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CURVED DOMAIN APPROXIMATION IN DIRICHLET'S PROBLEM

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Abstract. The purpose of this paper is to investigate the piecewise polynomial approximation for the curved boundary. We analyze the error of an approximated solution due to this approximation and then compare the approximation errors for the cases of polygonal and piecewise polynomial approximations for the curved boundary. Based on the results of analysis, p-version numerical methods for solving Dirichlet's problems are applied to any smooth curved domain.

1. Introduction

We consider the Poisson equation

$$-\Delta u = F \quad \text{in } \Omega, \tag{1.1}$$

with the Dirichlet boundary condition

$$u = 0 \quad \text{on } \partial\Omega,$$

where $F \in C^\infty(\Omega)$. Here Ω is a bounded domain in the plane with an infinitely differentiable curved boundary $\partial\Omega$ which may be neither polygonal nor polynomial.

It is usually assumed that the boundary $\partial\Omega$ of Ω is either polygonal structures or polynomials when some numerical methods are implemented and corresponding numerical estimations are obtained(see [1, 2, 3, 4]). Thus if a curved boundary is neither polygonal nor polynomial, these results may not be held.

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Strang and Berger [5] and Thomée [6] obtained the error estimates for the following equation

$$(1.2) \qquad -\Delta u_h = F \quad \text{in } \Omega_h, \qquad\qquad u_h = 0 \quad \text{on } \partial\Omega_h,$$

where a polygonal domain Ω_h approximates a curved domain Ω and h is the maximum length of the edges of $\partial\Omega_h$. Due to these theoretical results, the implementation of the numerical h-version method to a curved domain is meaningful.

To obtain more accurate results for the numerical solution on domains with curved boundary, we should apply a p-version method to the curved boundary. But we are aware of no studies that have examined p-version boundary approximation.

The purpose of this paper is to generalize the polygonal boundary approximation using polynomial approximation. We will approximate the curved boundary by a piecewise polynomial boundary and analyze the errors of the approximated solution due to this polynomial boundary approximation. In section 2, the error estimates in the maximum norm are discussed. In section 3, the L^2-error estimates for gradients of numerical solutions are discussed.

2. Error estimates in the maximum-norm

We assume that the boundary $\partial\Omega$ is a union of subboundaries $(f_i(s), g_i(s))$ such that

$$\partial\Omega = \cup_{1\le i\le k}\{(f_i(s), g_i(s)) : 0 \le s \le 1\}$$

with

$$(f_i(1), g_i(1)) = (f_{i+1}(0), g_{i+1}(0)), (f_k(1), g_k(1)) = (f_1(0), g_1(0)),$$

where $f_i,\ g_i \in C^\infty[0,1]$. Let Ω_p be the curved domain with boundary

$$\partial\Omega_p = \cup_{1\le i\le k}\{(\Pi_i(s), \Psi_i(s)) : 0 \le s \le 1\},$$

where Π_i and Ψ_i be the Lagrange interpolating polynomials of degree p_i satisfying

$$\Pi_i(\frac{\ell}{p_i}) = f_i(\frac{\ell}{p_i}), \quad \Psi_i(\frac{\ell}{p_i}) = g_i(\frac{\ell}{p_i}), \quad \ell = 0, \ldots, p_i.$$

Then the corresponding polynomial approximation problem for (1.1) becomes

$$(2.1)\qquad -\Delta u_p = F \quad \text{in } \Omega_p, \qquad u_p = 0 \quad \text{on } \partial\Omega_p.$$

Let $|\cdot|_\Omega$ denote the sup-norm in Ω and $\|\cdot\|_\Omega$ denote L_2-norm over Ω, $|u|_{m,\Omega} = \max_{|\alpha|\le m} |D^\alpha u|_\Omega$ and $|x|^2 = |(x_1, x_2)|^2 = x_1^2 + x_2^2$.

Using the maximum principle, we obtain the following theorem.

THEOREM 2.1. *Let $F \in C^\infty(\tilde{\Omega})$ where $\tilde{\Omega} \supsetneq \Omega$. Then there exists a constant C which depends only on Ω such that*

$$|u - u_p|_{\overline{\Omega\cap\Omega_p}} \le C \max_{1\le i\le k} \{ |f_i^{(p_i+1)}|_{[0,1]} \frac{1}{p_i+1} (\frac{1}{p_i})^{p_i+1}, \\ |g_i^{(p_i+1)}|_{[0,1]} \frac{1}{p_i+1} (\frac{1}{p_i})^{p_i+1} \} |F|_{\tilde{\Omega}},$$

where $\overline{\Omega\cap\Omega_p} = (\Omega\cap\Omega_p)\cup\partial(\Omega\cap\Omega_p)$ and $f_i^{(p_i+1)}$ is the (p_i+1)st derivative of f_i.

Proof. Let $d(x,\Omega)$ be the distance between Ω and x. Let

$$\Omega^\epsilon = \cup_{1\le i\le k} \{ x : d(x, \Omega_i) < \epsilon_i \},$$

where

$$\partial\Omega_i = \{ (f_i(s), g_i(s)) : 0 \le s \le 1 \},$$

$$\epsilon_i = \max\{ |f_i^{(p_i+1)}|_{[0,1]} \frac{1}{p_i+1} (\frac{1}{p_i})^{p_i+1},\ |g_i^{(p_i+1)}|_{[0,1]} \frac{1}{p_i+1} (\frac{1}{p_i})^{p_i+1} \}$$

so that $\Omega \cup \Omega_p \subset \Omega^\epsilon \subset \tilde{\Omega}$ for large p.

Let u^ϵ be the solution of

$$-\Delta u^\epsilon = F \qquad \text{in } \Omega^\epsilon, \qquad u^\epsilon = 0 \qquad \text{on } \partial\Omega^\epsilon.$$

Note that $\partial\Omega^\epsilon \in C^\infty$ except finite number of points. Then for $x \in \partial\Omega \cup \partial\Omega_p$, there exists $y_x \in \partial\Omega^\epsilon$ with $d(x, \partial\Omega^\epsilon) = d(x, y_x)$. Since $u^\epsilon - u$ is harmonic in Ω, we obtain for some C

$$\begin{aligned} |u^\epsilon - u|_\Omega &\le |u^\epsilon - u|_{\partial\Omega} \\ &= \sup_{x\in\partial\Omega} |u^\epsilon(y_x) - u^\epsilon(x)| \\ &\le \sup_{x\in\partial\Omega} |y_x - x| |u^\epsilon|_{1,\ \Omega^\epsilon} \\ &\le \max \epsilon_i |u^\epsilon|_{1,\ \Omega^\epsilon} \\ &\le C \max \epsilon_i |\Delta u^\epsilon|_{\Omega^\epsilon} \\ &\le C \max \epsilon_i |F|_{\tilde{\Omega}}. \end{aligned}$$

On the other hand

$$
\begin{aligned}
|u_p - u^\epsilon|_{\Omega_p} &\le |u_p - u^\epsilon|_{\partial\Omega_p} \\
&= \sup_{x\in\partial\Omega_p} |u^\epsilon(y_x) - u^\epsilon(x)| \\
&\le \sup_{x\in\partial\Omega_p} |y_x - x||u^\epsilon|_{1,\Omega^\epsilon} \\
&\le 2\max \epsilon_i |u^\epsilon|_{1,\Omega^\epsilon} \\
&\le C\max \epsilon_i |\Delta u^\epsilon|_{\Omega^\epsilon} \\
&\le C\max \epsilon_i |F|_{\tilde{\Omega}}.
\end{aligned}
$$

Using triangle inequality, we obtain the desired result. □

REMARK 2.1. Let $\{P_j\}_0^N$ be a sequence of points on $\partial\Omega$ with $P_0 = P_N$ and Ω_h be a polygonally approximated domain with the maximum length h of edges in a triangulation for Ω_h. Thomée [5] obtained for convex domain Ω

$$|u - u_h|_{\Omega_h} \le Ch^2.$$

Thus the errors of h-version and p-version are

$$
\begin{aligned}
|u - u_h|_{\Omega_h\cap\Omega_p} &= O(\frac{1}{N^2}), \\
|u - u_p|_{\Omega_h\cap\Omega_p} &= O(\frac{1}{(N-1)^{N+1}}),
\end{aligned}
$$

respectively, whenever $|\overrightarrow{P_iP_{i+1}}| = |\overrightarrow{P_{i+1}P_{i+2}}|$, $f^{(q)}$ and $g^{(q)}$ are bounded for $q = 0, \dots, N$. Therefore we can see that a curved boundary approximation is more accurate than a polygonal boundary approximation.

In most numerical studies on a curved domain, the curved boundary is polynomial. In this case, $\Omega = \Omega_p$ for some p. Thus the errors of h-version and p-version are, respectively,

$$
\begin{aligned}
|u - u_h|_{\Omega_h} &= O(\frac{1}{N^2}), \\
|u - u_p|_{\Omega_h} &= O(\frac{1}{(N-1)^{N+1}}).
\end{aligned}
$$

3. L_2-error estimates for gradients

Without loss of generality, we may assume that the origin belongs to Ω and there exists a positive constant μ such that

$$x \cdot \eta(x) \geq \mu > 0, \qquad x \in \partial\Omega,$$

where $\eta(x)$ is the outward unit normal vector to the boundary $\partial\Omega$ at a point x. For example, a convex domain satisfies the above condition. Let $s(x)$ be the unit tangent vector to the boundary $\partial\Omega$ at a point x.

Using the Green's identity, we obtain the following lemma.

LEMMA 3.1. *There exists a constant C such that for a harmonic function v in $\Omega \cap \Omega_p$*

$$\|\frac{\partial v}{\partial \eta}\|_{\partial(\Omega\cap\Omega_p)} \leq C|\frac{\partial v}{\partial s}|_{\partial(\Omega\cap\Omega_p)},$$

where $\frac{\partial}{\partial \eta}$ and $\frac{\partial}{\partial s}$ denote differentiations in the direction of normal and tangent vector, respectively.

Proof. Integrating the following identity

$$\sum_{j,k=1}^{2}\left[\frac{\partial}{\partial x_k}\{x_k(\frac{\partial v}{\partial x_j})^2\} - 2\frac{\partial}{\partial x_j}(x_k\frac{\partial v}{\partial x_j}\frac{\partial v}{\partial x_k})\right] + 2\sum_{k=1}^{2} x_k\frac{\partial v}{\partial x_k}\Delta v = 0$$

over $\Omega\cap\Omega_p$ and using Green's identity, we obtain for a harmonic function v

$$\int_{\partial(\Omega\cap\Omega_p)}\left[x\cdot\eta\sum_{j=1}^{2}(\frac{\partial v}{\partial x_j})^2 - 2\frac{\partial v}{\partial n}\frac{\partial v}{\partial r}\right]ds = 0, \tag{3.1}$$

where $\frac{\partial v}{\partial r}$ denotes the derivative in direction of $x = (x_1, x_2)$. Note that for unit vectors $\eta = (\eta_1, \eta_2)$ and $s = (-\eta_2, \eta_1)$, it holds that

$$\sum_{j=1}^{2}(\frac{\partial v}{\partial x_j})^2 = (\frac{\partial v}{\partial \eta})^2 + (\frac{\partial v}{\partial s})^2. \tag{3.2}$$

Setting

$$x\cdot\eta = \|x\|x_n, \qquad x\cdot s = \|x\|x_s, \qquad \frac{\partial v}{\partial r} = (x_n\frac{\partial v}{\partial \eta} + x_s\frac{\partial v}{\partial s})\|x\|,$$

and using (3.2), we may rewrite (3.1) as

$$\int_{\partial(\Omega\cap\Omega_p)} \left[x_n (\frac{\partial v}{\partial s})^2 - x_n (\frac{\partial v}{\partial \eta})^2 - 2x_s \frac{\partial v}{\partial s}\frac{\partial v}{\partial \eta} \right] \|x\| ds = 0.$$

Thus we obtain for $\epsilon > 0$

$$\begin{aligned} &\int_{\partial(\Omega\cap\Omega_p)} x_n (\frac{\partial v}{\partial \eta})^2 \|x\| ds \\ (3.3) \qquad &= \int_{\partial(\Omega\cap\Omega_p)} \left[x_n (\frac{\partial v}{\partial s})^2 - 2x_s \frac{\partial v}{\partial s}\frac{\partial v}{\partial \eta} \right] \|x\| ds \\ &\le \int_{\partial(\Omega\cap\Omega_p)} \left[(x_n + 2|x_s|\frac{1}{\epsilon})(\frac{\partial v}{\partial s})^2 + 2|x_s|\epsilon(\frac{\partial v}{\partial \eta})^2 \right] \|x\| ds. \end{aligned}$$

Since there exists a positive constant μ such that

$$x \cdot \eta \ge \mu > 0, \qquad x \in \partial\Omega,$$

and $\partial\Omega_p$ is any kind of polynomial approximation of $\partial\Omega$, there exists a positive constant λ such that for sufficiently large p

$$x \cdot \eta \ge \lambda > 0, \qquad x \in \partial\Omega_p.$$

Therefore we obtain

$$x \cdot \eta \ge \min\{\mu, \lambda\} > 0, \qquad x \in \partial(\Omega \cap \Omega_p).$$

Since there exist constants r_0 and r_1 such that for all $x \in \partial(\Omega \cap \Omega_p)$, $0 < r_0 \le \|x\| \le r_1$, we can choose ϵ in (3.3) such that

$$x_n\|x\| - 2|x_s|\epsilon\|x\| > \min\{\mu, \lambda\} - 2r_1\epsilon > 0, \qquad x \in \partial(\Omega \cap \Omega_p).$$

It completes the desired proof. □

Using the Green's formula, we obtain the following theorem.

THEOREM 3.1. *There exists a constant C depending on u such that*

$$\begin{aligned} &\|\nabla u - \nabla u_p\|_{\Omega\cap\Omega_p} \\ &\le C \max_{1\le i\le k} \{ |f_i^{(p_i+1)}|_{[0,1]} (\frac{1}{p_i})^{p_i+\frac{3}{2}},\ |g_i^{(p_i+1)}|_{[0,1]} (\frac{1}{p_i})^{p_i+\frac{3}{2}} \}. \end{aligned}$$

Proof. Since $e_p = u - u_p$ is a harmonic function in $\Omega \cap \Omega_p$, using Lemma 3.1, we obtain for some constant C

$$\|\nabla e_p\|^2_{\Omega\cap\Omega_p} = \int_{\partial(\Omega\cap\Omega_p)} e_p \frac{\partial e_p}{\partial \eta} ds \tag{3.4}$$

$$\leq \|e_p\|_{\partial(\Omega\cap\Omega_p)} \|\frac{\partial e_p}{\partial n}\|_{\partial(\Omega\cap\Omega_p)}$$

$$\leq C\|e_p\|_{\partial(\Omega\cap\Omega_p)} |\frac{\partial e_p}{\partial s}|_{\partial(\Omega\cap\Omega_p)}.$$

Note that

$$|\frac{\partial e_p}{\partial s}|_{\partial(\Omega\cap\Omega_p)} = |\frac{\partial e_p}{\partial s}|_{\partial(\Omega\cap\Omega_p)\cap\partial\Omega} + |\frac{\partial e_p}{\partial s}|_{\partial(\Omega\cap\Omega_p)\cap\partial\Omega_p} \tag{3.5}$$

and the second term in the righthand side of (3.5) is

$$|\frac{\partial e_p}{\partial s}|_{\partial(\Omega\cap\Omega_p)\cap\partial\Omega_p} = \sup_{x_p\in\partial(\Omega\cap\Omega_p)\cap\partial\Omega_p} |\nabla u(x_p)\cdot s(x_p)|.$$

Let $x_p = (\Pi_i(s_0), \Psi_i(s_0))$ and $\tilde{x} = (f_i(s_0), g_i(s_0))$. Then we obtain

$$|\nabla u(x_p)\cdot s(x_p)| \leq |\nabla u(\tilde{x})\cdot s(x_p)| + C\|x_p - \tilde{x}\| \tag{3.6}$$

$$\leq C\{|\cos(\frac{\pi}{2} \pm \theta)| + \|x_p - \tilde{x}\|\}$$

$$\leq C\{\sin\theta + \|x_p - \tilde{x}\|\},$$

where θ is the acute angle between the tangential vectors $s(x_p)$ on $\partial\Omega_p$ and $s(\tilde{x})$ on $\partial\Omega$.

Using the smoothness of functions f_i, g_i, Π_i, Ψ_i, the inequality (3.6) becomes

$$|\nabla u(x_p)\cdot s(x_p)|$$

$$\leq C\{\sin\theta + \|x_p - \tilde{x}\|\}$$

$$\leq C\{\sqrt{(f_i'(s_0) - \Pi_i'(s_0))^2 + (g_i'(s_0) - \Psi_i'(s_0))^2} + \|x_p - \tilde{x}\|\}.$$

Thus we obtain

(3.7)

$$|\frac{\partial e_p}{\partial s}|_{\partial(\Omega\cap\Omega_p)\cap\partial\Omega_p}$$

$$\leq C\Big[\max_{1\leq i\leq k}\{|f_i^{(p_i+1)}|_{[0,1]}(\frac{1}{p_i})^{p_i+1},\ |g_i^{(p_i+1)}|_{[0,1]}(\frac{1}{p_i})^{p_i+1}\}$$

$$+ \max_{1\leq i\leq k}\{|f_i^{(p_i+1)}|_{[0,1]}\frac{1}{p_i+1}(\frac{1}{p_i})^{p_i+1},\ |g_i^{(p_i+1)}|_{[0,1]}\frac{1}{p_i+1}(\frac{1}{p_i})^{p_i+1}\}\Big]$$

$$\leq C\max_{1\leq i\leq k}\{|f_i^{(p_i+1)}|_{[0,1]}(\frac{1}{p_i})^{p_i+1},\ |g_i^{(p_i+1)}|_{[0,1]}(\frac{1}{p_i})^{p_i+1}\}.$$

Similarly we obtain the inequality (3.7) for $|\frac{\partial e_p}{\partial s}|_{\partial(\Omega\cap\Omega_p)\cap\partial\Omega}$. Applying (3.7) and Theorem 2.1 to (3.4), we obtain the desired results. □

REMARK 3.1. Thomée [5] obtained for a convex domain Ω

$$|\nabla u - \nabla u_h|_{\Omega_h} \leq Ch^{\frac{3}{2}},$$

where h is the maximum length of the edges in a triangulation for Ω_h. Thus we obtain similar results as in Remark 2.1.

4. Concluding remark

We approximated a curved boundary by polynomials of variable degrees and obtained the corresponding error estimates. Since a piecewise polynomial boundary approximation for a curved boundary is more accurate than that a polygonal boundary approximation, we were able to show that the numerical solution by p-version methods is more accurate than that obtained by using the usual boundary approximation for a curved boundary.

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