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A NEW INTERPRETATION OF SUBHYPERGROUPS OF A HYPERGROUP

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ABSTRACT. In this paper we present a new and natural interpretation of subhypergroups in a partially ordered algebra. Then we study their connection with corresponding crisp concepts through their newly defined Q-cuts. The theorems proved also highly generalized the existing ones.

1. BASIC DEFINITIONS

We will be concerned primarily with a basic non-empty set H, elements of which are denoted by $x, y, z, \ldots$. A hyperoperation $\circ$ on H is a mapping of $H \times H$ into the family of non-empty subsets of H. If $(x, y) \in H \times H$, its image under $\circ$ is denoted by $x \circ y$. If $A, B \subseteq H$ then $A \circ B$ is given by $A \circ B = \cup\{x \circ y | \ x \in A, \ y \in B\}$; $x \circ A$ is used for $\{x\} \circ A$ and $A \circ x$ for $A \circ \{x\}$.

Let $\circ$ be a hyperoperation on H then $(H, \circ)$ is called a *hypergroupoid.* A hypergroup is a hypergroupoid $(H, \circ)$, that satisfies:

1) $x \circ (y \circ z) = (x \circ y) \circ z$ for all $x, y, z \in H$,
2) $x \circ H = H \circ x = H$ for all $x \in H$.

The second condition is frequently used in the form: Given $x, y \in H$, there exist $u, v \in H$ such that $y \in x \circ u$ and $y \in v \circ x$. A non-empty subset K of a hypergroup $(H, \circ)$ is called a *subhypergroup* if $x \circ K = K \circ x = K$ for all $x \in K$. A comprehensive review of the theory of hyperstructures appears in Corsini [1] and Vougiouklis [7].

Let $P = (P, *, 1, \leq)$ be a partially ordered algebra. Therefore $(P, *)$ is a monoid, where 1 is the unity for $*$, and $*$ is isotone in both variables; and $(P, \leq)$ is a complete

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lattice, *i. e.*, $\leq$ is a partial order on P such that for any $S \subseteq P$, infimum and supremum of S exist and these will be denoted by $\bigwedge_{s \in S}\{s\}$ and $\bigvee_{s \in S}\{s\}$, respectively. Forther on P always denotes such a structure. Let X be a non-empty set. A map $\mu : X \longrightarrow P$ is called a *P-subset of X* (*cf.* Filep [5]). If P is the unit interval $[0,1]$, then μ is a fuzzy set. The notion of fuzzy set was formulated by Zadeh [8]. Recall that a subset Q of an ordered set $(P, \leq)$ is called a *right segment of* $(P, \leq)$ if and only if

$$\forall q \in Q,\ \forall p \in P:\ (q \leq p \Longrightarrow p \in Q).$$

Clearly any closed interval $[p, 1]$ in P is a right segment of $(P, \leq)$.

2. P-SUBHYPERGROUPS AND P-STRONGLY REGULAR RELATIONS

In Davvaz [2], we introduced the concept of fuzzy subhypergroup of a hypergroup which is a generalization of fuzzy subgroups (*cf.* Rosenfeld [6, Definition]). In Davvaz [4], we introduced the concept of an interval-valued fuzzy subhypergroup of a hypergroup which is an extended notion of a fuzzy subhypergroup. In fact, we replaced $D[0,1]$, the family of all closed subintervals of $[0,1]$, instead of unit interval $[0,1]$. Now, we unify and generalize these definitions.

Definition 1. Let H be a hypergroup and μ a P-subset of H. Then μ is called a *P-subhypergroup of H*, if

1) $\mu(x) * \mu(y) \leq \bigwedge_{z \in x \circ y}\{\mu(z)\}$ for all $x, y \in H$,
2) for all $x, a \in H$ there exists $y \in H$ such that $x \in a \circ y$ and

$$\mu(a) * \mu(x) \leq \mu(y),$$

3) for all $x, a \in H$ there exists $z \in H$ such that $x \in z \circ a$ and

$$\mu(x) * \mu(a) \leq \mu(z).$$

Definition 1 is a generalization of Davvaz ([2, Definition 1] and [4, Definition 3.1]).

For example, consider $H = \{e, a, b\}$ and define $\circ$ on H with the help of the following table:

$\circ$	e	a	b
e	e	H	H
a	H	a	a
b	H	a	$\{a, b\}$

Then $(H, \circ)$ is a hypergroup. Suppose $(P, *, 1, \leq)$ is a partially ordered algebra such that P is a lattice and $*$ = meet. Now, we define $\mu : H \longrightarrow P$ by $\mu(a) = \mu(b) \leq \mu(e)$. Then μ is a P-subhypergroup of H.

Definition 2. Let Q be a right segment of P. Then by the Q-cut μ_Q of some P-subset μ of X we mean the following subset of X:

$$\mu_Q = \{x \in X |\ \mu(x) \in Q\}.$$

Theorem 3. *Let H be a hypergroup and μ a P-subset of H. If each non-empty Q-cut μ_Q of μ is a subhypergroup of H, then μ is a P-subhypergroup of H.*

Proof. Take any element $x, y \in H$ and consider the right segment of P as follows:

$$Q = [\mu(x) * \mu(y), 1].$$

Since

$$\mu(x) * \mu(y) \leq \mu(x) * 1 = \mu(x) \leq 1,$$
$$\mu(x) * \mu(y) \leq 1 * \mu(y) = \mu(y) \leq 1,$$

we get $\mu(x), \mu(y) \in Q$, therefore $x, y \in \mu_Q$. Since μ_Q is a subhypergroup of H, hence for every $z \in x \circ y$ we have $z \in \mu_Q$ and so $\mu(z) \in Q$. Therefore $\mu(x) * \mu(y) \leq \mu(z)$ which implies

$$\mu(x) * \mu(y) \leq \bigwedge_{z \in x \circ y} \{\mu(z)\},$$

and in this way the condition (1) of Definition 1 is verified. To verify the second condition, if $x, a \in H$, we consider the right segment of P as follows:

$$Q = [\mu(a) * \mu(x), 1].$$

Then $\mu(a), \mu(x) \in Q$, and so $x, a \in \mu_Q$. Hence there exists $y \in \mu_Q$ such that $x \in a \circ y$. Since $y \in \mu_Q$, we get $\mu(a) * \mu(x) \leq \mu(y)$. In the similar way the third condition of Definition 1 is valid. □

Theorem 4. *Let H be a hypergroup, μ a P-subhypergroup of H, and Q a right segment of P. If Q is closed under $*$, then μ_Q is a subhypergroup of H.*

Proof. Consider a right segment Q satisfying the given condition. Then for any elements $x, y \in \mu_Q$ we have $\mu(x), \mu(y) \in Q$, and so $\mu(x) * \mu(y) \in Q$. For every $z \in x \circ y$, we have $\mu(x) * \mu(y) \leq \mu(z)$. Since Q is a right segment, we get $\mu(z) \in Q$ or $z \in \mu_Q$ which means that $x \circ y \subseteq \mu_Q$.

Now, let $x, a \in \mu_Q$, we have $\mu(x), \mu(a) \in Q$ and so $\mu(a) * \mu(x) \in Q$. Since $x, a \in H$, then there exists $y \in H$ such that $x \in a \circ y$ and $\mu(a) * \mu(x) \leq \mu(y)$. Since Q is a right segment, we get $\mu(y) \in Q$ or $y \in \mu_Q$. Therefore we have $a \circ \mu_Q = \mu_Q$. Similarly we can show that $\mu_Q \circ a = \mu_Q$. Hence μ_Q is a subhypergroup of H. □

Theorems 3 and 4 are generalizations of Davvaz ([2, Thorem 1] and [4, Theorem 3.3]).

Let $(H_1, \circ)$ and $(H_2, \bullet)$ be two hypergroups. Then we can define a hyperproduct on $H_1 \times H_2$ as follows:

$$(x_1, x_2) \otimes (y_1, y_2) = \{(a, b) | a \in x_1 \circ x_2,\ b \in y_1 \bullet y_2\}.$$

Clearly $H_1 \times H_2$ equipped with $\otimes$ is a hypergroup.

Definition 5. Let H_1, H_2 be two hypergroups and μ, λ be P-subsets of H_1, H_2 respectively. The P-product of μ, λ is defined as follows:

$$(\mu \times \lambda)(x, y) = \mu(x) * \lambda(y).$$

In the above definition, if we condider $P = [0, 1]$ and $*$ =some t-norm, then we obtain the definition of t-product between two hypergroups where studied in Davvaz [3].

Theorem 6. *Let H_1, H_2 be two hypergroups and μ, λ be P-subhypergroups of H_1, H_2 respectively. If P is an abelian monoid, then the P-product $\mu \times \lambda$ is a P-subhypergroup of $H_1 \times H_2$.*

Proof. Suppose $(x_1, x_2), (y_1, y_2) \in H_1 \times H_2$. For every $(z_1, z_2) \in (x_1, x_2) \otimes (y_1, y_2)$ we have

$$\begin{aligned} \mu \times \lambda(x_1, x_2) * \mu \times \lambda(y_1, y_2) &= \mu(x_1) * \lambda(x_2) * \mu(y_1) * \lambda(y_2) \\ &= (\mu(x_1) * \mu(y_1)) * (\lambda(x_2) * \lambda(y_2)) \\ &\leq \mu(z_1) * \lambda(z_2) \\ &= \mu \times \lambda(z_1, z_2). \end{aligned}$$

Therefore the condition (1) of Definition 1 is satisfied. Now, for every (x_1, x_2) and $(a_1, a_2) \in H_1 \times H_2$ there exists $(y_1, y_2) \in H_1 \times H_2$ such that $x_1 \in a_1 \circ y_1$, $x_2 \in a_2 \bullet y_2$ and $\mu(a_1) * \mu(x_1) \leq \mu(y_1)$, $\lambda(a_2) * \lambda(x_2) \leq \lambda(y_2)$. Therefore we have

$(x_1, x_2) \in (a_1, a_2) \otimes (y_1, y_2)$ and

$$\begin{aligned}\mu \times \lambda(a_1, a_2) * \mu \times \lambda(x_1, x_2) &= \mu(a_1) * \lambda(a_2) * \mu(x_1) * \lambda(x_2) \\ &= (\mu(a_1) * \mu(x_1)) * (\lambda(a_2) * \lambda(x_2)) \\ &\le \mu(y_1) * \lambda(y_2) \\ &= \mu \times \lambda(y_1, y_2).\end{aligned}$$

The proof of condition (3) of Definition 1 is similar to the proof of second condition. □

Corollary 7. *Let H_1, H_2 be two hypergroups, μ, λ be P-subhypergroups of H_1 H_2 respectively. If Q_1, Q_2 are right segments of P and closed under $*$, then*

$$(\mu \times \lambda)_{Q_1 \times Q_2} = \mu_{Q_1} \times \lambda_{Q_2}.$$

Definition 8 (Corsini [1]). If H is a hypergroup and $R \subseteq H \times H$ is an equivalence relation, we set

$$A\overline{\overline{R}}B \Longleftrightarrow aRb \text{ for all } a \in A,\ b \in B,$$

for all pairs (A, B) of non-empty subsets of H. The relation R is said to be strongly regular to the right (*resp.* to the left) if

$$xRy \Longrightarrow x \circ a\overline{\overline{R}}y \circ a \text{ (resp. } xRy \Longrightarrow a \circ x\overline{\overline{R}}a \circ y)$$

for all $x, y, a \in H$. Moreover, R is called *strongly regular* if it is strongly regular to the right and to the left.

Definition 9 (Filep [5]). A P-subset $r : X \times X \longrightarrow P$ is called a *P-relation on P-subset* μ, if it satisfies the following property:

$$r(x, y) \le \mu(x) * \mu(y) \text{ for all } x, y \in X.$$

A P-relation r on a P-subset μ is said to be

1) *reflexive*, if $r(x, y) = \mu(x) * \mu(y)$ for all $x \in X$;
2) *symmetric*, if $r(x, y) = r(y, x)$ for all $x, y \in X$;
3) *transitive*, if for any $x, z \in X$

$$r(x, y) * r(y, z) \le r(x, z) \text{ for all } y \in X.$$

A reflexive, symmetric and transitive P-relation r on a P-subset μ is called *P-similarity.*

Definition 10. Let H be a hypergroup and μ a P-subhypergroup of H. A P-relation r on μ is called a *P-compatible relation on μ* if

$$r(x_1, x_2) * r(y_1, y_2) \leq \bigwedge_{\substack{z_1 \in x_1 \circ y_1 \\ z_2 \in x_2 \circ y_2}} \{r(z_1, z_2)\} \quad \text{for all } x_1, x_2, y_1, y_2 \in H.$$

A P-compatible P-similarity relation is called a *P-strongly regular relation* on the P-subhypergroup μ.

Theorem 11 (Filep [5]). *Let r be a P-relation on a P-subset μ. If each Q-cut r_Q is an equivalence relation on μ_Q for any right segment Q of P, then r is a P-similarity on μ.*

Now, we consider the inverse of Theorem 11.

Theorem 12 (Filep [5]). *Let r be a P-similarity on a P-subset μ, and let Q be a right segment of P. If Q is closed under $*$, then r_Q is an equivalence relation on μ_Q.*

Theorem 13. *Let H be a hypergroup, μ a P-subset of H, and r a P-relation on μ. If for all non-empty right segment Q of P, μ_Q is a subhypergroup and r_Q is a strongly regular relation on μ_Q, then r is a P-strongly regular relation on μ.*

Proof. Using Theorem 11, it is enough to show that r is a P-compatible relation. Suppose $x_1, x_2, y_1, y_2 \in H$, we put

$$Q = [r(x_1, x_2) * r(y_1, y_2), 1],$$

then $r(x_1, x_2), r(y_1, y_2) \in Q$ and so $(x_1, x_2) \in r_Q$, $(y_1, y_2) \in r_Q$. Since

$$r(x_1, x_2) \leq \mu(x_1) * \mu(x_2) \leq \mu(x_1) * 1 = \mu(x_1),$$
$$r(x_1, x_2) \leq \mu(x_1) * \mu(x_2) \leq 1 * \mu(x_2) = \mu(x_2),$$

we get $x_1, x_2 \in \mu_Q$, similarly we obtain $y_1, y_2 \in \mu_Q$. Since r_Q is a strongly regular relation on the subhypergroup μ_Q, therefore $x_1 \circ y_1 \overline{\overline{R}} x_2 \circ y_2$, and consequently for all $z_1 \in x_1 \circ y_1$ and $z_2 \in x_2 \circ y_2$ we have $z_1 r_Q z_2$ or $(z_1, z_2) \in r_Q$. Therefore $r(x_1, x_2) * r(y_1, y_2) \leq r(z_1, z_2)$, and so

$$r(x_1, x_2) * r(y_1, y_2) \leq \bigwedge_{\substack{z_1 \in x_1 \circ y_1 \\ z_2 \in x_2 \circ y_2}} \{r(z_1, z_2)\}. \qquad \square$$

Theorem 14. *Let H be a hypergroup, μ a P-subhypergroup of H, and r a P-strongly regular relation on μ. If a right segment Q of P is closed under $*$, then r_Q is a strongly regular relation on μ_Q.*

Proof. Suppose Q is some right segment satisfying the given condition. By Theorem 4, μ_Q is a subhypergroup of H and using Theorem 12, r_Q is an equivalence relation on μ_Q. Suppose $(x_1, x_2), (y_1, y_2) \in r_Q$ then $r(x_1, x_2) \in Q$ and $r(y_1, y_2) \in Q$ where x_1, x_2, y_1, y_2 are elements of H. Since Q is closed under $*$, it follows that

$$r(x_1, x_2) * r(y_1, y_2) \in Q.$$

Since r is a P-strongly regular relation, then

$$\bigwedge_{\substack{z_1 \in x_1 \circ y_1 \\ z_2 \in x_2 \circ y_2}} \{r(z_1, z_2)\} \in Q$$

and so $r(z_1, z_2) \in Q$ for all $z_1 \in x_1 \circ y_1$ and $z_2 \in x_2 \circ y_2$, which implies $(z_1, z_2) \in r_Q$. Hence $x_1 \circ y_1 \overline{\overline{r_Q}} x_2 \circ y_2$. On the other hand $(x_1, x_2), (y_1, y_2) \in r_Q$ imply that $x_1, x_2, y_1, y_2 \in \mu_Q$. Therefore r_Q is a strongly regular relation on μ_Q. □

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