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BOUNDEDNESS AND INVERSION PROPERTIES OF CERTAIN CONVOLUTION TRANSFORMS

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ABSTRACT. For a fixed function h we deal with a class of convolution transforms $f \longrightarrow f * h$, where

$$(f * h)(x) = \frac{1}{2x} \int_{\mathbf{R}^2_+} e^{-\frac{1}{2}\left(x\frac{u^2+y^2}{uy} + \frac{yu}{x}\right)} f(u)h(y)dudy, x \in \mathbf{R}_+$$

as integral operators $L_p(\mathbf{R}_+; xdx) \to L_r(\mathbf{R}_+; xdx)$, $p, r \geq 1$. The Young type inequality is proved. Boundedness properties are investigated. Certain examples of these operators are considered and inversion formulas in $L_2(\mathbf{R}_+; xdx)$ are obtained.

1. Introduction

As it was shown in [6], Ch. 4, the convolution transform

$$(1.1)\quad (f * h)(x) = \frac{1}{2x} \int_0^\infty \int_0^\infty e^{-\frac{1}{2}\left(x\frac{u^2+y^2}{uy} + \frac{yu}{x}\right)} f(u)h(y)dudy, \ x > 0$$

is well defined in the Banach ring $L^0(\mathbf{R}_+) \equiv L_1(\mathbf{R}_+; K_0(x)dx)$, where the space $L^0(\mathbf{R}_+)$ is equipped with the norm

$$(1.2)\qquad ||f||_{L^0(\mathbf{R}_+)} = \int_0^\infty K_0(x)|f(x)|dx.$$

Here $K_0(x)$ is the Macdonald function of the index zero [1]. It satisfies the following norm inequality (see [6], Th. 4.9)

$$(1.3)\qquad ||f * h||_{L^0(\mathbf{R}_+)} \leq ||f||_{L^0(\mathbf{R}_+)} ||h||_{L^0(\mathbf{R}_+)}.$$

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The convolution transform (1.1) is related to the Kontorovich-Lebedev transformation [3], [5] and [6]

$$K_{i\tau}[f] = \int_0^\infty K_{i\tau}(x)f(x)dx, \ \tau \in \mathbf{R}_+, \tag{1.4}$$

by means of the factorization identity, i.e.

$$K_{i\tau}[f * h] = K_{i\tau}[f]K_{i\tau}[h], \tau \in \mathbf{R}_+. \tag{1.5}$$

It is proved in [5] and [6] that transformation (1.4) is a bounded operator from $L^0(\mathbf{R}_+)$ into the space of bounded continuous functions vanishing at infinity. Its kernel consists of the Macdonald function $K_\nu(x)$ of the pure imaginary index $\nu = i\tau$. This function (cf. [1]) satisfies the differential equation

$$z^2\frac{d^2u}{dz^2} + z\frac{du}{dz} - (z^2+\nu^2)u = 0. \tag{1.6}$$

It is the solution that remains bounded as z tends to infinity on the real line. The Macdonald function has the asymptotic behavior [1]

$$K_\nu(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}[1 + O(1/z)], \qquad z \to \infty, \tag{1.7}$$

and near the origin

$$z^{|\nu|}K_\nu(z) = 2^{|\nu|-1}\Gamma(|\nu|) + o(1), \ z \to 0, \tag{1.8}$$

$$K_0(z) = -\log z + O(1), \ z \to 0. \tag{1.9}$$

Furthermore, it satisfies the inequality $|K_{i\tau}(x)| \leq K_0(x)$, $\tau \geq 0$ and therefore we have

$$|K_{i\tau}[f]| \leq ||f||_{L^0(\mathbf{R}_+)}.$$

We note here that the Macdonald function $K_\nu(z)$ has the following integral representations (see [4], Vol. I, relations (2.4.18.4), (2.3.16.1))

$$K_\nu(z) = \int_0^\infty e^{-z\cosh t}\cosh \nu t dt = \frac{1}{2}\left(\frac{z}{2}\right)^\nu \int_0^\infty e^{-t-\frac{z^2}{4t}} t^{-\nu-1}dt. \tag{1.10}$$

According to Lemma 2.3 [6] we see that the space $L^0(\mathbf{R}_+)$ contains all spaces $L_p(\mathbf{R}_+; xdx)$, $p > 2$, which are normed by

$$||f||_{L_p(\mathbf{R}_+;xdx)} = \left(\int_0^\infty |f(x)|^p xdx\right)^{1/p}. \tag{1.11}$$

Therefore integrals (1.1) and (1.4) exist as Lebesgue integrals when $f \in L_p(\mathbf{R}_+; xdx)$. However, if we define transformation (1.4) in $L_2(\mathbf{R}_+; xdx)$ as

$$K_{i\tau}[f] = \lim_{N\to\infty} \int_{1/N}^{N} K_{i\tau}(x) f(x) dx, \tag{1.12}$$

where the limit is taken in mean square sense with respect to the norm of $L_2(\mathbf{R}_+; \frac{2}{\pi^2}\tau \sinh \pi\tau d\tau)$, then (cf. [6], section 2.3) $K_{i\tau} : L_2(\mathbf{R}_+; xdx) \leftrightarrow L_2(\mathbf{R}_+; \frac{2}{\pi^2}\tau \sinh \pi\tau d\tau)$ is bounded and forms an isometric isomorphism between these spaces with the Parseval identity [3], [6] of the form

$$\int_0^\infty x|f(x)|^2 dx = \frac{2}{\pi^2} \int_0^\infty \tau \sinh \pi\tau |K_{i\tau}[f]|^2 d\tau. \tag{1.13}$$

Two definitions (1.4) and (1.12) are equivalent, if $f \in L_2(\mathbf{R}_+; xdx) \cap L^0(\mathbf{R}_+; xdx)$. The inverse operator in the latter case is given by the formula

$$f(x) = \frac{2}{\pi^2} \lim_{N\to\infty} \int_0^N \tau \sinh \pi\tau \frac{K_{i\tau}(x)}{x} K_{i\tau}[f] d\tau, \tag{1.14}$$

where the limit is meant in mean square sense with respect to the norm of the space $L_2(\mathbf{R}_+; xdx)$. It can be written for almost all $x \in \mathbf{R}_+$ in the equivalent form (see [6], formula (2.70))

$$f(x) = \frac{2}{x\pi^2} \frac{d}{dx} \int_0^\infty \int_0^x \tau \sinh \pi\tau K_{i\tau}(y) K_{i\tau}[f] dy d\tau. \tag{1.15}$$

The main goal of the paper is to establish boundedness properties of the convolution transform (1.1) with respect to $f \in L_p(\mathbf{R}_+; xdx), p \geq 2$ and $h \in L^0(\mathbf{R}_+; xdx)$. To do this we prove the Young type inequality [2] and represent (1.1) in terms of the Kontorovich-Lebedev transformation (1.14). Finally we give certain examples of such transforms and find their inversions in $L_2(\mathbf{R}_+; xdx)$ as compositions of the convolution transform and differential operator of the infinite order.

2. Young type inequality

We have

THEOREM 1. *Let* $0 < \gamma, \beta < 1$, $\gamma + \beta \leq 1$ *and* $f \in L_{1/\beta}(\mathbf{R}_+; xdx)$, $h \in L_{1/\gamma}(\mathbf{R}_+; xdx)$. *Then convolution transform* (1.1) *exists as a double*

Lebesgue integral for all $x > 0$ *and belongs to* $L_{1/\mu}(\mathbf{R}_+; xdx)$, *where* $\frac{\gamma+\beta+|\gamma-\beta|}{2} < \mu \le 1$. *Moreover, it satisfies*

$$||f * h||_{L_{1/\mu}(\mathbf{R}_+;xdx)} \le C_{\mu,\gamma,\beta}||f||_{L_{1/\beta}(\mathbf{R}_+;xdx)}||h||_{L_{1/\gamma}(\mathbf{R}_+;xdx)}, \tag{2.1}$$

where

$$C_{\mu,\gamma,\beta} = 2^{\gamma+\beta-1}\left(\int_0^\infty x^{1-\frac{\gamma+\beta}{\mu}} K^{\frac{1-\gamma-\beta}{\mu}}_{\frac{\gamma-\beta}{1-\gamma-\beta}}(x) K_0^{\frac{\gamma+\beta}{\mu}}(x)dx\right)^\mu. \tag{2.2}$$

When $\gamma \ne \beta$, $\mu \le \gamma+\beta < 1$, *then* (2.2) *satisfies*

$$\begin{aligned} &C_{\mu,\gamma,\beta} \\ &\le \pi^{\frac{\gamma+\beta}{2}} 2^{|\gamma-\beta|+\gamma+\beta-2}\left(\frac{\gamma+\beta}{\mu}\right)^{\gamma+\beta+|\gamma-\beta|-2\mu}\Gamma^\mu\left(2-\frac{\gamma+\beta+|\gamma-\beta|}{\mu}\right) \\ &\times\left[\Gamma\left(\frac{|\gamma-\beta|}{1-\gamma-\beta}\right)\right]^{1-\gamma-\beta}\left[\frac{\Gamma\left(\frac{2\mu-\gamma-\beta-|\gamma-\beta|}{2(\gamma+\beta)}\right)}{\Gamma\left(\frac{2\mu-|\gamma-\beta|}{2(\gamma+\beta)}\right)}\right]^{\gamma+\beta}. \end{aligned} \tag{2.3}$$

Finally, when $\gamma = \beta = \nu \le \frac{1}{2}$, $\nu < \mu \le 1$, *then* (2.2) *is estimated by*

$$C_{\mu,\nu,\nu} \le 2^{2(\nu-1)}\mu^{2(\mu-\nu)}\sqrt{\pi}\frac{\Gamma(\mu-\nu)}{\Gamma\left(\mu-\nu+\frac{1}{2}\right)}. \tag{2.4}$$

Proof. Taking $0 < \alpha, \beta, \gamma < 1$, $\alpha+\beta+\gamma = 1$ we begin to estimate (1.1) by using Hőlder inequality for three functions. Thus via (1.10) and the elementary inequality $K_0(\sqrt{x^2+u^2}) \le K_0(x)$ we obtain

$$\begin{aligned} &|(f*h)(x)| \\ &\le \frac{1}{2x}\left(\int_0^\infty\int_0^\infty e^{-\frac{1}{2}\left(x\frac{u^2+y^2}{uy}+\frac{yu}{x}\right)}|f(u)|^{\frac{1}{\beta}}u\frac{dudy}{y}\right)^\beta \\ &\times\left(\int_0^\infty\int_0^\infty e^{-\frac{1}{2}\left(x\frac{u^2+y^2}{uy}+\frac{yu}{x}\right)}|h(y)|^{\frac{1}{\gamma}}y\frac{dudy}{u}\right)^\gamma \\ &\times\left(\int_0^\infty\int_0^\infty e^{-\frac{1}{2}\left(x\frac{u^2+y^2}{uy}+\frac{yu}{x}\right)}u^{\frac{\gamma-\beta}{\alpha}}y^{\frac{\beta-\gamma}{\alpha}}dudy\right)^\alpha \\ &= \frac{2^{\gamma+\beta-1}}{x}\left(\int_0^\infty K_0(\sqrt{x^2+u^2})|f(u)|^{\frac{1}{\beta}}udu\right)^\beta \end{aligned} \tag{2.5}$$

$$\times \left(\int_0^\infty K_0(\sqrt{x^2+y^2})|h(y)|^{\frac{1}{\gamma}} y dy\right)^\gamma$$

$$\times \left(\int_0^\infty \int_0^\infty e^{-\frac{1}{2}\left(x\frac{u^2+y^2}{uy}+\frac{yu}{x}\right)} u^{\frac{\gamma-\beta}{\alpha}} y^{\frac{\beta-\gamma}{\alpha}} dudy\right)^\alpha$$

$$\leq \frac{2^{\gamma+\beta-1}}{x} K_0^{\gamma+\beta}(x) \left(\int_0^\infty \int_0^\infty e^{-\frac{1}{2}\left(x\frac{u^2+y^2}{uy}+\frac{yu}{x}\right)} u^{\frac{\gamma-\beta}{\alpha}} y^{\frac{\beta-\gamma}{\alpha}} dudy\right)^\alpha$$

$$\times ||f||_{L_{1/\beta}(\mathbf{R}_+;xdx)} ||h||_{L_{1/\gamma}(\mathbf{R}_+;xdx)}.$$

The latter double integral in (2.5) we calculate employing polar coordinates $u = r\cos\varphi,\ y = r\sin\varphi,\ r \geq 0, 0 \leq \varphi \leq \frac{\pi}{2}$ and making then the substitution $t = \tan\varphi$. Appealing again to the second integral representation of the Macdonald function in (1.10) we have the result

$$\left(\int_0^\infty \int_0^\infty e^{-\frac{1}{2}\left(x\frac{u^2+y^2}{uy}+\frac{yu}{x}\right)} u^{\frac{\gamma-\beta}{\alpha}} y^{\frac{\beta-\gamma}{\alpha}} dudy\right)^\alpha = x^\alpha K^\alpha_{\frac{\gamma-\beta}{\alpha}}(x).$$

By letting $\alpha = 1-\beta-\gamma$ inequality (2.5) takes the form

$$|(f*h)(x)|$$
$$\leq 2^{\gamma+\beta-1} x^{-\gamma-\beta} K^{1-\beta-\gamma}_{\frac{\gamma-\beta}{1-\beta-\gamma}}(x) K_0^{\gamma+\beta}(x) ||f||_{L_{1/\beta}(\mathbf{R}_+;xdx)} ||h||_{L_{1/\gamma}(\mathbf{R}_+;xdx)}.$$

Hence (see (1.11))

$$||f*h||_{L_{1/\mu}(\mathbf{R}_+;xdx)}$$
$$= \left(\int_0^\infty |(f*h)(x)|^{\frac{1}{\mu}} xdx\right)^\mu$$
$$\leq 2^{\gamma+\beta-1}\left(\int_0^\infty x^{1-\frac{\gamma+\beta}{\mu}} K^{\frac{1-\gamma-\beta}{\mu}}_{\frac{\gamma-\beta}{1-\gamma-\beta}}(x) K_0^{\frac{\gamma+\beta}{\mu}}(x)dx\right)^\mu$$
$$\times ||f||_{L_{1/\beta}(\mathbf{R}_+;xdx)} ||h||_{L_{1/\gamma}(\mathbf{R}_+;xdx)}$$

and we arrive at (2.1), where integral (2.2) is convergent under conditions of the theorem, which follow correspondingly from the asymptotic behavior (1.7), (1.8), (1.9) of the Macdonald function in the neighborhood of the origin and infinity.

In order to prove inequality (2.3) we appeal to the second integral representation (1.10) (cf. (1.8)) and easily deduce for all $x > 0$ the following inequality

$$x^{|\nu|} K_\nu(x) \leq 2^{|\nu|-1}\Gamma(|\nu|),\ \nu > 0.$$

Hence we have

$$K^{\frac{1-\gamma-\beta}{\mu}}_{\frac{\gamma-\beta}{1-\gamma-\beta}}(x) \le x^{-\frac{|\gamma-\beta|}{\mu}} 2^{\frac{|\gamma-\beta|-1+\gamma+\beta}{\mu}} \left[\Gamma\left(\frac{|\gamma-\beta|}{1-\gamma-\beta}\right)\right]^{\frac{1-\gamma-\beta}{\mu}}.$$

Then applying (1.10), the generalized Minkowski inequality and calculating the values of elementary integrals we may estimate integral (2.2). Indeed, we obtain

$$\begin{aligned}
&C_{\mu,\gamma,\beta} \\
&\le 2^{2(\gamma+\beta-1)+|\gamma-\beta|}\left[\Gamma\left(\frac{|\gamma-\beta|}{1-\gamma-\beta}\right)\right]^{1-\gamma-\beta} \\
&\times \left(\int_0^\infty x^{1-\frac{\gamma+\beta}{\mu}-\frac{|\gamma-\beta|}{\mu}}\left(\int_0^\infty e^{-x\cosh t}dt\right)^{\frac{\gamma+\beta}{\mu}}dx\right)^{\mu} \\
&\le 2^{2(\gamma+\beta-1)+|\gamma-\beta|}\left[\Gamma\left(\frac{|\gamma-\beta|}{1-\gamma-\beta}\right)\right]^{1-\gamma-\beta} \\
&\times \left(\int_0^\infty dt\left(\int_0^\infty x^{1-\frac{\gamma+\beta}{\mu}-\frac{|\gamma-\beta|}{\mu}} e^{-x\frac{(\gamma+\beta)}{\mu}\cosh t}dx\right)^{\frac{\mu}{\gamma+\beta}}\right)^{\gamma+\beta} \\
&= 2^{2(\gamma+\beta-1)+|\gamma-\beta|} \\
&\times \left[\Gamma\left(\frac{|\gamma-\beta|}{1-\gamma-\beta}\right)\right]^{1-\gamma-\beta}\left(\frac{\gamma+\beta}{\mu}\right)^{\gamma+\beta+|\gamma-\beta|-2\mu} \\
&\times \Gamma^{\mu}\left(2-\frac{\gamma+\beta+|\gamma-\beta|}{\mu}\right)\left(\int_0^\infty \frac{dt}{\cosh^{\frac{2\mu-|\gamma-\beta|-\gamma-\beta}{\gamma+\beta}} t}\right)^{\gamma+\beta} \\
&= 2^{2(\mu-1)-\gamma-\beta}\left(\frac{\gamma+\beta}{\mu}\right)^{\gamma+\beta+|\gamma-\beta|-2\mu} \\
&\times \Gamma^{\mu}\left(2-\frac{\gamma+\beta+|\gamma-\beta|}{\mu}\right)\left[\Gamma\left(\frac{|\gamma-\beta|}{1-\gamma-\beta}\right)\right]^{1-\gamma-\beta} \\
&\times \left[\frac{\Gamma^2\left(\frac{2\mu-\gamma-\beta-|\gamma-\beta|}{2(\gamma+\beta)}\right)}{\Gamma\left(\frac{2\mu-|\gamma-\beta|-\gamma-\beta}{\gamma+\beta}\right)}\right]^{\gamma+\beta}.
\end{aligned}$$

Employing the Gauss-Legendre duplication formula for Gamma functions (see [6], formula (1.30)) in the latter Gamma-ratio we get immediately inequality (2.3).

In the case (2.4) we find directly

$$(2.6)\quad \begin{aligned} &C_{\mu,\nu,\nu}\\ &= 2^{2\nu-1}\left(\int_0^\infty x^{1-\frac{2\nu}{\mu}} K_0^{\frac{1}{\mu}}(x)dx\right)^\mu\\ &\le 2^{2\nu-1}\int_0^\infty dt\left(\int_0^\infty x^{1-\frac{2\nu}{\mu}} e^{-\frac{x}{\mu}\cosh t}dx\right)^\mu\\ &= 2^{2(\nu-1)}\mu^{2(\mu-\nu)}\sqrt{\pi}\frac{\Gamma(\mu-\nu)}{\Gamma\left(\mu-\nu+\frac{1}{2}\right)} \end{aligned}$$

and complete the proof of Theorem 1. □

COROLLARY 1. *Let* $1 < s < \infty$ *and* $h \in L_s(\mathbf{R}_+; xdx)$. *Then operator* $\mathcal{K}_h : L_p(\mathbf{R}_+; xdx) \to L_r(\mathbf{R}_+; xdx)$, $(\mathcal{K}_h f)(x) = (f * h)(x)$ *is bounded, where* $1 < p, r < \infty$ *and* $\frac{1}{r} > \max\left(\frac{1}{p}, \frac{1}{s}\right)$.

In particular, let $h(x) = e^{-x\cos\delta}x^{\xi-1}$, $0 \le \delta < \frac{\pi}{2}$, $\xi > 0$. Then it is not difficult to see that $h \in L_s(\mathbf{R}_+; xdx)$, when $\xi > 1 - \frac{2}{s}$. Substituting h in (1.1) and calculating the inner integral with respect to y (see [6], section 4.5) we arrive at the following transformation

$$(2.7)\quad (\mathcal{K}f)(x) = x^{\xi-1}\int_0^\infty \frac{u^\xi K_\xi(\sqrt{x^2+u^2+2xu\cos\delta})}{(x^2+u^2+2xu\cos\delta)^{\xi/2}} f(u)du.$$

So if, for instance, $s > 2$, then $L_s(\mathbf{R}_+; xdx) \subset L^0(\mathbf{R}_+)$ and (2.7) is a bounded operator from $L_2(\mathbf{R}_+; xdx)$ into $L_r(\mathbf{R}_+; xdx)$, $1 < r < 2$. Theorem 1 excludes the case $r = 2$. Nevertheless, we prove in the next section that (1.1) keeps be bounded as the operator from $L_2(\mathbf{R}_+; xdx)$ into $L_2(\mathbf{R}_+; xdx)$ for any $h \in L^0(\mathbf{R}_+)$. Moreover, we establish the generalized Parseval equality for the convolution transform (1.1) and obtain its inversion formula in certain particular cases.

3. Boundedness properties in $L_2(\mathbf{R}_+; xdx)$

Let us extend the norm inequality (2.1) for $(f * h)(x)$ if one of the functions, say h, belongs to $L^0(\mathbf{R}_+)$ and $f \in L_2(\mathbf{R}_+; xdx)$.

We have

LEMMA 1. *Let* $f \in L_2(\mathbf{R}_+; xdx)$ *and* $h \in L^0(\mathbf{R}_+)$. *Then convolution transform* (1.1) *exists for each* $x > 0$ *as the Lebesgue double integral and belongs to* $L_2(\mathbf{R}_+; xdx)$. *Moreover,*

$$(3.1)\quad ||f * h||_{L_2(\mathbf{R}_+;xdx)} \le ||f||_{L_2(\mathbf{R}_+;xdx)}||h||_{L^0(\mathbf{R}_+)}.$$

Proof. Indeed, with Schwarz's inequality we deduce

$$|(f*h)(x)|^2 \le \frac{1}{4x^2}\int_0^\infty\int_0^\infty e^{-\frac{1}{2}\left(x\frac{u^2+y^2}{uy}+\frac{yu}{x}\right)}|h(y)|\frac{dudy}{u}$$
$$\times\int_0^\infty\int_0^\infty u|f(u)|^2 e^{-\frac{1}{2}\left(x\frac{u^2+y^2}{uy}+\frac{yu}{x}\right)}|h(y)|dudy.$$

Since (see (1.10))

$$\int_0^\infty e^{-\frac{1}{2}\left(x\frac{u^2+y^2}{uy}+\frac{yu}{x}\right)}\frac{du}{u} = 2K_0(\sqrt{x^2+y^2}), \tag{3.2}$$

it follows that

(3.3)

$$|(f*h)(x)|^2$$
$$\le \frac{1}{2x^2}\int_0^\infty K_0(\sqrt{x^2+y^2})|h(y)|dy$$
$$\times\int_0^\infty\int_0^\infty u|f(u)|^2 e^{-\frac{1}{2}\left(x\frac{u^2+y^2}{uy}+\frac{yu}{x}\right)}|h(y)|dudy$$
$$\le \frac{1}{2x^2}\int_0^\infty K_0(y)|h(y)|dy\int_0^\infty\int_0^\infty u|f(u)|^2 e^{-\frac{1}{2}\left(x\frac{u^2+y^2}{uy}+\frac{yu}{x}\right)}|h(y)|dudy.$$

Hence multiplying both sides of (3.2) by x we integrate with respect to $x \in \mathbf{R}_+$. Inverting the order of integration by the Fubini theorem we invoke (3.2) to obtain

(3.4)

$$\int_0^\infty x|(f*h)(x)|^2 dx$$
$$\le \int_0^\infty u|f(u)|^2 du\int_0^\infty K_0(y)|h(y)|dy\int_0^\infty K_0(\sqrt{u^2+y^2})|h(y)|dy$$
$$\le \int_0^\infty u|f(u)|^2 du\left(\int_0^\infty K_0(y)|h(y)|dy\right)^2.$$

Now we recall norm (1.11) and write (3.4) in equivalent form (3.1). Lemma 1 is proved. □

The relationship between operator (1.1) and the Kontorovich-Lebedev transformation (1.4) under the conditions of Lemma 1 is given by

THEOREM 2. *Let f, h be under conditions of Lemma 1. Then the transform $(f*h)(x)$ satisfies factorization identity* (1.5). *Furthermore,* *for almost all $x > 0$ the generalized Parseval equality holds*

$$(3.5)\qquad (f*h)(x) = \frac{2}{\pi^2}\lim_{N\to\infty}\int_0^N \tau\sinh\pi\tau\frac{K_{i\tau}(x)}{x}K_{i\tau}[f]K_{i\tau}[h]d\tau,$$

where the limit is taken with respect to the norm of $L_2(\mathbf{R}_+; xdx)$. In particular, it can be written in the form

$$(3.6)\qquad (f*h)(x) = \frac{2}{\pi^2}\int_0^\infty \tau\sinh\pi\tau\frac{K_{i\tau}(x)}{x}K_{i\tau}[f]K_{i\tau}[h]d\tau, x > 0,$$

if the latter integral converges absolutely and uniformly on $x \ge x_0 > 0$.

Proof. By virtue of the formula (1.15) we have for almost all $x > 0$

$$(3.7)\qquad x(f*h)(x) = \frac{2}{\pi^2}\frac{d}{dx}\int_0^\infty\int_0^x \tau\sinh\pi\tau K_{i\tau}(y)K_{i\tau}[f*h]dyd\tau.$$

Meanwhile, since $h \in L^0(\mathbf{R}_+)$ then it follows that $|K_{i\tau}[h]| \le ||h||_{L^0(\mathbf{R}_+)}$. Further, as it is proved in [6], Lemma 2.5, the kernel $\int_0^x K_{i\tau}(y)dy \in L_2(\mathbf{R}_+; \tau\sinh\pi\tau d\tau)$ for each $x > 0$. Therefore we have that the product $K_{i\tau}[h]\int_0^x K_{i\tau}(y)dy \in L_2(\mathbf{R}_+; \tau\sinh\pi\tau d\tau)$. In the same time if we denote by

$$\theta_x(y) = \begin{cases} 1, & \text{if } y \in [0,x], \\ 0, & \text{if } y \in (x,\infty), \end{cases}$$

we see that $\theta_x(y) \in L^0(\mathbf{R}_+)$ and the factorization property (1.5) takes place in the Banach ring $L^0(\mathbf{R}_+)$ for the functions $h, \theta_x(y)$, namely

$$K_{i\tau}[h]K_{i\tau}[\theta_x] = K_{i\tau}[h*\theta_x].$$

Thus Lemma 1 and the Parseval equality (1.13) yield

$$(3.8)\qquad \begin{aligned} &\frac{2}{\pi^2}\int_0^\infty\int_0^x \tau\sinh\pi\tau K_{i\tau}(y)K_{i\tau}[h]K_{i\tau}[f]dyd\tau \\ &= \frac{2}{\pi^2}\int_0^\infty \tau\sinh\pi\tau K_{i\tau}[f]K_{i\tau}[h*\theta_x]d\tau \\ &= \int_0^\infty uf(u)(h*\theta_x)(u)du. \end{aligned}$$

Substituting the double integral (1.1) for $(h*\theta_x)(u)$ in (3.8) and inverting the order of integration by using Fubini's theorem, we get

$$\int_0^\infty uf(u)(h*\theta_x)(u)du = \int_0^x v(f*h)(v)dv.$$

Consequently, for almost all $x > 0$, we obtain

$$(3.9) \quad x(f * h)(x) = \frac{2}{\pi^2} \frac{d}{dx} \int_0^\infty \int_0^x \tau \sinh \pi\tau K_{i\tau}(y) K_{i\tau}[f] K_{i\tau}[h] dy d\tau$$

and comparing with (3.7) we verify the factorization equality (1.5) for $(f * h)(x)$ under conditions of the theorem.

It is possible to justify the differentiation under the integral sign in (3.9) by the absolute and uniform convergence of the differentiated integral. Therefore in this case we write (3.9) in the form (3.6). However, comparing with (1.15) we find, that for the sequence f_N of $L_2(\mathbf{R}_+; xdx)$-functions, which is defined by (1.14) the differentiation in (3.9) is performed and

$$(3.10) \quad \begin{aligned} (f_N * h)(x) &= \frac{2}{\pi^2} \int_0^\infty \tau \sinh \pi\tau \frac{K_{i\tau}(x)}{x} K_{i\tau}[f_N] K_{i\tau}[h] d\tau \\ &= \frac{2}{\pi^2} \int_0^N \tau \sinh \pi\tau \frac{K_{i\tau}(x)}{x} K_{i\tau}[f] K_{i\tau}[h] d\tau, \end{aligned}$$

where

$$K_{i\tau}[f_N] = \begin{cases} K_{i\tau}[f], & \text{if } \tau \in [0, N], \\ 0, & \text{if } \tau \in (N, \infty). \end{cases}$$

Indeed, the integral (3.10) converges uniformly with respect to x over any finite interval $(0, N)$. Further, invoking (3.1) we derive

$$\begin{aligned} &||f * h - f_N * h||_{L_2(\mathbf{R}_+;xdx)} = ||(f - f_N) * h||_{L_2(\mathbf{R}_+;xdx)} \\ &\leq ||f - f_N||_{L_2(\mathbf{R}_+;xdx)} ||h||_{L^0(\mathbf{R}_+)} \to 0, N \to \infty. \end{aligned}$$

Thus the limit of $(f_N * h)(x)$ with respect to the norm in $L_2(\mathbf{R}_+; xdx)$ coincides with $(f * h)(x)$ and we prove (3.5) and complete the proof of Theorem 2. □

COROLLARY 1. *Under conditions of Lemma 1 the Parseval equality* (1.13) *takes the form*

$$\int_0^\infty x |(f * h)(x)|^2 dx = \frac{2}{\pi^2} \int_0^\infty \tau \sinh \pi\tau |K_{i\tau}[f] K_{i\tau}[h]|^2 d\tau.$$

In particular, for $f \in L^0(\mathbf{R}_+) \cap L_2(\mathbf{R}_+; xdx)$ it gives

$$\int_0^\infty x |(f * \overline{f})(x)|^2 dx = \frac{2}{\pi^2} \int_0^\infty \tau \sinh \pi\tau |K_{i\tau}[f]|^4 d\tau.$$

If $h \in L^0 \cap L_2(\mathbf{R}_+; xdx)$, then $K_{i\tau}[h] \in L_2(\mathbf{R}_+; \frac{2}{\pi^2} \tau \sinh \pi\tau d\tau)$. Therefore via Schwarz's inequality we see that convolution transform (1.1) can be written in the form (3.6). Invoking relation (2.16.2.1) from [4], Vol. II we obtain

$$\begin{aligned}
&\int_0^\infty x|(f*h)(x)|dx \\
(3.11)\qquad &\le \frac{2}{\pi^2}\int_0^\infty \tau\sinh\pi\tau|K_{i\tau}[f]K_{i\tau}[h]|\left|\int_0^\infty K_{i\tau}(x)dx\right|d\tau \\
&= \frac{2}{\pi}\int_0^\infty \tau\sinh\left(\frac{\pi}{2}\tau\right)|K_{i\tau}[f]K_{i\tau}[h]|d\tau.
\end{aligned}$$

Hence by virtue of (1.13) we find

$$\begin{aligned}
&\frac{2}{\pi}\int_0^\infty \tau\sinh\left(\frac{\pi}{2}\tau\right)|K_{i\tau}[f]K_{i\tau}[h]|d\tau \\
&\le \frac{1}{\pi}\int_0^\infty \tau\sinh\pi\tau|K_{i\tau}[f]K_{i\tau}[h]|d\tau \\
&\le \frac{\pi}{2}\left(\frac{2}{\pi^2}\int_0^\infty \tau\sinh\pi\tau|K_{i\tau}[f]|^2d\tau\right)^{1/2} \\
&\times\left(\frac{2}{\pi^2}\int_0^\infty \tau\sinh\pi\tau|K_{i\tau}[h]|^2d\tau\right)^{1/2} \\
&= \frac{\pi}{2}||K_{i\tau}[f]||_{L_2(\mathbf{R}_+;\frac{2}{\pi^2}\tau\sinh\pi\tau d\tau)}||K_{i\tau}[h]||_{L_2(\mathbf{R}_+;\frac{2}{\pi^2}\tau\sinh\pi\tau d\tau)} \\
&= \frac{\pi}{2}||f||_{L_2(\mathbf{R}_+;xdx)}||h||_{L_2(\mathbf{R}_+;xdx)}.
\end{aligned}$$

Combining with (3.11) we arrive at the following norm inequality for operator (1.1) in $L_1(\mathbf{R}_+;xdx)$

$$||f*h||_{L_1(\mathbf{R}_+;xdx)} \le \frac{\pi}{2}||f||_{L_2(\mathbf{R}_+;xdx)}||h||_{L_2(\mathbf{R}_+;xdx)},$$

which is a case of (2.4) when $\nu=\frac{1}{2}$, $\mu=1$.

4. Examples

In this section we study boundedness and inversion properties of certain convolution transforms of type (2.7) in the space $L_2(\mathbf{R}_+;xdx)$ by letting concrete values of parameters ξ, δ. Indeed, taking into account a convergence of the inner integral with respect to y in (1.1) we may put in (2.7), for instance, $\delta=\frac{\pi}{2}$, $\xi=1$ and $\delta=0$, $\xi=\frac{1}{2}$. Then for each case we obtain, correspondingly, $h_1(x)=1$, $h_2(x)=\frac{e^{-x}}{\sqrt{x}}$ and we consider the following operators $(\mathcal{K}_{h_i} f)(x) = (f * h_i)(x), x > 0,\ i = 1, 2$. According to (2.7) the operator $(\mathcal{K}_{h_1} f)(x)$ takes the form

$$(\mathcal{K}_{h_1} f)(x) = \int_0^\infty \frac{K_1(\sqrt{x^2+u^2})}{\sqrt{x^2+u^2}} u f(u) du. \tag{4.1}$$

Meanwhile, since the Macdonald function of the index $\frac{1}{2}$ reduces to

$$K_{1/2}(z) = e^{-z}\sqrt{\frac{\pi}{2z}}$$

(see [1]), we may write $(\mathcal{K}_{h_2} f)(x)$ as

$$(\mathcal{K}_{h_2} f)(x) = \sqrt{\frac{\pi}{2x}} \int_0^\infty \frac{e^{-x-u}}{x+u} \sqrt{u} f(u) du. \tag{4.2}$$

The result is stated by

THEOREM 3. *Operators $\mathcal{K}_{h_i} : L_2(\mathbf{R}_+; xdx) \to L_2(\mathbf{R}_+; xdx),\ i = 1, 2$ are bounded and exist for all $x > 0$ as Lebesgue integrals* (4.1), (4.2), *respectively. Moreover, the following norm inequalities*

$$||\mathcal{K}_{h_1} f||_{L_2(\mathbf{R}_+;xdx)} \le \frac{\pi}{2} ||f||_{L_2(\mathbf{R}_+;xdx)}, \tag{4.3}$$

$$||\mathcal{K}_{h_2} f||_{L_2(\mathbf{R}_+;xdx)} \le \pi\sqrt{\frac{\pi}{2}} ||f||_{L_2(\mathbf{R}_+;xdx)}, \tag{4.4}$$

hold. Finally, an arbitrary $f \in L_2(\mathbf{R}_+; xdx)$ is to be determined by the corresponding inversion formulas

$$f(x) = \lim_{N\to\infty} \frac{2}{\pi x} \prod_{n=1}^{N} \left(1 + \frac{x\left(x - \frac{d}{dx} - x\frac{d^2}{dx^2}\right)}{(2n-1)^2}\right) x(\mathcal{K}_{h_1} f)(x), \tag{4.5}$$

(4.6)

$$f(x) = \frac{\sqrt{2}}{\pi\sqrt{\pi}} \lim_{N\to\infty} \frac{1}{x} \prod_{n=1}^{N} \left(1 + \frac{4x\left(x - \frac{d}{dx} - x\frac{d^2}{dx^2}\right)}{(2n-1)^2}\right) x(\mathcal{K}_{h_2} f)(x),$$

where the limit is taken with respect to the norm in $L_2(\mathbf{R}_+; xdx)$.

Proof. By using relations (2.16.2.1) and (2.16.6.4) from [4], Vol. II we find that $K_{i\tau}[h_1] = \frac{\pi}{2\cosh(\pi\tau/2)}$, $K_{i\tau}[h_2] = \frac{\pi\sqrt{\pi}}{\sqrt{2}\cosh \pi\tau}$. Hence since $h_1, h_2 \in L^0(\mathbf{R}_+)$ we may calculate their norms by (1.11) and the above mentioned values of integrals. Thus via Lemma 1 we immediately arrive at the inequalities (4.3), (4.4). Appealing to Theorem 2 and Schwarz's inequality it is not difficult to verify that operators (4.1), (4.2) may be written as

$$(4.7)\qquad (\mathcal{K}_{h_1}f)(x) = \frac{2}{\pi}\int_0^\infty \tau\sinh\left(\frac{\pi\tau}{2}\right)\frac{K_{i\tau}(x)}{x}K_{i\tau}[f]d\tau,$$

$$(4.8)\qquad (\mathcal{K}_{h_2}f)(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \tau\tanh\pi\tau\frac{K_{i\tau}(x)}{x}K_{i\tau}[f]d\tau.$$

Let us set now

$$(4.9)\qquad f_N(x) = \frac{2}{\pi x}\prod_{n=1}^{N}\left(1+\frac{x\left(x-\frac{d}{dx}-x\frac{d^2}{dx^2}\right)}{(2n-1)^2}\right)x(\mathcal{K}_{h_1}f)(x),$$

where the product means a composition of differential operators. We will show that $||f_N - f||_{L_2(\mathbf{R}_+;xdx)} \to 0,\ N\to\infty$ and therefore establish inversion formula (4.5) of the convolution transform (4.1). In the similar manner we may prove (4.6).

In view of (4.7) we have

(4.10)

$$f_N(x)$$
$$= \frac{4}{\pi^2 x}\prod_{n=1}^{N}\left(1+\frac{x\left(x-\frac{d}{dx}-x\frac{d^2}{dx^2}\right)}{(2n-1)^2}\right)\int_0^\infty \tau\sinh\left(\frac{\pi}{2}\tau\right)K_{i\tau}(x)K_{i\tau}[f]d\tau$$
$$= \frac{4}{\pi^2 x}\int_0^\infty \tau\sinh\left(\frac{\pi}{2}\tau\right)\prod_{n=1}^{N}\left(1+\frac{x\left(x-\frac{d}{dx}-x\frac{d^2}{dx^2}\right)}{(2n-1)^2}\right)K_{i\tau}(x)K_{i\tau}[f]d\tau.$$

The change of the order of operators in (4.10) is due to the absolute and uniform convergence on $x \geq x_0 > 0$ of the latter integral. Indeed, invoking (1.6) we find that

$$\prod_{n=1}^{N}\left(1+\frac{x\left(x-\frac{d}{dx}-x\frac{d^2}{dx^2}\right)}{(2n-1)^2}\right)K_{i\tau}(x) = K_{i\tau}(x)\prod_{n=1}^{N}\left(1+\frac{\tau^2}{(2n-1)^2}\right).$$

Consequently,

(4.11)

$$f_N(x) = \frac{4}{\pi^2 x}\int_0^\infty \tau\sinh\left(\frac{\pi}{2}\tau\right)K_{i\tau}(x)\prod_{n=1}^{N}\left(1+\frac{\tau^2}{(2n-1)^2}\right)K_{i\tau}[f]d\tau.$$

Splitting (4.11) on two integrals $\int_0^E$ and $\int_E^\infty$ it is not difficult to verify the interchange in the first integral over any finite interval $[0,E]$ since the integrand is analytic with respect to $x > 0$. To change the order in the second integral we show that

$$\left| \int_E^\infty \tau \sinh\left(\frac{\pi}{2}\tau\right) K_{i\tau}(x) \prod_{n=1}^{N} \left(1 + \frac{\tau^2}{(2n-1)^2}\right) K_{i\tau}[f] d\tau \right| \to 0, E \to \infty,$$

uniformly for all $x > 0$. With Schwarz's inequality we obtain
(4.12)

$$\left| \int_E^\infty \tau \sinh\left(\frac{\pi}{2}\tau\right) K_{i\tau}(x) \prod_{n=1}^{N} \left(1 + \frac{\tau^2}{(2n-1)^2}\right) K_{i\tau}[f] d\tau \right|$$
$$\leq \frac{1}{\sqrt{2}} \left(\int_E^\infty \tau \sinh \pi\tau |K_{i\tau}[f]|^2 d\tau \right)^{1/2}$$
$$\times \left(\int_0^\infty \tau \tanh\left(\frac{\pi\tau}{2}\right) |K_{i\tau}(x)|^2 \left[\prod_{n=1}^{N} \left(1 + \frac{\tau^2}{(2n-1)^2}\right) \right]^2 d\tau \right)^{1/2}.$$

Further we use the uniform estimate (1.100) in [6] for the Macdonald function and the latter integral in (4.12) is majorized by

$$\left(\int_0^\infty \tau \tanh\left(\frac{\pi\tau}{2}\right) |K_{i\tau}(x)|^2 \left[\prod_{n=1}^{N} \left(1 + \frac{\tau^2}{(2n-1)^2}\right) \right]^2 d\tau \right)^{1/2}$$
$$\leq K_0(x_0 \cos\delta) \left(\int_0^\infty \tau \tanh\left(\frac{\pi\tau}{2}\right) (1+\tau^2)^{2N} e^{-2\delta\tau} d\tau \right)^{1/2} < \infty$$

when $x \geq x_0 > 0$, $\delta \in \left(0, \frac{\pi}{2}\right)$. Thus via (1.13) the right-hand side of (4.12) tends to zero when $E \to \infty$. Moreover, employing elementary infinite product

$$\cosh\left(\frac{\pi\tau}{2}\right) = \prod_{n=1}^{\infty} \left(1 + \frac{\tau^2}{(2n-1)^2}\right), \tag{4.13}$$

and the inequality

$$|K_{i\tau}[f]| \frac{\prod_{n=1}^{N} \left(1 + \frac{\tau^2}{(2n-1)^2}\right)}{\cosh(\pi\tau/2)} \leq |K_{i\tau}[f]|, \tag{4.14}$$

where the right-hand side of (4.14) belongs to $L_2\left(\mathbf{R}_+; \frac{2}{\pi^2}\tau \sinh \pi\tau d\tau\right)$ we deduce that $f_N(x) \in L_2(\mathbf{R}_+; x dx)$. Combining with (4.11) and (1.13) it follows that

$$\int_0^\infty x|f_N(x)|^2 dx \tag{4.15}$$
$$= \frac{4}{\pi^2}\int_0^\infty \tau \tanh\left(\frac{\pi\tau}{2}\right)\left|K_{i\tau}[f]\prod_{n=1}^{N}\left(1+\frac{\tau^2}{(2n-1)^2}\right)\right|^2 d\tau.$$

Appealing to Levi's theorem and invoking (4.13) we have

$$\lim_{N\to\infty}\frac{4}{\pi^2}\int_0^\infty \tau \tanh\left(\frac{\pi\tau}{2}\right)\left|K_{i\tau}[f]\prod_{n=1}^{N}\left(1+\frac{\tau^2}{(2n-1)^2}\right)\right|^2 d\tau$$
$$= \frac{2}{\pi^2}\int_0^\infty \tau\sinh\pi\tau\,|K_{i\tau}[f]|^2\,d\tau = \int_0^\infty x|f(x)|^2dx.$$

Consequently,

$$\lim_{N\to\infty}\int_0^\infty x|f_N(x)|^2dx = \int_0^\infty x|f(x)|^2dx.$$

It is clear (see (4.11)) that

$$K_{i\tau}[f_N] = K_{i\tau}[f]\frac{\prod_{n=1}^{N}\left(1+\frac{\tau^2}{(2n-1)^2}\right)}{\cosh(\pi\tau/2)}$$

and obviously converges pointwisely to a function $K_{i\tau}[f]$. But

$$\int_0^\infty x|f_N(x)-f(x)|^2dx = \frac{2}{\pi^2}\int_0^\infty \tau\sinh\pi\tau\,|K_{i\tau}[f_N]-K_{i\tau}[f]|^2\,d\tau. \tag{4.16}$$

The right-hand side of (4.16) tends to zero when $N\to\infty$ via the dominated convergence theorem since $|K_{i\tau}[f_N]-K_{i\tau}[f]| \le 2|K_{i\tau}[f]|$ (see (4.14)). Thus we establish that $f_N(x)\to f(x)$ in $L_2(\mathbf{R}_+;xdx)$ and formula (4.5) is proved.

Concerning (4.6) we have accordingly

$$f_N(x) = \frac{\sqrt{2}}{\pi\sqrt{\pi x}}\prod_{n=1}^{N}\left(1+\frac{4x\left(x-\frac{d}{dx}-x\frac{d^2}{dx^2}\right)}{(2n-1)^2}\right)x(\mathcal{K}_{h_2}f)(x).$$

Hence after substitution the representation (4.8), we change the order of N-th product and the integral by the same motivation as above. Then

invoking (1.6) it becomes
(4.17)

$$f_N(x) = \frac{2}{\pi^2 x}\int_0^\infty \tau \tanh(\pi\tau) K_{i\tau}(x) \prod_{n=1}^{N}\left(1+\frac{4\tau^2}{(2n-1)^2}\right) K_{i\tau}[f] d\tau.$$

Passing to the limit in (4.17) with respect to the norm in $L_2(\mathbf{R}_+; xdx)$, when $N \to \infty$ we use the fact that the product tends to its value

$$\cosh \pi\tau = \prod_{n=1}^{\infty}\left(1+\frac{4\tau^2}{(2n-1)^2}\right)$$

and establish (4.6). Theorem 3 is proved. □

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