ANALYTICITY FOR THE STOKES OPERATOR IN BESOV SPACES

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ABSTRACT. We first show the analyticity of Stokes operator in Besov spaces $B_{p,q}^{a}(\mathbb{R}_{+}^{n})$. Then, we estimate the asymptotic behavior of the Stokes solutions. We also show the Hodge decomposition.

1. Introduction

We consider the Stokes equations in the half spaces:

(1.1)
$$\begin{aligned}
\mathbf{u}_t - \Delta\mathbf{u} + \nabla p &= 0, && \text{in } \mathbb{R}_+^3 \times (0,\infty), \\
\nabla \cdot \mathbf{u} &= 0, && \text{in } \mathbb{R}_+^3 \times (0,\infty), \\
\mathbf{u}(x,0) &= \mathbf{u}_0, && \text{for } x \in \mathbb{R}_+^3, \\
\mathbf{u}(x,t) &= 0, && \text{for } x_3 = 0,\ t \in (0,\infty).
\end{aligned}$$

Here, $\mathbb{R}_+^3 = \{x \in \mathbb{R}^3 : x = (\bar{x}, x_3) := (x_1, x_2, x_3), x_3 > 0\}$ is the upper half space of $\mathbb{R}^3$, and $\mathbf{u}_0$ is given initial data. The velocity $\mathbf{u} = (u_1, u_2, u_3)$, and the pressure p are unknown. This paper consists of three parts. We first estimate the solutions in Besov spaces to show the analyticity of the semigroup generated by the Stokes operator. Secondly, we estimate the decay rates of the Stokes solutions, finally we show the Hodge decomposition.

In the first part, we show the semigroup obtained by the Stokes solutions on $\mathbb{R}_+^n$ is analytic in Besov spaces in time. From Theorem 5.2 in Pazy [5], it is enough to check that the semigroup $T(t)$ generated by the

Stokes operator is uniformly bounded C^0 semigroup, and $T(t)$ is differentiable for $t > 0$ and there is a constant C such that $\|\Delta T(t)\| \leq C/t$ for $t > 0$.

Let $\mathcal{P}_{p,q}^a$ denote the projection operator

$$\mathcal{P}_{p,q}^a : B_{p,q}^a(\mathbb{R}_+^n) \rightarrow B_{p,q}^{a,\sigma}(\mathbb{R}_+^n),$$

where $B_{p,q}^a$ and $B_{p,q}^{a,\sigma}$ will be defined later. In Section 5, we show the projection operator $\mathcal{P}_{p,q}^a$ is bounded. In Sections 3, we show the operator $-\mathcal{A}^a := -\mathcal{P}_{p,q}^a \Delta$ generates an analytic semigroup. Since $(\lambda + \mathcal{A}^a)^{-1}$ depends analytically on λ, if $(\lambda + \mathcal{A}^a)^{-1}$ exists for all $\lambda = |\lambda| e^{i\theta}$ with $\theta < \pi$, then, given θ with $|\lambda| > 0$ and $|\theta| < \theta_0 < \pi$, we will have $\|(\lambda + \mathcal{A}^a)^{-1}\| \leq (1/|\lambda|) C_a(\theta_0)$, where $C_a(\theta_0)$ depends on only a and θ_0.

In the second part, we estimate the asymptotic behavior of the solutions for the nonstationary Stokes equations (1.1). The decay problems are studied by many authors, for example, Leray [4], Schonbek [6], Wiegner [9], Borchers and Miyakawa [3], Bae and Choe [2], and Bae[1]. In Section 4 we study the decay rate in Besov space.

2. Besov spaces

We review the definitions of Besov spaces in Triebel [7].

DEFINITION 1. Let $\Phi(\mathbb{R}^n)$ be the collection of all systems $\varphi = \{\varphi_j(x)\}_{j=0}^\infty \subset S(\mathbb{R}^n)$ such that

$$(2.1) \quad \begin{array}{l} \text{supp } \varphi_0 \subset \{x : |x| \leq 2\}, \\ \text{supp } \varphi_j \subset \{x : 2^{j-1} \leq |x| \leq 2^{j+1}\} \quad \text{if } j = 1, 2, 3, \ldots, \end{array}$$

for every multi-index α there exists a positive number c_α such that

$$2^{j|\alpha|} |D^\alpha \varphi_j(x)| \leq c_\alpha \quad \text{for all } j = 0, 1, 2, \ldots \text{ and all } x \in \mathbb{R}^n$$

and

$$\sum_{j=0}^\infty \varphi_j(x) = 1 \quad \text{for all } x \in \mathbb{R}^n.$$

Here, $S(\mathbb{R}^n)$ denotes the Schwartz space on $\mathbb{R}^n$.

DEFINITION 2. Let $-\infty < a < \infty$ and $0 < q \leq \infty$. Let $\varphi = \{\varphi_j\}_{j=0}^{\infty} \in \Phi(\mathbb{R}^n)$. Then, for $0 < p \leq \infty$,

$$B_{p,q}^a(\mathbb{R}^n) := \{f \in S'(\mathbb{R}^n) : \|f\|_{B_{p,q}^a(\mathbb{R}^n)}$$

$$:= \Big[\sum_{j=0}^{\infty} \Big(2^{aj} \|\mathcal{F}^{-1}(\varphi_j \mathcal{F}f)\|_{L^p(\mathbb{R}^n)} \Big)^q \Big]^{1/q} < \infty\},$$

where $S'(\mathbb{R}^n)$ denotes the space of tempered distributions on $\mathbb{R}^n$, and $\mathcal{F}$, $\mathcal{F}^{-1}$ denote the Fourier and the inverse Fourier transforms, respectively. For $q = \infty$,

$$\|f\|_{B_{p,q}^a(\mathbb{R}^n)} := \sup_j \Big(2^{aj} \|\mathcal{F}^{-1}(\varphi_j \mathcal{F}f)\|_{L^p(\mathbb{R}^n)} \Big).$$

Let N be a natural number. Denote $\|m\|_N$ by

$$(2.2) \qquad \|m\|_N := \sup_{|\alpha| \leq N} \sup_{\xi \in \mathbb{R}^n} |\xi|^{|\alpha|} |D^\alpha m(\xi)|,$$

where $D^\alpha := (\partial/\partial\xi_1)^{\alpha_1} \cdots (\partial/\partial\xi_n)^{\alpha_n}$, and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Then the following theorem is given in Triebel [7].

THEOREM 2.1. *Let* $-\infty < a < \infty$ *and* $0 < q \leq \infty$ *and* $0 < p \leq \infty$. *If* N *is sufficiently large, then there exists a positive number* c *such that*

$$\|\mathcal{F}^{-1}m\mathcal{F}f\|_{B_{p,q}^a(\mathbb{R}^n)} \leq c\|m\|_N \|f\|_{B_{p,q}^a(\mathbb{R}^n)}$$

holds for all infinitely differentiable functions m *and all* $f \in B_{p,q}^a(\mathbb{R}^n)$.

DEFINITION 3. Let $-\infty < a < \infty$ and $0 < p, q \leq \infty$. Then, $B_{p,q}^a(\mathbb{R}_+^n)$ is the collection of all restrictions of functions in $B_{p,q}^a(\mathbb{R}^n)$. If f is the restriction of g on $\mathbb{R}_+^n$, its norm is defined by

$$\|f\|_{B_{p,q}^a(\mathbb{R}_+^n)} := \inf \|g\|_{B_{p,q}^a(\mathbb{R}^n)},$$

where the infimum is to be taken over all g with $g|_{\mathbb{R}_+^n} = f$.

3. Analyticity of the Stokes operator

In this section we show the semigroup obtained by the Stokes solutions on $\mathbb{R}_+^n$ is analytic in Besov spaces in time. From Theorem 5.2 in Pazy [5], it is enough to check that (1) the semigroup $T(t)$ generated by the Stokes operator is uniformly bounded C^0 semigroup, and (2) $T(t)$ is differentiable for $t > 0$ and (3) there is a constant C such that $\|\Delta T(t)\| \leq C/t$ for $t > 0$.

We adopt the solution formula of the Stokes equations in $\mathbb{R}_+^n$ given by Ukai [8]. We review the formula. Setting $\phi_j = \mathcal{F}^{-1}\varphi_j$, we have $\mathcal{F}^{-1}\varphi_j \mathcal{F}f = \phi_j * f$. We denote $\bar{x} := (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$, and $x = (\bar{x}, x_n) \in \mathbb{R}_+^n$. We denote by $K(\bar{x}, x_n, t)$ the heat kernel in the whole space $\mathbb{R}^n$,

$$K_t(x) = K_t(\bar{x}, x_n) := (4\pi t)^{-n/2} e^{-\frac{|\bar{x}|^2}{4t}} e^{-\frac{|x_n|^2}{4t}}.$$

Then the solution $v(x, t)$ of heat equation $v_t - \Delta v = 0$ with the boundary condition $v(\bar{x}, 0, t) = 0, t > 0$ and the initial condition $v(x, 0) = g(x)$ in the half space $\mathbb{R}_+^n$ has a potential expression

$$E(t)g(x) := v_g(x, t) := \int_{\mathbb{R}_+^n} \big(K_t(\bar{x}-\bar{y}, x_n-y_n)$$

$$- K_t(\bar{x}-\bar{y}, x_n+y_n) \big) g(y)\, dy.$$

Denote by the Riesz' operators, $R_j, j = 1, \ldots, n$, and $S_j, j = 1, \ldots, n-1$, which are the singular integral operators with the symbols

$$\sigma(R_j) = i\xi_j/|\xi|, \qquad j = 1, \ldots, n,$$

$$\sigma(S_j) = i\xi_j/|\bar{\xi}|, \qquad j = 1, \ldots, n-1,$$

where $\xi = (\xi_1, \ldots, \xi_n) = (\bar{\xi}, \xi_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ is the dual variable to $x \in \mathbb{R}^n$. Then, the Riesz transform R_j are defined in $\mathbb{R}^n$ by

$$R_j f(x) = \text{p.v.} \int_{\mathbb{R}^n} R_j(x-y) f(y)\, dy,$$

where p.v. means the principal value of the integral and

$$R_j(x) = c_n x_j/|x|^{n+1}, \qquad c_n = 2^{1-n/2}\sqrt{\pi}\Gamma(\frac{1}{2}(n-1)).$$

Here, Γ is the gamma function. The Riesz transform S_j are defined similarly,

$$S_j f(x) = \text{p.v.} \int_{\mathbb{R}^{n-1}} S_j(\bar{x} - \bar{y}) f(\bar{y})\, d\bar{y},$$

where $S_j(x) = c_{n-1} x_j/|\bar{x}|^n$. Set

$$\bar{R} = (R_1, R_2, \ldots, R_{n-1}), \qquad S = (S_1, S_2, \ldots, S_{n-1}),$$

and define the operators V_1 and V_2 by

$$V_1 \mathbf{u}_0 = -S \cdot \bar{\mathbf{u}}_0 + u_{0,n},$$

$$V_2 \mathbf{u}_0 = \bar{\mathbf{u}}_0 + S u_{0,n},$$

where $\mathbf{u}_0 = (u_{0,1}, u_{0,2}, \ldots, u_{0,n}) = (\bar{\mathbf{u}}_0, u_{0,n})$. Furthermore, let h be the restriction operator from $\mathbb{R}^n$ to $\mathbb{R}^n_+$, that is,

$$hf = f|_{\mathbb{R}^n_+},$$

and e the extension operator from $\mathbb{R}^n_+$ over $\mathbb{R}^n$ with value 0:

$$ef = \begin{cases} f & \text{for } x_n > 0, \\ 0 & \text{for } x_n < 0. \end{cases}$$

We also define the operator U by

$$Uf = h\bar{R} \cdot S(\bar{R} \cdot S + R_n)ef.$$

By Ukai [8], the solution u of (1.1) is represented as

$$u_n = UE(t)V_1\mathbf{u}_0,$$
$$\bar{\mathbf{u}} = E(t)V_2\mathbf{u}_0 - SUE(t)V_1\mathbf{u}_0.$$

Consider the Riesz operator S:

$$\left(\int_{\mathbb{R}^n} \left| \mathcal{F}^{-1}\big(\sigma(S_j)\varphi_k\mathcal{F}f\big)(x) \right|^p dx \right)^{1/p}$$

$$= \left(\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^{n-1}} \frac{x_j - y_j}{|\bar{x} - \bar{y}|^n} \mathcal{F}^{-1}\big(\varphi_k\mathcal{F}f\big)(\bar{y}, x_n)d\bar{y} \right|^p dx \right)^{1/p}$$

$$\leq C\left(\int_{\mathbb{R}^n} \left| \mathcal{F}^{-1}\big(\varphi_k\mathcal{F}f\big)(x) \right|^p dx \right)^{1/p}$$

by the Calderon-Zygmund inequality. Therefore, the Riesz' operators R and S are bounded on $B^a_{p,q}(\mathbb{R}^n_+)$ for $1 < p < \infty$.

THEOREM 3.1. *Let v_g be the solution of the heat equation with initial data g. Assume that $1 \leq r \leq p \leq \infty$, $0 < q \leq \infty$ and $-\infty < a < \infty$. Then, we have that for all $t > 0$,*

$$\|v_g(\cdot, t)\|_{B^a_{p,q}(\mathbb{R}^n_+)} \leq C(1 + t)^{-\frac{n}{2}\left(\frac{1}{r} - \frac{1}{p}\right)} \|g\|_{B^a_{r,q}(\mathbb{R}^n_+)}.$$

Furthermore, if g belongs to $B^a_{r,s}(\mathbb{R}^n_+)$ for $0 < s \leq q \leq \infty$, then we also have

$$\|v_g\|_{B^a_{p,q}(\mathbb{R}^n_+)} \leq C(1 + t)^{-n/2(1/r - 1/p)} \|g\|_{B^a_{r,s}(\mathbb{R}^n_+)}.$$

Proof. Let $a = eg$ be the extension of g to $\mathbb{R}^n$ by zero, that is, $a(z) = g(z)$ for $z \in \mathbb{R}^n_+$ and $a(z) = 0$ for $z \notin \mathbb{R}^n_+$, and denote $a'(z) = a(\bar{z}, -z_n)$

and $\tilde{a}(z) = a(z) - a'(z) = a(z) - a(\bar{z}, -z_n)$. Then we have that for $y \in \mathbb{R}_+^n$,

$$v_a(y,t) = (4\pi t)^{-n/2} \int_{\mathbb{R}^n} e^{-\frac{|\bar{y}-\bar{z}|^2}{4t}} \left(e^{-\frac{|y_n-z_n|^2}{4t}} - e^{-\frac{|y_n+z_n|^2}{4t}}\right) a(z)\, dz$$

$$(3.1) \qquad = (4\pi t)^{-n/2} \int_{\mathbb{R}^n} e^{-\frac{|y-z|^2}{4t}} \tilde{a}(z)\, dz = K_t * \tilde{a}(y).$$

We denote by the same notation v_a the extension of v_a defining by (3.1) for $y \in \mathbb{R}^n$. By taking $y - z = s$, we have

$$\phi_j * v_a(x,t) = \phi_j * (K_t * \tilde{a})(x) = K_t * (\phi_j * \tilde{a})(x)$$

$$= (4\pi t)^{-n/2} \int_{\mathbb{R}^n} \phi_j * \tilde{a}(x-s) e^{-\frac{|s|^2}{4t}}\, ds,$$

which is the heat solution with initial data $\phi_j * \tilde{a}(x)$. We have

$$\left(\int_{\mathbb{R}^n} |\phi_j * v_a(x,t)|^p dx\right)^{1/p}$$

$$\leq C(1+t)^{-n/2(1/r-1/p)} \left(\int |\phi_j * \tilde{a}(x)|^r dx\right)^{1/r}$$

since the estimation of the heat solution is well known. Hence, we have for $1 \leq r \leq p < \infty$,

$$\|v_a\|_{B_{p,q}^a(\mathbb{R}^n)} \leq C(1+t)^{-n/2(1/r-1/p)} \|a - a'\|_{B_{r,q}^a(\mathbb{R}^n)}$$

$$\leq C(1+t)^{-n/2(1/r-1/p)} \left(\|a\|_{B_{r,q}^a(\mathbb{R}^n)} + \|a'\|_{B_{r,q}^a(\mathbb{R}^n)}\right)$$

$$\leq C(1+t)^{-n/2(1/r-1/p)} \|a\|_{B_{r,q}^a(\mathbb{R}^n)}.$$

Therefore, by the definition of $B_{p,q}^a(\mathbb{R}_+^n)$, we have

$$\|v_g\|_{B_{p,q}^a(\mathbb{R}_+^n)} \leq C(1+t)^{-n/2(1/r-1/p)} \|g\|_{B_{r,q}^a(\mathbb{R}_+^n)}.$$

Since for $0 < s \leq q \leq \infty$, $l^s \subset l^q \subset l^\infty$, that is,

$$\|b\|_{l^\infty} \leq \|b\|_{l^q} \leq \|b\|_{l^s} \qquad \text{for } b \in l^s,$$

we also have for $0 < s \leq q$,

$$\|v\|_{B_{p,q}^a(\mathbb{R}_+^n)} \leq C(1+t)^{-n/2(1/r-1/p)} \|g\|_{B_{r,s}^a(\mathbb{R}_+^n)}$$

if g belongs to $B_{r,s}^a(\mathbb{R}_+^n)$. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Hence, the above theorem says the decay rate, and we may conclude that $E(t)$ is uniformly bounded in $B_{p,q}^a(\mathbb{R}_+^n)$. Since S and R is bounded, we have the following corollary.

CORALLARY 3.2. *Let* **u** *be the solution of* (1.1). *Assume that* $1 < r \leq p < \infty$ *and* $0 < s \leq q \leq \infty$. *Then, we have that for all* $t > 0$,

$$\|\mathbf{u}\|_{B^a_{p,q}(\mathbb{R}^n_+)} \leq C(1+t)^{-n/2(1/r-1/p)}\|\mathbf{u}_0\|_{B^a_{r,s}(\mathbb{R}^n_+)}.$$

Now let us consider the continuity at $t = 0$. As we saw, $\phi_j * v_a$ is the form of the heat solutions with initial data $\phi_j * (a - a')$.

THEOREM 3.3. $v_g \to g$ *in* $B^a_{p,q}(\mathbb{R}^n_+)$ *as* $t \to 0$ *with* $t > 0$.

Proof. If we assume that $v_g(y,t) = K_t * \tilde{a}(y)$ is also defined in the lower half space $\mathbb{R}^n_-$, then v_g is an odd function. We consider the odd extension of $v_g(y,t) - g(y)$ to $\mathbb{R}^n$, which can be denoted by $v_g(y,t) - \tilde{a}(y)$. Since the Schwartz space $S(\mathbb{R}^n) \subset B^a_{p,q}(\mathbb{R}^n)$ is dense (refer to p.48, Triebel [7]), it is enough to show the theorem for $\tilde{a} \in S(\mathbb{R}^n)$. Hence we may assume that $\tilde{a} \in S(\mathbb{R}^n)$.

Since $(4\pi t)^{-n/2} \int_{\mathbb{R}^n} e^{-\frac{|y-z|^2}{4t}} dz = 1$, we have

$$v_g(y,t) - \tilde{a}(y) = (4\pi t)^{-n/2} \int_{\mathbb{R}^n} e^{-\frac{|y-z|^2}{4t}} \left(\tilde{a}(z) - \tilde{a}(y)\right) dz.$$

Notice that for $1 < p$,

$$\phi_j * (v_g - \tilde{a})(x,t)$$

$$= (4\pi t)^{-n/2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \phi_j(x-y)e^{-\frac{|y-z|^2}{4t}} \left(\tilde{a}(z) - \tilde{a}(y)\right) dz dy$$

$$= \pi^{-n/2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \phi_j(x-z-\sqrt{4t}s)e^{-|s|^2} \left(\tilde{a}(z) - \tilde{a}(z+\sqrt{4t}s)\right) ds dz$$

$$= \pi^{-n/2} \int_{\mathbb{R}^n} \left[\phi_j * \tilde{a}(x-s\sqrt{4t}) - \phi_j * \tilde{a}(x)\right]e^{-|s|^2} ds,$$

and that

$$\int_{\mathbb{R}^n} |\phi_j * (v_g - \tilde{a})(x,t)|^p dx$$

$$= \int_{\mathbb{R}^n} \pi^{-n/2} \left| \int_{\mathbb{R}^n} \left[\phi_j * \tilde{a}(x-s\sqrt{4t}) - \phi_j * \tilde{a}(x)\right]e^{-|s|^2} ds \right|^p dx$$

$$\leq C \int \left[\int_{\mathbb{R}^n} |\phi_j * \tilde{a}(x-s\sqrt{4t}) - \phi_j * \tilde{a}(x)|^p e^{-p|s|^2/2} ds \right]$$

$$\times \left[\int_{\mathbb{R}^n} e^{-\frac{|s|^2 p}{2(p-1)}} ds \right]^{p-1} dx$$

$$\leq C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\phi_j * \tilde{a}(x - s\sqrt{4t}) - \phi_j * \tilde{a}(x)|^p e^{-p|s|^2/2} dx ds$$

$$\leq C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left| \int_0^1 \frac{d}{d\epsilon} \phi_j * \tilde{a}(x - \epsilon s\sqrt{4t}) d\epsilon \right|^p e^{-p|s|^2/2} dx ds$$

$$\leq C\sqrt{4t}^p \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\phi_j * \nabla\tilde{a}(\tilde{x})|^p s^p e^{-p|s|^2/2} d\tilde{x} ds$$

$$\leq C\sqrt{4t}^p \int_{\mathbb{R}^n} |\phi_j * \nabla\tilde{a}(\tilde{x})|^p d\tilde{x} \leq C\sqrt{4t}^p \|\phi_j * \nabla a\|_{L^p}^p,$$

by the fundamental theorem of calculus, which tends to zero as $t \to 0$. For $p = 1$, we can do in a similar way. $\quad\square$

Therefore, we conclude that the semigroup $E(t)$ generated by the heat operator in $B_{p,q}^a(\mathbb{R}_+^n)$ is a uniformly bounded C^0 semigroup. The semigroup $T(t)$ generated by the Stokes operator in $B_{p,q}^a(\mathbb{R}_+^n)$ is also a uniformly bounded C^0 semigroup.

CORALLARY 3.4. *For* $1 < p < \infty$ *and* $0 < q \leq \infty$, $\mathbf{u} \to \mathbf{u}_0$ *in* $B_{p,q}^a(\mathbb{R}_+^n)$ *as* $t \to 0$ *with* $t > 0$.

We now consider the differentiability of the semigroup for $t > 0$. Notice that

$$\phi_j * v_a(x) = K_t * (\phi_j * \tilde{a})(x) = (4\pi t)^{-n/2} \int e^{-\frac{|x-s|^2}{4t}} \phi_j * \tilde{a}(s) \, ds,$$

which is the solution with initial data $\phi_j * \tilde{a}$, and the heat solution in L^p is differentiable for $t > 0$. Since, in L^p

$$\frac{\partial}{\partial t}(K_t * \phi_j * \tilde{a})(x) = (4\pi t)^{-n/2} \int e^{-\frac{|x-s|^2}{4t}} \left(\frac{|x-s|^2}{4t^2} - \frac{n}{2t} \right) \phi_j * \tilde{a}(s) \, ds$$

$$= (4\pi t)^{-n/2} \sum_k \partial_{x_k} \int e^{-\frac{|x-s|^2}{4t}} \frac{x_k - s_k}{-2t} \phi_j * \tilde{a}(s) \, ds$$

$$= (4\pi t)^{-n/2} \Delta \int e^{-\frac{|x-s|^2}{4t}} \phi_j * \tilde{a}(s) \, ds,$$

we have $\frac{\partial}{\partial t}(K_t * \phi_j * \tilde{a})(x) = \Delta(K_t * \phi_j * \tilde{a})(x)$ in L^p, and we can say $v_{\tilde{a}}$ is differentiable for $t > 0$ and is equal to $\Delta v_{\tilde{a}}$ in $B_{p,q}^a(\mathbb{R}^n)$. Since, in L^p

$$(4\pi t)^{n/2} \phi_j * \partial_t(K_t * \tilde{a})(x)$$

$$= (4\pi t)^{n/2} \Delta(K_t * \phi_j * \tilde{a})(x)$$

$$= \int e^{-\frac{|s|^2}{4t}} \left(\frac{|s|^2}{4t^2} - \frac{n}{2t} \right) \phi_j * \tilde{a}(x - s) \, ds$$

$$= \int e^{-\frac{|s|^2}{4t}} \left(\frac{|s|^2}{4t^2} - \frac{n}{2t} \right) \left(\int \tilde{a}(y) \phi_j(x - s - y)\, dy \right) ds$$

$$= \int \int e^{-\frac{|z-y|^2}{4t}} \left(\frac{|z-y|^2}{4t^2} - \frac{n}{2t} \right) \phi_j(x - z) \tilde{a}(y)\, dz dy$$

$$= \int \phi_j(x - z) \left(\int \Delta_z e^{-\frac{|z-y|^2}{4t}} \tilde{a}(y)\, dy \right) dz$$

$$= \phi_j * (\Delta K_t * \tilde{a})(x),$$

we have $\phi_j * \partial_t (K_t * \tilde{a})(x) = \phi_j * (\Delta K_t * \tilde{a})$ in L^p. Since, by Minkowski's inequality,

$$\left(\int \left| \int e^{-\frac{|s|^2}{4t}} \frac{|s|^2}{4t^2} \phi_j * \tilde{a}(x - s)\, ds \right|^p dx \right)^{1/p}$$

$$\leq \int \left| \int |\phi_j * \tilde{a}(x - s)|^p dx \right|^{1/p} e^{-\frac{|s|^2}{4t}} \frac{|s|^2}{4t^2} ds$$

$$\leq \|\phi_j * \tilde{a}\|_{L^p} \frac{1}{t} \int e^{-\frac{|s|^2}{4t}} \frac{|s|^2}{4t} ds \leq t^{n/2-1} \|\phi_j * \tilde{a}\|_{L^p}$$

and

$$\left(\int \left| \int e^{-\frac{|s|^2}{4t}} \frac{n}{2t} \phi_j * \tilde{a}(x - s)\, ds \right|^p dx \right)^{1/p} \leq t^{n/2-1} \|\phi_j * \tilde{a}\|_{L^p},$$

we have

$$\|\phi_j * \partial_t (K_t * \tilde{a})\|_{L^p}$$

$$= \|\Delta (K_t * (\phi_j * \tilde{a}))\|_{L^p}$$

$$= \left\| (4\pi t)^{-n/2} \int e^{-\frac{|s|^2}{4t}} \left(\frac{|s|^2}{4t^2} - \frac{n}{2t} \right) \phi_j * \tilde{a}(x - s)\, ds \right\|_{L^p}$$

$$\leq C \frac{1}{t} \|\phi_j * \tilde{a}\|_{L^p}.$$

Hence, we have $\|\Delta v_g\|_{B^a_{p,q}} \leq \frac{C}{t} \|g\|_{B^a_{p,q}}$, therefore, the heat semigroup $E(t)$ is analytic in $B^a_{p,q}$ by the Theorem 5.2, [5].

THEOREM 3.5. *The semigroup generated by the heat equation in* $B^a_{p,q}(\mathbb{R}^n_+)$ *is analytic.*

CORALLARY 3.6. *The semigroup generated by the Stokes equations in* $B^a_{p,q}(\mathbb{R}^n_+)$ *is analytic for* $1 < p < \infty$, $0 < q \leq \infty$ *and* $-\infty < a < \infty$.

4. Decay rate of solutions for the Stokes equations

In this section, we study the asymptotic behavior of weak solutions of the Stokes equations in $\mathbb{R}^n_+$.

We consider the heat solutions again: from the fundamental theorem of calculus, we also have

$$
\phi_j * v_a(x,t)
$$

$$
= (4\pi t)^{-n/2} \int \int_{\mathbb{R}^n} \phi_j(x-y)a(z)e^{-\frac{|\bar{y}-\bar{z}|^2}{4t}}
$$

$$
\times \left[e^{-\frac{|y_n-z_n|^2}{4t}} - e^{-\frac{|y_n+z_n|^2}{4t}} \right] dzdy
$$

$$
= (4\pi t)^{-n/2} \int \int_{\mathbb{R}^n} \phi_j(x-y)a(z)e^{-\frac{|\bar{y}-\bar{z}|^2}{4t}}
$$

$$
\times \int_{-1}^{1} e^{-\frac{|y_n-sz_n|^2}{4t}} \frac{y_n - sz_n}{2t} z_n \, dsdzdy,
$$

taking $\bar{l} = \bar{y} - \bar{z}$ and $l_n = y_n - sz_n$, we have

$$
= Ct^{-n/2-1} \int_{-1}^{1} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \phi_j(\bar{x} - \bar{l} - \bar{z}, x_n - l_n - sz_n)
$$

$$
\times z_n a(z) l_n e^{-\frac{|l|^2}{4t}} \, dldzds.
$$

Taking $\bar{k} = \bar{z}$, $k_n = sz_n$, we have

$$
(4.1) \quad \phi_j * v_a(x,t)
$$

$$
= Ct^{-n/2-1} \int \int \phi_j(x-l-k) \left(\int_{-1}^{1} \frac{k_n}{s^2} a(\bar{k}, \frac{k_n}{s}) ds \right) l_n e^{-\frac{|l|^2}{4t}} \, dkdl.
$$

For short, we put

$$
\check{a}(k) = \left(\int_{-1}^{1} \frac{k_n}{s^2} a(\bar{k}, \frac{k_n}{s}) ds \right).
$$

THEOREM 4.1. *Let v be the solution of the heat equation with initial data g. Assume that $1 \le r \le p < \infty$, $0 < s \le q \le \infty$ and $-\infty < a < \infty$. Then, we have that for sufficiently large t,*

$$
\|v_g(\cdot, t)\|_{B^a_{p,q}(\mathbb{R}^n_+)} \le Ct^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{p})-\frac{1}{2}} \|\check{a}\|_{B^a_{r,s}(\mathbb{R}^n)}.
$$

Proof. From (4.1), we have for $r > 1$,

$$|\phi_j * v_a(x,t)|$$

$$\leq Ct^{-n/2-1} \int_{\mathbb{R}^n} \phi_j * \check{a}(x-l) l_n e^{-\frac{|l|^2}{8t}} e^{-\frac{|l|^2}{8t}} \, dl$$

$$\leq Ct^{-n/2-1} \left(\int_{\mathbb{R}^n} l_n^{\frac{r}{r-1}} e^{-\frac{r}{r-1}\frac{|l|^2}{8t}} \, dl \right)^{\frac{r-1}{r}}$$

$$\times \left(\int_{\mathbb{R}^n} e^{-r\frac{|l|^2}{8t}} |\phi_j * \check{a}(x-l)|^r \, dl \right)^{1/r}.$$

Taking $k = \frac{l}{\sqrt{8t}} \frac{\sqrt{r}}{\sqrt{r-1}}$, we have

$$\left(\int l_n^{\frac{r}{r-1}} e^{-\frac{r}{r-1}\frac{|l|^2}{8t}} \, dl \right)^{\frac{r-1}{r}}$$

$$= \left(\frac{r-1}{r} \right)^{\frac{n+1}{2}} (8t)^{\frac{n(r-1)}{2r}+\frac{1}{2}} \left(\int k_n^{\frac{r}{r-1}} e^{-|k|^2} \, dl \right)^{\frac{r-1}{r}}$$

$$\leq C \left(\frac{r-1}{r} \right)^{\frac{n+1}{2}} t^{\frac{n(r-1)}{2r}+\frac{1}{2}}.$$

So, we have

$$\left(\int |\phi_j * v_a(x,t)|^p dx \right)^{1/p}$$

$$\leq Ct^{-\frac{n}{2r}-\frac{1}{2}} \left(\int \left(\int_{\mathbb{R}^n} e^{-r\frac{|l|^2}{8t}} |\phi_j * \check{a}(x-l)|^r \, dl \right)^{p/r} dx \right)^{1/p}.$$

By Minkowski's inequlaity, we have

$$\leq Ct^{-\frac{n}{2r}-\frac{1}{2}} \left(\int \left(\int_{\mathbb{R}^n} e^{-p\frac{|l|^2}{8t}} |\phi_j * \check{a}(x-l)|^p \, dx \right)^{r/p} dl \right)^{1/r}.$$

$$= Ct^{-\frac{n}{2r}-\frac{1}{2}} \left(\int \left(\int_{\mathbb{R}^n} e^{-p\frac{|x-l|^2}{8t}} \, dx \right)^{r/p} |\phi_j * \check{a}(l)|^r dl \right)^{1/r}.$$

Taking $k = (x-l)\sqrt{p}/\sqrt{8t}$, we have

$$= Ct^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{p}\right)-\frac{1}{2}} \left(\int \left(\int_{\mathbb{R}^n} e^{-|k|^2} \, dk \right)^{r/p} |\phi_j * \check{a}(l)|^r dl \right)^{1/r}$$

$$\leq Ct^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{p}\right)-\frac{1}{2}} \left(\int |\phi_j * \check{a}(l)|^r \, dl \right)^{1/r},$$

for sufficiently large t. Hence we have our result.

We now let $r = 1$ and $p \geq 1$. By Minkowski's inequality, we have

$$\left(\int |\phi_j * v_a(x,t)|^p \right)^{1/p}$$

$$\leq Ct^{-n/2-1} \left(\int \left(\int_{\mathbb{R}^n} |\phi_j * \breve{a}(l)| \, |l_n| e^{-\frac{|l|^2}{4t}} \, dl \right)^p dx \right)^{1/p}$$

$$= Ct^{-n/2-1} \left(\int \left(\int_{\mathbb{R}^n} |\phi_j * \breve{a}(l)| \, |x_n - l_n| e^{-\frac{|x-l|^2}{4t}} \, dl \right)^p dx \right)^{1/p}$$

$$\leq Ct^{-n/2-1} \int \left(\int |x_n - l_n|^p e^{-p\frac{|x-l|^2}{4t}} dx \right)^{1/p} |\phi_j * \breve{a}(x - l)| \, dl$$

$$\leq Ct^{-\frac{n}{2}(1-\frac{1}{p})-\frac{1}{2}} \int |\phi_j * \breve{a}(x - l)| \, dl,$$

for sufficiently large t. $\square$

We put

$$\breve{a}_j(k) = \left(\int_{-1}^{1} \frac{k_n}{s^2} a_j(\bar{k}, \frac{k_n}{s}) ds \right),$$

where a_j is the extension of $u_{0,j}$ and $\breve{\mathbf{a}} := (\breve{a}_1, \ldots, \breve{a}_n)$.

COROLLARY 4.2. *Let* $\mathbf{u}$ *be the solution of the Stokes equation* (1.1) *with initial data* $\mathbf{u}_0$. *Assume that* $1 < r \leq p < \infty$ *and* $0 < s \leq q \leq \infty$. *Then, we have that for sufficiently large* t,

$$\|\mathbf{u}(\cdot, t)\|_{B_{p,q}^a(\mathbb{R}_+^n)} \leq Ct^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{p})-\frac{1}{2}} \|\breve{\mathbf{a}}\|_{B_{r,s}^a(\mathbb{R}^n)}, \quad \text{for } -\infty < a < \infty.$$

5. The Hodge decompositions

This section is devoted to the Hodge decomposition.

THEOREM 5.1. *Let* $1 < p < \infty$, $0 < q \leq \infty$, $-\infty < a < \infty$. $B_{p,q}^a(\mathbb{R}^n)$ *is written as* $B_{p,q}^{a,\sigma}(\mathbb{R}^n) \oplus G_{p,q}^a(\mathbb{R}^n)$. $B_{p,q}^a(\mathbb{R}_+^n)$ *is written as* $B_{p,q}^{a,\sigma}(\mathbb{R}_+^n) \oplus G_{p,q}^a(\mathbb{R}_+^n)$. *Here,* $B_{p,q}^{a,\sigma}(\mathbb{R}_+^n)$ *is the closure of* $\{\mathbf{v} \in C_0^\infty(\mathbb{R}_+^n) : \nabla \cdot \mathbf{v} = 0\}$ *in* $B_{p,q}^a(\mathbb{R}_+^n)$, *and* $G_{p,q}^a$ *is the closure in* $B_{p,q}^a$ *of* $\{\nabla \phi : \phi \in C^\infty, \nabla \phi \in B_{p,q}^a\}$.

Proof. It is enough to consider for $\mathbf{f} \in C_0^\infty(\mathbb{R}^n)$ by the density. Let's just consider for the case $n = 3$. For $\mathbf{f} \in C_0^\infty(\mathbb{R}^n)$, consider the problem

$$\Delta \phi = \nabla \cdot \mathbf{f},$$

of which solution is given by

$$\phi = -\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{1}{|x-y|}\nabla\cdot\mathbf{f}(y)dy.$$

By the divergence theorem, we have

$$\phi = \frac{1}{4\pi}\int_{\mathbb{R}^3}\sum_{i=1}^{3}\frac{x_i-y_i}{|x-y|^3}f_i(y)dy,$$

and

$$\frac{\partial}{\partial x_j}\phi(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3}\sum_{i=1}^{3}\left(\frac{\delta_{ij}}{|x-y|^3}-3\frac{(x_i-y_i)(x_j-y_j)}{|x-y|^5}\right)f_i(y)dy,$$

of which kernel is a constant multiple of the Riesz transforms. The symbol of the kernel in $\frac{\partial}{\partial x_j}\phi(x)$ has a form $c\frac{\xi_j}{|\xi|}$, which satisfies $\|m\|_N \le c$ for all $N \ge 0$. By Theorem 2.1, we have

$$\|\nabla\phi\|_{B_{p,q}^a} \le c\|\mathbf{f}\|_{B_{p,q}^a}.$$

Defining $\mathbf{v} = \mathbf{f} - \nabla\phi$, we have $\nabla\cdot\mathbf{v} = 0$ and $\|\mathbf{v}\|_{B_{p,q}^a} \le c\|\mathbf{f}\|_{B_{p,q}^a}$.

In $\mathbb{R}^3_+$, we denote by $x^* = (x_1, x_2, -x_3)$ the reflection of x with respect to $\{x_3 = 0\}$ plane. Then

$$\phi = -\frac{1}{4\pi}\int_{\mathbb{R}^3_+}\left(\frac{1}{|x-y|}+\frac{1}{|x^*-y|}\right)\nabla\cdot\mathbf{f}(y)dy,$$

is the solution of the Poisson problem $\Delta\phi = \nabla\cdot\mathbf{f}$, $\frac{\partial\phi}{\partial x_3}(x_1, x_2, 0) = f_3(x_1, x_2, 0)$.

By divergence theorem,

$$\phi(x)$$
$$= \frac{1}{4\pi}\int_{\mathbb{R}^3_+}\left(\frac{x_j-y_j}{|x-y|^3}+\frac{x_j^*-y_j}{|x^*-y|^3}\right)f_j(y)dy$$
$$= \frac{1}{4\pi}\int_{\mathbb{R}^3_+}\sum_{i=1,2}\frac{x_i-y_i}{|x-y|^3}(f_i(y)+f_i(y^*))+\frac{x_3-y_3}{|x-y|^3}(f_3(y)-f_3(y^*))\,dy,$$

if we extend $\mathbf{f}$ to be zero on $\mathbb{R}^3_-$. We have that $\nabla\phi$ belongs to $B_{p,q}^a(\mathbb{R}^n_+)$, and $\|\nabla\phi\|_{B_{p,q}^a} \le C\|\mathbf{f}\|_{B_{p,q}^a}$, and therefore, $\|\mathbf{v}\|_{B_{p,q}^a}+\|\nabla\phi\|_{B_{p,q}^a} \le C\|\mathbf{f}\|_{B_{p,q}^a}$.
□

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