goldblatt [2], Madanshekaf & tavakoli [5] and Ebahimi & Mahmoudi [1].

Definition 2.6. A topos is called *well-pointed* if it satisfies the extensionality principle for morphism, *i. e.*, if $f, g : A \to B$ are a pair of distinct parallel morphisms, then there is an element $a : 1 \to A$ of A such that $f \circ a \neq g \circ a$.

Lemma 2.7. *In a well-pointed topos $\mathcal{E}$, every object is an absolute retract.*

Proof. Since $\mathcal{E}$ is well-pointed, for any monomorphism $m : A \to B$ there exists a monomorphism $-m : -A \to B$ such that the following square

$$\begin{array}{ccc} 0 & \longrightarrow & A \\ \downarrow & & \downarrow m \\ -A & \xrightarrow[-m]{} & B \end{array}$$

is a pushout.

For $i_A : A \to A$ and $a \circ ! : -A \to A$ where $! : -A \to 1$ and $a : 1 \to A$, the following square

$$\begin{array}{ccc} 0 & \longrightarrow & A \\ \downarrow & & \downarrow i_A \\ -A & \xrightarrow[a\circ !]{} & A \end{array}$$

commutes.

Thus, there exist unique morphism $t : B \to A$ such that $t \circ m = i_A$. □

Definition 2.8. A topos $\mathcal{E}$ satisfies *well-ordering* (WO) if every noninitial object of $\mathcal{E}$ has an ordering with minimal choice, *i. e.*, for any P and $q : U \to \Omega^P$, if there exist $\alpha : V \to U$ and $p : V \to P$ such that $(q\alpha, p)$ factors through $\in_P \to \Omega^P \times P$, then there exists $\alpha_0 : V_0 \to U$ and $p_0 : V_0 \to P$ such that $(q\alpha_0, p_0)$ factors through $\in_P$, and such that for all $\beta : W \to V_0$ and all $p_1 : W \to P$, if $(q\alpha_0\beta, p_1)$ factors through $\in_P$, then $(p_0\beta, p_1)$ factors through a monomorphism $P_1 \to P \times P$.

3. Main part

Theorem 3.1. *If every object is an absolute retract in a topos $\mathcal{E}$, then* (WO) *implies* (AC2).

Proof. Let $f : A \to B$ be a morphism in $\mathcal{E}$, then there exist an epimorpmism $e : A \to X$ and a monomorphism $m : X \to B$ such that $f = m \circ e$. By hypothesis, there exists a morphism $t : B \to X$ such that $t \circ m = i_X$. We only show that there is a morphism $s : X \to A$ such that $f = f \circ (s \circ t) \circ f = f$. Since $e : A \to X$ is an epimorphism, there is a morphism $q : X \to \Omega^A$ which is the interpretation of the term $\{a | e(a) = x\}$. By definition of (WO), we can find an epimorphism $r : V \to X$ and a morphism $n : V \to A$ such that n is a minimal choice of qr. Since every epimorphism is a coequalizer, there are morphisms $u, v : W \to V$ such that the following square

$$\begin{array}{ccc} W & \xrightarrow{u} & V \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle r} \\ V & \xrightarrow[r]{} & X \end{array}$$

commutes.

Thus we get $q \circ r \circ v = q \circ r \circ u$. Also nu, nv are both minimal choice of $q \circ r \circ v = q \circ r \circ u$. By definition of (WO), we can find $nu = nv$. Since every epimorphism is a coequalizer, there is a morphism $s : X \to A$ such that $s \circ r = n$. Also there is a morphism $c : X \to \in_A$ such that $k \circ s = c$ where $k : A \to \in_A$ and $\in_A$ is the subobject classified by $ev : \Omega^A \times A \to \Omega$. Then we have $(q, s) = l \circ c = l \circ k \circ s = (q \circ e \circ s, s)$ where $l : \in_A \to \Omega^A \times A$. Since q is a monomorphism, we have that $f = f \circ (s \circ t) \circ f = f$. □

Theorem 3.2. *In a Boolean topos $\mathcal{E}$,* (AC2) *implies* (WO).

Proof. Let X_0 be a noninitial object in $\mathcal{E}$. Since $\mathcal{E}$ satisfies (AC2), there is a morphism $\psi : NX_0 \to X_0$ such that $\psi \circ g_i \in g_i'$ where NX_0 is the object of noninitial subobjects of X_0 with the usual ordering, $g_i : U \to NX_0$ is a morphism and $g_i' : X_0' \to X_0$ is a monomorphism. Since $\mathcal{E}$ is Boolean, we get that $-(\psi \circ g_0) \equiv g_1$ where the pullback of $\psi \circ g_0$ and $-(\psi \circ g_0)$ is the initial object, $-(\psi \circ g_1) \equiv g_2$ where the pullback of $\psi \circ g_1$ and $-(\psi \circ g_1)$ is the initial object, etc. Generally, we get that $-(\psi \circ g_{n-1}) \equiv g_n$ where the pullback of $\psi \circ g_{n-1}$ and $-(\psi \circ g_{n-1})$ is the initial object.

Thus we construct $\phi : X_0 \to NX_0$ such that $\mathrm{Im}(\phi)$ is the subobject of NX_0 consisting of $g_0, -(\psi \circ g_0), -(\psi \circ g_1), \cdots$ and $-(\psi \circ g_{m-1})$, where $-g_m$ is an initial object, and $\psi \circ \phi = i_{X_0}$. Then $\mathrm{Im}(\phi)$ is linearly ordered with minimal choice. Since ϕ is a monomorphism, X_0 has an ordering with minimal choice. □

Theorem 3.3. *In a bivalent topos $\mathcal{E}$,* (WO) *implies* (AC3).

Proof. For a product object $\Omega^A \times \Omega$ together with two projections $q_1 : \Omega^A \times \Omega \to \Omega^A$ and $q_2 : \Omega^A \times \Omega \to \Omega$ in $\mathcal{E}$, there are morphisms $\langle p_1, ev \rangle : \Omega^A \times A \to \Omega^A \times \Omega$ and $\langle i_{\Omega^A}, t \rangle : \Omega^A \to \Omega^A \times \Omega$ where $t : \Omega^A \to \Omega$ is a morphism such that $t \circ d = \top$ for all $d : 1 \to \Omega^A$ and $p_1 : \Omega^A \times A \to \Omega^A$ is a projection. By bivalency, $\langle p_1, ev \rangle$ is an epimorphism. Since every epimorphism has a right inverse, there is a morphism $h : \Omega^A \times \Omega \to \Omega^A \times A$ such that $\langle p_1, ev \rangle \circ h = i_{\Omega^A \times \Omega}$. We construct a morphism $p_2 \circ h \circ \langle i_{\Omega^A}, t \rangle : \Omega^A \to A$ where $p_2 : \Omega^A \times A \to A$. We show that $p_2 \circ h \circ \langle i_{\Omega^A}, t \rangle \circ c \in$

k where $c : 1 \to \Omega^A$ is any morphism and $k : B \to A$ is a monomorphism. Since, for some $\langle d, a \rangle : 1 \to \Omega^A \times A$,

$$h \circ \langle i_{\Omega^A}, t \rangle \circ c = h \circ \langle c, \top \rangle = \langle d, a \rangle : 1 \to \Omega^A \times A,$$

it yields $d = p_1 \circ \langle d, a \rangle = p_1 \circ h \circ \langle c, \top \rangle = q_1 \circ \langle p_1, ev \rangle \circ h \circ \langle c, \top \rangle = q_1 \circ \langle c, \top \rangle = c$. We only show that $a \in k$. Since $ev \circ \langle c, a \rangle = ev \circ h \circ \langle c, \top \rangle = ev \circ h \circ \langle p_1, ev \rangle \circ \langle c, k \circ b \rangle = q_2 \circ \langle p_1, ev \rangle \circ h \circ \langle p_1, ev \rangle \circ \langle c, k \circ b \rangle = ev \circ \langle c, k \circ b \rangle$ where $b : 1 \to B$, it implies $a \in k$. Therefore it turns out that $p_2 \circ h \circ \langle i_{\Omega^A}, t \rangle \circ c = p_2 \circ \langle d, a \rangle = a \in k$. □

Theorem 3.4. *In a Boolean topos $\mathcal{E}$,* (AC3) *implies* (WO).

Proof. Since (AC1) is equivalent to (WO) in a Boolean topos (*cf.* Mawanda [6]), we only claim that (AC3) implies (AC1) in a Boolean topos. Let $e : X \to Y$ be an epimorphism and $x : 1 \to X$ be any morphism in $\mathcal{E}$. Since (AC3) holds in $\mathcal{E}$, we construct the following morphism

$$\sigma \circ \Omega^e \circ \{\} : Y \to X$$

where $\{\} : Y \to \Omega^Y$, $\Omega^e : \Omega^Y \to \Omega^X$ and $\sigma : \Omega^X \to X$.

We claim that $e \circ \sigma \circ \Omega^e \circ \{\} \circ e \circ x = e \circ x$. Since $e \circ x : 1 \to Y$ is a monomorphism, the terminal object 1 is a pullback of the $\top : 1 \to \Omega$ and $\chi_{e\circ x} : Y \to \Omega$, and V is a pullback of $\top : 1 \to \Omega$ and $\chi_{e\circ x} \circ e : X \to \Omega$ where $k : V \to X$ is a monomorphism.

$$\begin{array}{ccccc} V & \xrightarrow{\ !\ } & 1 & \xrightarrow{\ i_1\ } & 1 \\ {\scriptstyle k}\downarrow & & {\scriptstyle e\circ x}\downarrow & & \downarrow{\scriptstyle \top} \\ X & \xrightarrow[e]{} & Y & \xrightarrow[\chi_{e\circ x}]{} & \Omega \end{array}$$

By (AC3), for any $\sigma \circ \Omega^e \circ \{\} \circ e \circ x : 1 \to X$ where $\sigma : \Omega^X \to X$ and $\Omega^e \circ \{\} \circ e \circ x : 1 \to \Omega^X$, there exists a morphism $t : 1 \to V$ such that $k \circ t = \sigma \circ \Omega^e \circ \{\} \circ e \circ x$. And by the property of pullback, the left square of the above diagram is also a pullback square. Hence it yields

$$e \circ \sigma \circ \Omega^e \circ \{\} \circ e \circ x = e \circ k \circ t = e \circ x \circ ! \circ t = e \circ x \circ i_1 = e \circ x.$$

Since 1 is a generator and e is an epimorphism, we get $e \circ \sigma \circ \Omega^e \circ \{\} = i_Y$. □

Corollary 3.5. *In a well-pointed topos $\mathcal{E}$,* (AC2), (AC3) *and* (WO) *are equivalent.*

Proof. By Lemma 2.7, Theorem 3.1, Theorem 3.2, Theorem 3.3 and Theorem 3.4. □

Remark. Investigate the relationships between the forms of the axiom of choices and well-ordering in a weak topos which has a weak subobject classifier.

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