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NOTES ON KÄHLER SURFACES WITH DISTINCT CONSTANT RICCI EIGENVALUES

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ABSTRACT. In this paper, we study a connected, simply connected homogeneous Kähler surface with distinct constant Ricci eigenvalues, and specify the local structure of them.

1. Introduction

Recently, V. Apostolov, T. Drăghici and A. Moroianu studied compact Kähler manifolds whose Ricci tensor Ric (or rather the symmetric endomorphism of the tangent bundle corresponding to Ric via the metric) has two distinct constant eigenvalues and proved the followings.

THEOREM A ([1]). *Let $M = (M, J, g)$ be a compact Kähler manifold whose Ricci tensor has two distinct non-negative eigenvalues λ and μ. Then, the universal covering of M is the product of two simply connected Kähler-Einstein manifolds of Ricci eigenvalues λ and μ respectively.*

THEOREM B ([1]). *Let $M = (M, J, g)$ be a compact Kähler surface whose Ricci tensor has two distinct constant eigenvalues. Then, one of the following alternatives holds:*
(i) *M is locally symmetric, i.e. M is locally the product of Riemannian surfaces of distinct constant Gaussian curvature;*
(ii) *if M is not as described in* (i), *then the eigenvalues of the Ricci tensor are both negative and (M, J) must be a minimal surface of general type with ample canonical bundle and with even and positive signature. Moreover, in this case, reversing the orientations, the manifold would admit an Einstein, strictly almost Kähler structure.*

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On one hand, H. Shima studied $2n$-dimensional homogeneous Kähler manifolds with non-degenerate Ricci form of signature $(2, 2(n-1))$ and obtained the following classification in dimension 4.

THEOREM C ([8]). *Let $M = (M, J, g)$ be a connected, simply connected homogeneous Kähler surface with non-degenerate Ricci form ρ. Then, the signature of ρ is $(4,0)$ or $(2,2)$ or $(0,4)$. Moreover,*
(i) *if the signature is $(4,0)$, M is either $P_1(\mathbb{C}) \times P_1(\mathbb{C})$ or $P_2(\mathbb{C})$, where $P_n(\mathbb{C})$ is a complex projective space;*
(ii) *if the signature is $(0,4)$, M is either $H_1(\mathbb{C}) \times H_1(\mathbb{C})$ or $H_2(\mathbb{C})$, where $H_n(\mathbb{C})$ is a complex hyperbolic space;*
(iii) *if the signature is $(2,2)$, M is a holomorphic fiber bundle over $H_1(\mathbb{C})$ with fiber $P_1(\mathbb{C})$.*

In this paper, we specify the local structure of a connected, simply connected homogeneous Kähler surface $M = (M, J, g)$ with distinct constant Ricci eigenvalues (§ 3). From the argument of § 3, we will also find that if the signature of Ricci form of M is $(2,2)$, then M is locally the product of Riemannian surfaces of constant Gaussian curvature.

We remark that the problem of existence of Kähler metrics whose Ricci tensor has two distinct constant eigenvalues is related to the Goldberg conjecture (in the open case, i.e. the negative scalar curvature). In fact, in [1], it was showed that the irreducible homogeneous Kähler manifolds whose Ricci tensor has two distinct constant negative eigenvalues give rise to complete (non-compact) Einstein strictly almost Kähler manifolds of any even dimension grater than 4. For example, $M^{2n} = SO(2,n)/(SO(2) \times SO(n))$, $n \geq 3$.

2. Local structures

Let (M, J, g) be a Kähler surface of the constant Ricci eigenvalues λ, μ $(\lambda < \mu)$, and E_λ, E_μ the corresponding J-invariant eigenspaces at peach point of M. E_λ and E_μ give rise to two smooth distributions on M. We denote by Ω the Kähler form of (J, g), which is given by $\Omega(\cdot,\cdot) = g(J\cdot,\cdot)$.

Now, we consider the 2-forms α, β defined respectively by

$$\alpha(X, Y) = \Omega(\mathrm{Pr}^\lambda(X), \mathrm{Pr}^\lambda(Y)), \tag{2.1}$$

for vector fields X, $Y \in TM$, and

$$\beta = \Omega - \alpha$$

where Pr^λ (resp. Pr^μ) denotes the orthogonal projection $\mathrm{Pr}^\lambda : TM \to E_\lambda$ (resp. $\mathrm{Pr}^\mu : TM \to E_\mu$). The Ricci form $\rho(\cdot,\cdot) = \mathrm{Ric}(J\cdot,\cdot)$ is given by

$$\rho = \lambda\alpha + \mu\beta. \tag{2.2}$$

Thus, we have

$$\alpha = \frac{1}{\lambda-\mu}(\rho - \mu\Omega), \qquad \beta = \frac{1}{\lambda-\mu}(\lambda\Omega - \rho). \tag{2.3}$$

Since ρ and Ω are both closed, so are α and β.

PROPOSITION 1. *The distributions E_λ and E_μ are both involutive.*

Proof. Let $\{e_1, e_2 = Je_1\}$ and $\{e_3, e_4 = Je_3\}$ be any local unitary frame fields for E_λ and E_μ respectively. We set

$$\nabla_{e_i} e_j = \sum_{k=1}^{4} \Gamma_{ijk} e_k.$$

Then, we have

$$\Gamma_{ijk} = -\Gamma_{ikj}, \qquad \Gamma_{i\bar{j}\bar{k}} = \Gamma_{ijk}, \tag{2.4}$$

where we adopt the notational convention $e_{\bar{i}} = Je_i$ $(i = 1, \ldots, 4)$. From the definition of α and β, we see that

$$\alpha(e_1, e_2) = -\alpha(e_2, e_1) = 1, \qquad \beta(e_3, e_4) = -\beta(e_4, e_3) = 1, \tag{2.5}$$

and the others are zero. We denote by A and B the linear endomorphisms of TM corresponding to the 2-forms α and β respectively. Since α and β are closed, we have

$$\underset{X,Y,Z}{\mathfrak{S}} g((\nabla_X A)Y, Z) = 0, \qquad \underset{X,Y,Z}{\mathfrak{S}} g((\nabla_X B)Y, Z) = 0,$$

where $\mathfrak{S}_{X,Y,Z}$ denotes the cyclic sum with respect to X, Y, Z. In particular, from (2.4) and (2.5), we get

$$\begin{aligned} 0 &= \underset{e_1,e_2,e_3}{\mathfrak{S}} g((\nabla_{e_1} A)e_2, e_3) = \underset{e_1,e_2,e_3}{\mathfrak{S}} g(\nabla_{e_1}(Ae_2) - A(\nabla_{e_1} e_2), e_3) \\ &= -\Gamma_{113} + \Gamma_{232} = \Gamma_{142} - \Gamma_{241}, \end{aligned}$$

and hence

$$\Gamma_{142} - \Gamma_{241} = 0. \tag{2.6}$$

Similarly, from $\mathfrak{S}_{e_1,e_2,e_4} g((\nabla_{e_1} A)e_2, e_4) = 0$, we get

$$\Gamma_{132} - \Gamma_{231} = 0. \tag{2.7}$$

Thus, E_λ is involutive. By the similar way, we obtain

$$\Gamma_{314} - \Gamma_{413} = \Gamma_{324} - \Gamma_{423} = 0. \tag{2.8}$$

Thus, we see also that E_μ is involutive. □

From Proposition 1, we may choose a local coordinate system (x_1, y_1, x_2, y_2) such that $\{\partial/\partial x_1, \partial/\partial y_1\}$ spans E_λ and $\{\partial/\partial x_2, \partial/\partial y_2\}$ spans E_μ. Since E_λ and E_μ are J-invariant, we may set

$$J\frac{\partial}{\partial x_1} = J_1^1\frac{\partial}{\partial x_1} + J_1^{\bar{1}}\frac{\partial}{\partial y_1}, \qquad J\frac{\partial}{\partial y_1} = J_{\bar{1}}^1\frac{\partial}{\partial x_1} + J_{\bar{1}}^{\bar{1}}\frac{\partial}{\partial y_1},$$
$$J\frac{\partial}{\partial x_2} = J_2^2\frac{\partial}{\partial x_2} + J_2^{\bar{2}}\frac{\partial}{\partial y_2}, \qquad J\frac{\partial}{\partial y_2} = J_{\bar{2}}^2\frac{\partial}{\partial x_2} + J_{\bar{2}}^{\bar{2}}\frac{\partial}{\partial y_2}.$$

Then, $J^2 = -1$ implies

$$\begin{aligned}
(J_1^1)^2 + J_{\bar{1}}^1 J_1^{\bar{1}} &= -1, & (J_{\bar{1}}^{\bar{1}})^2 + J_{\bar{1}}^1 J_1^{\bar{1}} &= -1,\\
J_1^{\bar{1}}(J_1^1 + J_{\bar{1}}^{\bar{1}}) &= 0, & J_{\bar{1}}^1(J_1^1 + J_{\bar{1}}^{\bar{1}}) &= 0,\\
(J_2^2)^2 + J_{\bar{2}}^2 J_2^{\bar{2}} &= -1, & (J_{\bar{2}}^{\bar{2}})^2 + J_{\bar{2}}^2 J_2^{\bar{2}} &= -1,\\
J_2^{\bar{2}}(J_2^2 + J_{\bar{2}}^{\bar{2}}) &= 0, & J_{\bar{2}}^2(J_2^2 + J_{\bar{2}}^{\bar{2}}) &= 0.
\end{aligned} \tag{2.9}$$

Thus, we have $J_1^{\bar{1}} \neq 0$, $J_{\bar{1}}^1 \neq 0$ and

$$J_{\bar{1}}^{\bar{1}} = -J_1^1. \tag{2.10}$$

Since $g(J(\partial/\partial x_1), \partial/\partial x_1) = 0$, we have

$$J_1^1 g_{11} + J_1^{\bar{1}} g_{1\bar{1}} = 0.$$

Thus, we may put

$$J_1^1 = -Fg_{1\bar{1}}, \qquad J_1^{\bar{1}} = Fg_{11}. \tag{2.11}$$

From $g(J(\partial/\partial x_1), \partial/\partial y_1) + g(\partial/\partial x_1, J(\partial/\partial y_1)) = 0$ and (2.10), we have

$$J_1^{\bar{1}} g_{\bar{1}\bar{1}} + J_{\bar{1}}^1 g_{11} = 0,$$

and hence,

$$J_{\bar{1}}^1 = -Fg_{\bar{1}\bar{1}}. \tag{2.12}$$

Taking account of (2.9)~(2.12), we have

$$\det\begin{pmatrix} g_{11} & g_{1\bar{1}} \\ g_{1\bar{1}} & g_{\bar{1}\bar{1}} \end{pmatrix} = \frac{1}{F^2}.$$

We put $e^{2\sigma_1} = 1/F^2$. Summing up the above arguments, we obtain

$$\begin{pmatrix} g_{11} & g_{1\bar{1}} \\ g_{1\bar{1}} & g_{\bar{1}\bar{1}} \end{pmatrix} = e^{-\sigma_1}\begin{pmatrix} J_1^{\bar{1}} & -J_1^1 \\ -J_1^1 & -J_{\bar{1}}^1 \end{pmatrix}. \tag{2.13}$$

Similarly, we get

(2.14)

$$\begin{pmatrix} g_{22} & g_{2\bar{2}} \\ g_{2\bar{2}} & g_{\bar{2}\bar{2}} \end{pmatrix} = e^{-\sigma_2} \begin{pmatrix} J_2^{\bar{2}} & -J_{\bar{2}}^{2} \\ -J_2^2 & -J_{\bar{2}}^{2} \end{pmatrix}, \quad \text{where} \quad e^{2\sigma_2} = \det \begin{pmatrix} g_{22} & g_{2\bar{2}} \\ g_{2\bar{2}} & g_{\bar{2}\bar{2}} \end{pmatrix}.$$

From (2.13) and (2.14), we have

$$\Omega = e^{-\sigma_1} dx_1 \wedge dy_1 + e^{-\sigma_2} dx_2 \wedge dy_2.$$

Since Ω is closed, it must follow that $\sigma_i = \sigma_i(x_i, y_i)$, $(i = 1, 2)$. Thus, we conclude the following.

PROPOSITION 2. *There exists a local Darboux coordinate system* (x_1, y_1, x_2, y_2) *such that* $\{\partial/\partial x_1, \partial/\partial y_1\}$ *and* $\{\partial/\partial x_2, \partial/\partial y_2\}$ *are local frame fields for* E_λ *and* E_μ *respectively.*

We remark that Propositions 1 and 2 do not necessarily guarantee the existence of local complex coordinate system $(z_1 = x_1 + \sqrt{-1}\, y_1, z_2 = x_2 + \sqrt{-1}\, y_2)$ which is compatible with the complex structure J such that $\{\partial/\partial x_1, \partial/\partial y_1\}$ and $\{\partial/\partial x_2, \partial/\partial y_2\}$ are local bases for the distributions E_λ and E_μ respectively. In fact, we may easily show that, for a Kähler surface $M = (M, J, g)$ of constant distinct Ricci eigenvalues λ and μ, if there exists a complex coordinate system $(z_1 = x_1 + \sqrt{-1}\, y_1, z_2 = x_2 + \sqrt{-1}\, y_2)$ around any point of M which is compatible with the complex structure J, then M is locally the product of two Riemann surfaces with Gaussian curvature λ and μ (compare with the Example in § 4).

3. Homogeneous Kähler surface with distinct constant Ricci eigenvalues

An almost Hermitian manifold $M = (M, J, g)$ is homogeneous if there exists a Lie subgroup G of the automorphism group $\mathrm{Aut}(M, J, g)$ which acts transitively on M. The following result is known.

THEOREM D ([5]). *Let* $M = (M, J, g)$ *be a homogeneous almost Hermitian manifold. Then, there exists a skew-symmetric tensor field* T *of type* $(1, 2)$ *satisfying the following conditions for any* $X \in TM$:

(1) $\nabla_X J = T(X) \cdot J$, (2) $\nabla_X R = T(X) \cdot R$, (3) $\nabla_X T = T(X) \cdot T$,

where $T(X)Y = T(X, Y)$ *and the symbol "* $T(X) \cdot$ *" means that* $T(X)$ *acts as a derivation of the tensor algebra. The tensor field* T *is called a homogeneous structure of* M.

Now, let $U(M)$ be the unitary frame bundle over a connected, simply connected, complete almost Hermitian manifold $M = (M, J, g)$ and T a skew-symmetric tensor field of type $(1,2)$ satisfying the conditions (1), (2) and (3) in Theorem D. Let u_0 be an arbitrary fixed point of $U(M)$ and ∇^* be the linear connection defined by $\nabla^*_X = \nabla_X - T(X)$ for $X \in TM$. Then, we may easily check that g and J are parallel with respect to ∇^*, i.e. $\nabla^* g = 0$ and $\nabla^* J = 0$. The holonomy bundle G of ∇^* through u_0 has a Lie group structure with identity u_0 and acts transitively on M as a group of automorphisms. This is the converse of Theorem D.

A homogeneous almost Hermitian manifold $M = (M, J, g)$ satisfies the following condition $H(m)$ for all non-negative integers m.
$H(m)$: for any x, $y \in M$, there exists a linear isometry $\Phi : (T_xM, g_x) \to (T_yM, g_y)$ such that

$$\Phi \circ J_x = J_y \circ \Phi, \quad \text{and} \quad \Phi(\nabla^k R)_x = (\nabla^k R)_y \quad \text{for } k = 0,\, 1,\, \ldots,\, m.$$

In fact, Φ is given by putting $\Phi = d\varphi_x$, the differential mapping of a holomorphic isometry φ of M with $\varphi(x) = y$. For a Riemannian manifold $M = (M, g)$, we denote by $P(m)$ the following condition:

$P(m)$: for x, $y \in M$, there exists a linear isometry $\Phi : (T_xM, g_x) \to (T_yM, g_y)$ such that

$$\Phi(\nabla^k R)_x = (\nabla^k R)_y \quad \text{for } k = 0,\, 1,\, \ldots,\, m.$$

A Riemannian manifold M satisfying the condition $P(0)$ is called curvature homogeneous. In general, a curvature homogeneous Riemannian manifold is not homogeneous even if it is connected and simply connected ([4]), and the following result is known.

THEOREM E ([7]). *Let $M = (M, g)$ be a 4-dimensional connected, simply connected, complete Riemannian manifold satisfying the condition $P(1)$. Then, M is homogeneous, i.e. M is a locally symmetric space or a group space.*

Now, let $M = (M, J, g)$ be a $2n$-dimensional homogeneous almost Hermitian manifold with the automorphisms group $\hat{G} = \mathrm{Aut}(M, J, g)$ which acts transitively on M. For a point $u_0 = (x; e_1, Je_1, \ldots, e_n, Je_n) \in U(M)$, we set $G = \{\, \hat{\gamma}(u_0) \mid \hat{\gamma} \in \hat{G} \,\}$. Then, G is a principal subbundle of $U(M)$ and has a Lie group structure with the identity $\varepsilon = u_0$ which is isomorphic to $\hat{G}$ by the canonical correspondence $\hat{\gamma} \mapsto \gamma = \hat{\gamma}(\varepsilon)$. Moreover, G acts transitively on M as a group of automorphism by $\gamma(x) = \pi(\gamma\gamma')$, $\gamma \in G$, $x \in M$, where $\pi : U(M) \to M$ is the bundle projection and $\pi(\gamma') = x$. Then, relating to G (or $\hat{G}$), we have a skew-symmetric tensor field T of type $(1, 2)$ on M satisfying the conditions (1), (2) and (3) in Theorem D. We define functions $f_{h_s \cdots h_1; ijkl}$ and T_{ijk} on $U(M)$ respectively by

$$f_{h_s \cdots h_1; ijkl}(u) = g((\nabla^s_{e_{h_s} \cdots e_{h_1}} R)(e_i, e_j)e_k, e_l), \; T_{ijk}(u) = g(T(e_i, e_j), e_k),$$

for $u = (x; e_1, Je_1, \dots, e_n, Je_n) \in U(M, J, g)$. Then, we have the following.

PROPOSITION 3. *On G, the functions $f_{h_s \cdots h_1; ijkl}$ and T_{ijk} are constant.*

For each point $x \in M$, we define a Lie group G^x_s by

$$G^x_s = \left\{ a \in U(T_x M, J_x, g_x) \,\middle|\, t_i \circ a = t_i \text{ for } i = 0, \dots, s \right\},$$

where t_i and $t_i \circ a$ are defined respectively by

$$\begin{aligned} t_i(X_i, \dots, X_1, Y, Z) &= (\nabla^i_{X_i \cdots X_1} R)(Y, Z), \\ (t_i \circ a)(X_i, \dots, X_1, Y, Z) &= a^{-1}(\nabla^i_{aX_i \cdots aX_1} R)(aY, aZ)a, \end{aligned}$$

for $X_i, \dots, X_1, Y, Z \in T_x M$. We denote by $\mathfrak{g}^x_s$ the Lie algebra of G^x_s and put $\mathfrak{k} = \mathfrak{g}^x_1$. We note that $\mathfrak{g}^x_1$ and $\mathfrak{g}^y_1$ are isomorphic for any x, $y \in M$.

In the sequel, we assume that (M, J, g) is a connected, simply connected homogeneous Kähler surface with distinct constant Ricci eigenvalues λ, μ $(\lambda < \mu)$. Then, the following cases are possible.

Case I: $\mathfrak{k} = \{0\}$,

Case II: $\mathfrak{k} = \mathfrak{u}(1) \oplus \{0\}$ or Case II′: $\mathfrak{k} = \{0\} \oplus \mathfrak{u}(1)$,

Case III: $\mathfrak{k} = \mathfrak{u}(1) \oplus \mathfrak{u}(1)$.

Case I

In this case, the corresponding Lie group G of automorphisms acts simply transitively on M. We denote by ∇^* the linear connection on the principal subbundle G of $U(M, J, g)$ through $u_0 \in U(M, J, g)$. The tensor field T of type (1,2) defined by $T = \nabla^* - \nabla$ satisfies the condition (1), (2) and (3) in Theorem D. If $\gamma = (x; e_1, e_2 = Je_1, e_3, e_4 = Je_3) \in G$, then we see that $\nabla^*_{e_i} e_j = 0$ and hence $T_{ijk} = \Gamma_{ijk}$ are regarded as constant-valued functions on M. Then, taking account of (2.4) and

(2.6)∼(2.8), we have

$$
(3.1)\qquad
\begin{aligned}
R_{1313} &= \Gamma_{341}(2\Gamma_{134} - \Gamma_{112}) - \Gamma_{132}(2\Gamma_{312} - \Gamma_{334}) \\
&\quad + \Gamma_{132}^2 + \Gamma_{142}^2 + \Gamma_{341}^2 + \Gamma_{342}^2, \\
R_{1414} &= -\Gamma_{341}(2\Gamma_{134} - \Gamma_{112}) - \Gamma_{142}(2\Gamma_{412} - \Gamma_{434}) \\
&\quad + \Gamma_{132}^2 + \Gamma_{142}^2 + \Gamma_{341}^2 + \Gamma_{342}^2, \\
R_{2424} &= \Gamma_{142}(2\Gamma_{412} - \Gamma_{434}) - \Gamma_{342}(2\Gamma_{234} - \Gamma_{212}) \\
&\quad + \Gamma_{132}^2 + \Gamma_{142}^2 + \Gamma_{341}^2 + \Gamma_{342}^2, \\
R_{2323} &= \Gamma_{132}(2\Gamma_{312} - \Gamma_{334}) + \Gamma_{342}(2\Gamma_{234} - \Gamma_{212}) \\
&\quad + \Gamma_{132}^2 + \Gamma_{142}^2 + \Gamma_{341}^2 + \Gamma_{342}^2.
\end{aligned}
$$

Since $R_{1313} = R_{2424}$ and $R_{1414} = R_{2323}$, we have

$$
\begin{aligned}
&\Gamma_{341}(2\Gamma_{134} - \Gamma_{112}) - \Gamma_{132}(2\Gamma_{312} - \Gamma_{334}) \\
&= \Gamma_{142}(2\Gamma_{412} - \Gamma_{434}) - \Gamma_{342}(2\Gamma_{234} - \Gamma_{212}), \\
&-\Gamma_{341}(2\Gamma_{134} - \Gamma_{112}) - \Gamma_{142}(2\Gamma_{412} - \Gamma_{434}) \\
&= \Gamma_{132}(2\Gamma_{312} - \Gamma_{334}) + \Gamma_{342}(2\Gamma_{234} - \Gamma_{212}),
\end{aligned}
$$

and hence,

$$
(3.2)\qquad
\begin{aligned}
\Gamma_{132}(2\Gamma_{312} - \Gamma_{334}) + \Gamma_{142}(2\Gamma_{412} - \Gamma_{434}) &= 0, \\
\Gamma_{341}(2\Gamma_{134} - \Gamma_{112}) + \Gamma_{342}(2\Gamma_{234} - \Gamma_{212}) &= 0.
\end{aligned}
$$

On one hand, we have

$$
\begin{aligned}
R_{1314} &= -\Gamma_{142}(2\Gamma_{312} - \Gamma_{334}) - \Gamma_{342}(2\Gamma_{134} - \Gamma_{112}), \\
R_{1413} &= -\Gamma_{132}(2\Gamma_{412} - \Gamma_{434}) - \Gamma_{342}(2\Gamma_{134} - \Gamma_{112}).
\end{aligned}
$$

Since $R_{1314} = R_{1413}$, we have

$$
(3.3)\qquad \Gamma_{132}(2\Gamma_{412} - \Gamma_{434}) - \Gamma_{142}(2\Gamma_{312} - \Gamma_{334}) = 0.
$$

Similarly, from $R_{1323} = R_{2313}$, we obtain

$$
(3.4)\qquad \Gamma_{341}(2\Gamma_{234} - \Gamma_{212}) - \Gamma_{342}(2\Gamma_{134} - \Gamma_{112}) = 0.
$$

From the equalities (3.2), (3.3) and (3.4), we see that the following cases are possible.

Case(I-1): $(\Gamma_{132}, \Gamma_{142}) \neq (0,0)$ and $(\Gamma_{341}, \Gamma_{342}) \neq (0,0)$,

Case(I-2): $(\Gamma_{132}, \Gamma_{142}) = (0,0)$ and $(\Gamma_{341}, \Gamma_{342}) \neq (0,0)$,

Case(I-3): $(\Gamma_{132}, \Gamma_{142}) \neq (0,0)$ and $(\Gamma_{341}, \Gamma_{342}) = (0,0)$,

Case(I-4): $(\Gamma_{132}, \Gamma_{142}) = (0,0)$ and $(\Gamma_{341}, \Gamma_{342}) = (0,0)$.

Case (I-1). In this case, (3.2), (3.3) and (3.4) are reduced to

(3.5)
$$2\Gamma_{134}-\Gamma_{112}=0,\ 2\Gamma_{234}-\Gamma_{212}=0,\ 2\Gamma_{312}-\Gamma_{334}=0,\ 2\Gamma_{412}-\Gamma_{434}=0.$$

Taking account of (3.5), we have

$$\begin{aligned}R_{1213} &= -\Gamma_{132}(2\Gamma_{212}-\Gamma_{234})-\Gamma_{142}(2\Gamma_{112}-\Gamma_{134})\\ &= -3(\Gamma_{132}\Gamma_{234}+\Gamma_{142}\Gamma_{134}),\\ R_{3413} &= -\Gamma_{341}(2\Gamma_{434}-\Gamma_{412})-\Gamma_{342}(2\Gamma_{334}-\Gamma_{312})\\ &= -3(\Gamma_{341}\Gamma_{412}+\Gamma_{342}\Gamma_{312}).\end{aligned}$$

Since $R_{1213}+R_{3413}=0$, we have

$$\Gamma_{132}\Gamma_{234}+\Gamma_{142}\Gamma_{134}+\Gamma_{341}\Gamma_{412}+\Gamma_{342}\Gamma_{312}=0. \tag{3.6}$$

Similarly, from $R_{1214}+R_{3414}=0$, we obtain

$$\Gamma_{132}\Gamma_{134}-\Gamma_{142}\Gamma_{234}+\Gamma_{342}\Gamma_{412}-\Gamma_{341}\Gamma_{312}=0. \tag{3.7}$$

On one hand, we get

$$\begin{aligned}R_{1412} &= -2\Gamma_{341}\Gamma_{132}+2\Gamma_{342}\Gamma_{142}+\Gamma_{132}\Gamma_{112}-\Gamma_{142}\Gamma_{212}\\ &\quad+\Gamma_{134}\Gamma_{312}+\Gamma_{412}\Gamma_{212}-\Gamma_{341}\Gamma_{312}+\Gamma_{342}\Gamma_{412},\\ R_{1434} &= 2\Gamma_{341}\Gamma_{132}-2\Gamma_{342}\Gamma_{142}+\Gamma_{132}\Gamma_{134}-\Gamma_{142}\Gamma_{234}\\ &\quad+\Gamma_{134}\Gamma_{334}+\Gamma_{412}\Gamma_{234}-\Gamma_{341}\Gamma_{334}+\Gamma_{342}\Gamma_{434}.\end{aligned}$$

Since $R_{1412}+R_{1434}=0$, we have

$$\Gamma_{132}\Gamma_{134}-\Gamma_{142}\Gamma_{234}+\Gamma_{312}(\Gamma_{134}-\Gamma_{341})+\Gamma_{412}(\Gamma_{234}+\Gamma_{342})=0. \tag{3.8}$$

Similarly, from $R_{1312}+R_{1334}=0$, we obtain

$$\Gamma_{132}\Gamma_{234}+\Gamma_{142}\Gamma_{134}-\Gamma_{312}(\Gamma_{234}-\Gamma_{342})+\Gamma_{412}(\Gamma_{134}+\Gamma_{341})=0. \tag{3.9}$$

Thus, from (3.6)$\sim$(3.9), we have

$$\Gamma_{312}\Gamma_{134}+\Gamma_{412}\Gamma_{234}=0, \qquad \Gamma_{412}\Gamma_{134}-\Gamma_{312}\Gamma_{234}=0. \tag{3.10}$$

By direct calculation, we get

$$R_{1212}=\Gamma_{112}^2+\Gamma_{212}^2-2(\Gamma_{132}^2+\Gamma_{142}^2)=4(\Gamma_{134}^2+\Gamma_{234}^2)-2(\Gamma_{132}^2+\Gamma_{142}^2),$$
$$R_{1234}=2(\Gamma_{132}^2+\Gamma_{142}^2+\Gamma_{134}^2+\Gamma_{234}^2).$$

Thus, we have

$$-\lambda=R_{1212}+R_{1234}=6(\Gamma_{134}^2+\Gamma_{234}^2). \tag{3.11}$$

Similarly, we have

$$-\mu=R_{3412}+R_{3434}=6(\Gamma_{312}^2+\Gamma_{412}^2)(\geq 0). \tag{3.12}$$

Since $\lambda \neq 0$, we see that $(\Gamma_{134}, \Gamma_{234}) \neq (0,0)$ by virtue of (3.11). Thus, (3.10) implies $(\Gamma_{312}, \Gamma_{412}) = (0,0)$. Then, (3.8) and (3.9) are reduced to

$$\Gamma_{142}\Gamma_{134} + \Gamma_{132}\Gamma_{234} = 0, \qquad \Gamma_{132}\Gamma_{134} - \Gamma_{142}\Gamma_{234} = 0.$$

Since $(\Gamma_{134}, \Gamma_{234}) \neq (0,0)$, it must follow that $(\Gamma_{132}, \Gamma_{142}) = (0,0)$. But, this contradicts to the assumption of case (I-1). Thus, case (I-1) cannot occur.

Case (I-2). In this case, from (3.2) and (3.4), we have

$$2\Gamma_{134} - \Gamma_{112} = 0, \qquad 2\Gamma_{234} - \Gamma_{212} = 0. \tag{3.13}$$

Further, we get

$$R_{1212} = \Gamma_{112}^2 + \Gamma_{212}^2, \qquad R_{1313} = \Gamma_{341}^2 + \Gamma_{342}^2, \qquad R_{1414} = \Gamma_{341}^2 + \Gamma_{342}^2. \tag{3.14}$$

Thus, we have

$$-\lambda = R_{1212} + R_{1313} + R_{1414} = \Gamma_{112}^2 + \Gamma_{212}^2 + 2(\Gamma_{341}^2 + \Gamma_{342}^2), \tag{3.15}$$

and hence, $\lambda < 0$. By direct calculation, we get also

$$R_{1213} = 0, \qquad R_{3413} = -\Gamma_{341}(2\Gamma_{434} - \Gamma_{412}) - \Gamma_{342}(2\Gamma_{334} - \Gamma_{312}),$$
$$R_{1214} = 0, \qquad R_{3414} = \Gamma_{342}(2\Gamma_{434} - \Gamma_{412}) - \Gamma_{341}(2\Gamma_{334} - \Gamma_{312}).$$

Thus, form $R_{1213} + R_{3413} = 0$ and $R_{1214} + R_{3414} = 0$, we have

$$-\Gamma_{341}(2\Gamma_{434} - \Gamma_{412}) - \Gamma_{342}(2\Gamma_{334} - \Gamma_{312}) = 0,$$
$$\Gamma_{342}(2\Gamma_{434} - \Gamma_{412}) - \Gamma_{341}(2\Gamma_{334} - \Gamma_{312}) = 0.$$

Since $(\Gamma_{341}, \Gamma_{342}) \neq (0,0)$, we get

$$2\Gamma_{334} - \Gamma_{312} = 0, \qquad 2\Gamma_{434} - \Gamma_{412} = 0. \tag{3.16}$$

Thus, we have also

$$R_{3413} = R_{3414} = 0.$$

Further, we have

$$R_{1234} = 2(\Gamma_{134}^2 + \Gamma_{234}^2), \qquad R_{3434} = -2(\Gamma_{341}^2 + \Gamma_{342}^2) + \Gamma_{334}^2 + \Gamma_{434}^2, \tag{3.17}$$

and hence,

$$-\mu = 2(\Gamma_{134}^2 + \Gamma_{234}^2) - 2(\Gamma_{341}^2 + \Gamma_{342}^2) + \Gamma_{334}^2 + \Gamma_{434}^2. \tag{3.18}$$

On one hand, we have

$$R_{1314} = 0, \qquad R_{3412} = 2(\Gamma_{341}^2 + \Gamma_{342}^2 + \Gamma_{334}^2 + \Gamma_{434}^2).$$

Since $R_{1234} = R_{3412}$, we have

$$\Gamma_{134}^2 + \Gamma_{234}^2 = \Gamma_{341}^2 + \Gamma_{342}^2 + \Gamma_{334}^2 + \Gamma_{434}^2. \tag{3.19}$$

Substituting this equality into (3.18), we get

$$-\mu = 3(\Gamma_{334}^2 + \Gamma_{434}^2), \tag{3.20}$$

and hence, $\mu \leq 0$. Further, we have

$$R_{1212} = \Gamma_{112}^2 + \Gamma_{212}^2 = 4(\Gamma_{134}^2 + \Gamma_{234}^2) = 2R_{1234},$$

and hence,

$$-\lambda = R_{1212} + R_{1234} = 3R_{1234}.$$

Therefore, we obtain

$$R_{1212} = -\frac{2}{3}\lambda, \qquad R_{1234} = -\frac{\lambda}{3}, \qquad R_{1313} = R_{1414} = -\frac{\lambda}{6}. \tag{3.21}$$

Since $R_{1412} = R_{1214} = 0$ and $R_{1312} = R_{1213} = 0$, we have

$$\Gamma_{312}(\Gamma_{134} - \Gamma_{341}) + \Gamma_{412}(\Gamma_{212} + \Gamma_{342}) = 0,$$
$$\Gamma_{312}(\Gamma_{212} - \Gamma_{342}) - \Gamma_{412}(\Gamma_{134} + \Gamma_{341}) = 0.$$

Thus, we see that $(\Gamma_{312}, \Gamma_{412}) \neq (0,0)$ or $(\Gamma_{312}, \Gamma_{412}) = (0,0)$. First, we assume that $(\Gamma_{312}, \Gamma_{412}) \neq (0,0)$. Then, the equality

$$\Gamma_{341}^2 + \Gamma_{342}^2 = \Gamma_{134}^2 + \Gamma_{212}^2$$

holds. On one hand, from $R_{2412} = R_{1213} = 0$ and $R_{2312} = -R_{1214} = 0$, we have

$$(\Gamma_{234} - \Gamma_{342})\Gamma_{312} - (\Gamma_{341} + \Gamma_{112})\Gamma_{412} = 0,$$
$$(\Gamma_{341} - \Gamma_{112})\Gamma_{312} - (\Gamma_{234} + \Gamma_{342})\Gamma_{412} = 0.$$

Since $(\Gamma_{312}, \Gamma_{412}) \neq (0,0)$, we have

$$\Gamma_{341}^2 + \Gamma_{342}^2 = \Gamma_{112}^2 + \Gamma_{234}^2 = 4\Gamma_{134}^2 + \frac{1}{4}\Gamma_{212}^2.$$

Thus, we have $\Gamma_{134}^2 = \Gamma_{212}^2/4$, and hence $\Gamma_{112}^2 = \Gamma_{212}^2$. If we put $\gamma = \Gamma_{112}^2$, then

$$R_{1212} = 2\gamma, \qquad R_{1234} = \gamma, \qquad R_{1313} = R_{1414} = \frac{5}{4}\gamma.$$

Thus, we have $\gamma = R_{1234} = R_{1313} + R_{1414} = 5\gamma/2$, and hence $\gamma = 0$. Then, we have $\lambda = 0$, but this is a contradiction. Next, we consider the case $(\Gamma_{312}, \Gamma_{412}) = (0,0)$. Then, from (3.16), we have $\Gamma_{334} = \Gamma_{434} = 0$. Thus, from (3.20), we have $\mu = 0$. Further, from (3.19), we have $\Gamma_{134}^2 + \Gamma_{234}^2 = \Gamma_{341}^2 + \Gamma_{342}^2$. Thus, from (3.14), (3.17) and (3.21), we have

$$R_{3434} = -2(\Gamma_{341}^2 + \Gamma_{342}^2) = -2R_{1313} = \frac{\lambda}{3}. \tag{3.22}$$

Case (I-3). This case is essentially same as Case (I-2).

Case (I-4). In this case, M is locally the product of Riemannian surfaces of constant Gaussian curvature λ and μ.

Case II

In this case, taking account of Proposition 3, by direct calculation, we see that

$$
\begin{aligned}
&R_{1213}=0, \quad R_{1214}=0, \quad R_{1224}=0, \quad R_{1223}=0,\\
&R_{3413}=0, \quad R_{3414}=0, \quad R_{3424}=0, \quad R_{3423}=0,\\
&R_{1314}=0, \quad R_{2324}=0, \quad R_{1323}=0, \quad R_{1424}=0,
\end{aligned}
\tag{3.23}
$$

and R_{1212}, $R_{1313}=R_{1414}$, $R_{2424}=R_{2323}$, R_{3434} are constant. Further, we see that T_{312}, T_{412}, T_{334}, T_{434}, T_{132}, T_{231}, T_{142}, T_{241} are constant, and

$$
T_{132}+T_{231}=0, \qquad T_{142}+T_{241}=0, \qquad T_{\bar{i}\bar{j}\bar{k}}=T_{ijk} \quad (1\le i,j,k\le 4).
\tag{3.24}
$$

Thus, from direct calculation, we have

$$
\begin{aligned}
&\nabla_1 R_{1213}=-\Gamma_{132}(R_{1212}-4R_{1313}), && \nabla_1 R_{1213}=-T_{132}(R_{1212}-4R_{1313}),\\
&\nabla_1 R_{1214}=-\Gamma_{142}(R_{1212}-4R_{1313}), && \nabla_1 R_{1214}=-T_{142}(R_{1212}-4R_{1313}),\\
&\nabla_2 R_{1213}=\Gamma_{241}(R_{1212}-4R_{1313}), && \nabla_2 R_{1213}=T_{241}(R_{1212}-4R_{1313}),\\
&\nabla_2 R_{1214}=-\Gamma_{231}(R_{1212}-4R_{1313}), && \nabla_2 R_{1214}=-T_{231}(R_{1212}-4R_{1313}),
\end{aligned}
$$

and hence,

$$
\begin{aligned}
&(T_{132}-\Gamma_{132})(R_{1212}-4R_{1313})=0,\\
&(T_{142}-\Gamma_{142})(R_{1212}-4R_{1313})=0,\\
&(T_{241}-\Gamma_{241})(R_{1212}-4R_{1313})=0,\\
&(T_{231}-\Gamma_{231})(R_{1212}-4R_{1313})=0.
\end{aligned}
\tag{3.25}
$$

Further, we get

$$
\begin{aligned}
&\nabla_1 R_{3413}=\Gamma_{132}(R_{3434}-4R_{1313}), && \nabla_1 R_{3413}=T_{132}(R_{3434}-4R_{1313}),\\
&\nabla_1 R_{3414}=\Gamma_{142}(R_{3434}-4R_{1313}), && \nabla_1 R_{3414}=T_{142}(R_{3434}-4R_{1313}),\\
&\nabla_2 R_{3413}=-\Gamma_{241}(R_{3434}-4R_{1313}), && \nabla_2 R_{3413}=-T_{241}(R_{3434}-4R_{1313}),\\
&\nabla_2 R_{3414}=\Gamma_{231}(R_{3434}-4R_{1313}), && \nabla_2 R_{3414}=T_{231}(R_{3434}-4R_{1313}),
\end{aligned}
$$

and hence,

$$
\begin{aligned}
&(T_{132}-\Gamma_{132})(R_{3434}-4R_{1313})=0,\\
&(T_{142}-\Gamma_{142})(R_{3434}-4R_{1313})=0,\\
&(T_{241}-\Gamma_{241})(R_{3434}-4R_{1313})=0,\\
&(T_{231}-\Gamma_{231})(R_{3434}-4R_{1313})=0.
\end{aligned}
\tag{3.26}
$$

Thus, from (3.25) and (3.26), we see that $(R_{1212} - 4R_{1313}, R_{3434} - 4R_{1313}) = (0,0)$ or $(R_{1212} - 4R_{1313}, R_{3434} - 4R_{1313}) \neq (0,0)$. First, we assume that $(R_{1212} - 4R_{1313}, R_{3434} - 4R_{1313}) = (0,0)$. Then, we have

$$\lambda = -R_{1212} - R_{1313} - R_{1414} = -6R_{1313} = -R_{3434} - 2R_{1313} = \mu.$$

But, this is a contradiction, and hence $(R_{1212} - 4R_{1313}, R_{3434} - 4R_{1313}) \neq (0,0)$. In this case, we have

$$\Gamma_{132} = T_{132}, \qquad \Gamma_{231} = T_{231}, \qquad \Gamma_{142} = T_{142}, \qquad \Gamma_{241} = T_{241},$$

and hence, from (2.6), (2.7) and (3.24), we have

$$\Gamma_{132} = \Gamma_{231} = \Gamma_{142} = \Gamma_{241} = 0.$$

Therefore, (M, J, g) is locally the product of Riemannian surfaces of constant Gaussian curvature λ and μ.

Case II$'$

This case is essentially same as Case II.

Case III

In this case, M is locally the product of Riemannian surfaces of constant Gaussian curvature λ and μ.

4. Example

In this section, we introduce an Example established by O. Kowalski ([3]). Let $M = \mathbb{R}^4_+ = \{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 > 0 \}$ and put

$$e_1 = -x_1 \frac{\partial}{\partial x_1}, \qquad e_2 = -\frac{1}{x_1}\frac{\partial}{\partial y_1},$$
$$e_3 = \sqrt{x_1}\, y_1 \frac{\partial}{\partial y_2} + \sqrt{x_1}\frac{\partial}{\partial x_2}, \qquad e_4 = \frac{1}{\sqrt{x_1}}\frac{\partial}{\partial y_2}.$$

We define an almost Hermitian structure (J, g) by $Je_1 = e_2$, $Je_3 = e_4$, and $g(e_i, e_j) = \delta_{ij}$ $(1 \leq i, j \leq 4)$. Let $\{e^i\}$ be the dual basis of $\{e_i\}$. Then, we have

$$e^1 = -\frac{1}{x_1}dx_1,\ e^2 = -x_1 dy_1,\ e^3 = \frac{1}{\sqrt{x_1}}dx_2,\ e^4 = \sqrt{x_1}\, dy_2 - \sqrt{x_1}y_1 dx_2,$$

and

$$\Omega = e^1 \wedge e^2 + e^3 \wedge e^4 = dx_1 \wedge dy_1 + dx_2 \wedge dy_2.$$

Namely, (x_1, y_1, x_2, y_2) is a Darboux coordinate system of M. With respect to the coordinate system (x_1, y_1, x_2, y_2), the almost complex structure J is expressed by

$$J\frac{\partial}{\partial x_1} = \frac{1}{x_1^2}\frac{\partial}{\partial y_1}, \ J\frac{\partial}{\partial y_1} = -x_1^2\frac{\partial}{\partial x_1},$$

$$J\frac{\partial}{\partial x_2} = x_1 y_1\frac{\partial}{\partial x_2} + \left(x_1 y_1^2 + \frac{1}{x_1}\right)\frac{\partial}{\partial y_2}, \ J\frac{\partial}{\partial y_2} = -x_1 y_1\frac{\partial}{\partial y_2} - x_1\frac{\partial}{\partial x_2}.$$

By direct calculation, we get

$$[e_1, e_2] = e_2, \qquad [e_1, e_3] = -\frac{1}{2}e_3, \qquad [e_1, e_4] = \frac{1}{2}e_4,$$
$$[e_2, e_3] = -e_4, \qquad [e_2, e_4] = 0, \qquad [e_3, e_4] = 0,$$

and

$$\Gamma_{112} = 0, \quad \Gamma_{113} = 0, \quad \Gamma_{114} = 0, \quad \Gamma_{123} = 0, \quad \Gamma_{124} = 0, \quad \Gamma_{134} = 0,$$
$$\Gamma_{212} = -1, \quad \Gamma_{213} = 0, \quad \Gamma_{214} = 0, \quad \Gamma_{223} = 0, \quad \Gamma_{224} = 0, \quad \Gamma_{234} = -\frac{1}{2},$$
$$\Gamma_{312} = 0, \quad \Gamma_{313} = \frac{1}{2}, \quad \Gamma_{314} = 0, \quad \Gamma_{323} = 0, \quad \Gamma_{324} = \frac{1}{2}, \quad \Gamma_{334} = 0,$$
$$\Gamma_{412} = 0, \quad \Gamma_{413} = 0, \quad \Gamma_{414} = -\frac{1}{2}, \quad \Gamma_{423} = \frac{1}{2}, \quad \Gamma_{424} = 0, \quad \Gamma_{434} = 0.$$

We have also

$$R_{1212} = 1, \ R_{1234} = \frac{1}{2}, \ R_{1313} = \frac{1}{4}, \ R_{1324} = \frac{1}{4},$$
$$R_{1414} = \frac{1}{4}, \ R_{1423} = -\frac{1}{4}, \ R_{2323} = \frac{1}{4}, \ R_{2424} = \frac{1}{4}, \ R_{3434} = -\frac{1}{2},$$

and otherwise being zero up to sign. Thus, the Ricci eigenvalues are $\lambda = -3/2$ and $\mu = 0$. We may easily check that (M, J, g) is a Kähler manifold. This example corresponds to the case (I-2).

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