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FOCAL POINT IN THE C^0-LORENTZIAN METRIC

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ABSTRACT. In this paper we address focal points and treat manifolds (M, g) whose Lorentzian metric tensors g have a spacelike C^0-hypersurface Σ [10]. We apply Jacobi fields for such manifolds, and check the local length maximizing properties of C^1-geodesics. The condition of maximality of timelike curves(geodesics) passing C^0-hypersurface is studied.

1. Introduction

The concept of a warped product manifold was introduced by Bishop and O'Neill in ([1]), where it served to provide a class of complete Riemannian manifolds with negative curvature. The connection with general relativity was first made by Beem, Ehrich, and Powell ([2], [3]) who pointed out that several of the well-known exact solutions to Einstein's field equations are pseudo-Riemannian warped products. Beem and Ehrich ([4], [5]) further explored the extent to which certain causal and completeness properties of a space-time maybe determined by the presence of a warped product structure.

Many authors ([6]-[11]) have dealt with Lorentzian manifolds with non-smooth metric tensors from various view points. Of particular interest are space-times which have a metric tensor which fails to be C^1 across a hypersurface, and is C^∞ off the hypersurface. A space-time which, in an admissible coordinate system, the metric tensor is continuous but has a jump in its first and second derivatives across a submanifold will have a curvature tensor containing a Dirac delta function ([12]). The support of this distribution may be of three, two, or one dimensional or may even consist of a single event. Lichnerowicz's formalism ([13]) for dealing with such tensors is modified so as to obtain

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a formalism in which the Riemannian curvature tensor and Ricci curvature tensor exist in the sense of distributions. Thus, warped product spaces are extended to a richer class of spaces involving multiply products. Multiply warped products spaces were studied by Flores, J, L. and M. Sánchez ([14]). The conditions of spacelike boundaries in the multiply warped products spacetimes were studied by Steven G. Harris. The Kasner metric ([15]) was studied as a cosmological model by Schücking and Heckmann(1958). J. Choi investigated the curvature of a multiply warped product with C^0-warping functions [16]. Very recently, we have studied a multiply warped product manifold associated with the BTZ (de Sitter) black holes to evaluate the Ricci curvature components inside (outside) the black hole horizons. We have also shown that all the Ricci components and the Einstein scalar curvatures are identical both in the exterior and interior of the event horizons without discontinuities for both the BTZ and dS black holes [17].

In this work, we study focal points and treat manifolds (M, g) whose C tensors g have a spacelike C^0-hypersurface Σ ([9]). We apply Jacobi fields for such manifolds, and check the local length maximizing properties of C^1-geodesics. The condition of maximality of timelike curves(geodesics) is studied. Conjugate points and cut points play large roles in Lorentzian geometry and general relativity ([10], [18]). The existence of conjugate points along non-spacelike geodesics in a physically realistic space-time is an essential part of the proof ([19]) of the Hawking-Penrose singularity theorems ([20]).

2. Focal point in the C^0-Lorentzian metric

The concept of a conjugate point along a geodesic can be generalized to the notion of a focal point of the submanifold. Let H be a nondegenerate submanifold of the space-time (M, g). At a each point $p \in H$ the tangent space T_pH may be naturally identified with the vectors of T_pM which are tangent to H at p. The normal space $T_p^{\perp}H$ consists of all vectors orthogonal to H at p. Since H is nondegenerate, $T_p^{\perp}H \cap T_pH = \{0\}$ for all $p \in H$. We denote the exponential map restricted to the normal bundle $T_p^{\perp}H$ by $\exp^{\perp}$. Then the vector $X \in T_p^{\perp}H$ is said to be a *focal point* of H if $(\exp^{\perp})_*$ is singular at X. The corresponding point $\exp^{\perp}(X)$ of M is said to be a *focal point of H along the geodesic segment* $\exp^{\perp}$. When H is a single point, then $T_p^{\perp}H = T_pM$ and a focal point is just an ordinary conjugate point. Focal points may also defined using Jacobi fields and the second fundamental form. Jacobi fields are used to measure the separation of nearby geodesics. If p is a focal point along a geodesic c which is orthogonal to the submanifold H, then some geodesics close to c and orthogonal to H tend to focus at p.

We recall the definition of a regularly embedded hypersurface. Let M be a smooth manifold of dimension n. Then subset S of M is a regularly embedded hypersurface of M if for all $p \in S$, there exists a coordinate neighborhood $U(p)$ with coordinates $(x_1, \ldots, x_n)$ such that $S \cap U = \{(x_1, \ldots, x_n) \in U \mid x_n = p\}$. For convenience, we say that such a neighborhood U is partitioned by S. We denote $\{x \in U \mid x_n > p\}$ and $\{x \in U \mid x_n < p\}$ by U_p^+ and U_p^-, respectively. Now let M be a smooth manifold with a regularly embedded hypersurface S. Let S^c denote the complement of S. We define the concept of a C^0-Lorentzian metric on M.

DEFINITION 2.1. [9] C^0-Lorentzian metric on M is a nondegenerate (0,2) tensor of Lorentzian signature such that:

(1) $g \in C^0$ on S,

(2) $g \in C^\infty$ on $M \cap S^c$,

(3) For all $p \in S$, and $U(p)$ partitioned by S, $g|_{U_p^+}$ and $g|_{U_p^-}$ have smooth extensions to U. We call S a C^0-singular hypersurface of (M, g).

Let M be a manifold with a spacelike C^0-hypersurface S as above. Also, let $c : [a, b] \to M$ be a piecewise timelike geodesic for the partitions $a < 0 < b$ and $c(0) \in S$. Let $\alpha : [a, b] \times (-\epsilon, \epsilon) \to (M, g)$ be a piecewise smooth variation of c such that $\alpha = \alpha(t, s)$ where $T = \alpha_* \frac{\partial}{\partial t}$, $W = \alpha_* \frac{\partial}{\partial s}$ and satisfying

(1) $\alpha(t, 0) = c(t)$ for $t \in [a, b]$,

(2) $\alpha(0, s) \in S$ for all $s \in (-\epsilon, \epsilon)$.

The variation vector field $W(t) = \alpha_* \frac{\partial}{\partial s}$ of a variation α along c may have discontinuities in its derivative at $t = 0$. If N is the normal component of W along c such that $N = W + \langle W, c' \rangle c'$, N may also fail to be smooth at $t = 0$. We derive the second variation formula for $L''(0)$ in terms of N and W.

PROPOSITION 2.2. *Let $c : [a, b] \to (M, g)$ be a unit speed timelike geodesic segment and $\alpha : [a, b] \times (-\epsilon, \epsilon) \to (M, g)$ be a variation of c for the partition $t_0 = a < t_1 = 0 < t_2 = b$, T, W, N and L as above, then*

$$L'(0) = - \langle W, T \rangle|_{t=b}$$

$$L''(0) = \int_a^b \langle N'' + R(W,c')c', N\rangle|_t dt + \sum_{i=0}^{2} \langle N(t_i), \triangle_{t_i} N'\rangle$$
$$- \langle \nabla_W W, c'\rangle|_a^b + \Big[\langle \nabla_W W, T\rangle\Big]_0.$$

Proof. Recalling that

$$\frac{dL_i}{ds} = \int_{t_{i-1}}^{t_i} (-\langle T,T\rangle)^{-1/2}(-\langle \nabla_T W, T\rangle)dt,$$
$$\frac{d^2L_i}{ds^2} = \int_{t_{i-1}}^{t_i} \frac{\partial}{\partial s}\Big\{(-\langle T,T\rangle)^{-1/2}(-\langle \nabla_T W, T\rangle)\Big\}dt,$$

differentiating the expression under the integral sign, and using the identity

$$\nabla_W T - \nabla_T W = [W,T] = 0$$

we have

$$\begin{aligned}
&\frac{\partial}{\partial s}\Big\{(-\langle T,T\rangle)^{-1/2}(-\langle \nabla_T W, T\rangle)\Big\}\\
&= \frac{-\langle \nabla_W \nabla_T W, T\rangle - \langle \nabla_T W, \nabla_W T\rangle}{(-\langle T,T\rangle)^{1/2}}\\
&\quad + \frac{\langle \nabla_T W, T\rangle\langle \nabla_T W, T\rangle}{(-\langle T,T\rangle)^{1/2}\langle T,T\rangle}\\
&= -\langle \nabla_W \nabla_T W, T\rangle - \langle \nabla_T W, \nabla_W T\rangle - \langle \nabla_T W, T\rangle^2
\end{aligned}$$

for $(t,s) \in (t_{i-1}, t_i) \times (-\epsilon, \epsilon)$.

From $W = N - \langle W,T\rangle T$ we have

$$\begin{aligned}
&\langle \nabla_T W, \nabla_T W\rangle|_{(t,0)}\\
&= \langle \nabla_T N, \nabla_T N\rangle - 2\Big\langle \nabla_T N, \nabla_T(\langle W,T\rangle T)\Big\rangle\\
&\quad + \Big\langle \nabla_T(\langle W,T\rangle T), \nabla_T(\langle W,T\rangle T)\Big\rangle\\
&= \langle \nabla_T N, \nabla_T N\rangle - 2\Big(\frac{d}{dt}(\langle W,T\rangle)|_{(t,0)}\langle \nabla_T N, T\rangle|_{(t,0)}\Big)\\
&\quad + \Big\langle \frac{d}{dt}(\langle W,T\rangle)T + \langle W,T\rangle \nabla_T T,\\
&\quad\quad \frac{d}{dt}(\langle W,T\rangle)T + \langle W,T\rangle \nabla_T T\Big\rangle|_{(t,0)}\\
&= \langle \nabla_T N, \nabla_T N\rangle + \Big\{\frac{d}{dt}(\langle W,T\rangle)^2\langle T,T\rangle\Big\}\\
&= \langle \nabla_T N, \nabla_T N\rangle - (\langle \nabla_T W, T\rangle)^2
\end{aligned}$$

and obtain

$$\begin{aligned}&\frac{\partial}{\partial s}\Big\{(-\langle T,T\rangle)^{-1/2}(-\langle\nabla_T W,T\rangle)\Big\}\\ &= -\langle\nabla_W\nabla_T W,T\rangle-\langle\nabla_T N,\nabla_T N\rangle\end{aligned}$$

this yields

$$L_i''(0)=\int_{t_{i-1}}^{t_i}(-\langle\nabla_W\nabla_T W,T\rangle-\langle\nabla_T N,\nabla_T N\rangle)dt.$$

Also, using $[W,T]=0$ one obtains

$$R(T,W)W=\nabla_T\nabla_W W-\nabla_W\nabla_T W$$

and

$$\langle\nabla_T\nabla_W W,T\rangle=\frac{d}{dt}\langle\nabla_W W,T\rangle$$

therefore

$$\begin{aligned}L_i''(0)=&\int_{t_{i-1}}^{t_i}(\langle R(T,W)W,T\rangle-\langle\nabla_T N,\nabla_T N\rangle)|_{(t,0)}dt\\ &-\langle\nabla_W W,T\rangle|_{t_{i-1}}^{t_i}.\end{aligned}$$

From

$$\begin{aligned}-\langle\nabla_T N,\nabla_T N\rangle|_{(t,0)}=&-\frac{d}{dt}\langle N,\nabla_T N\rangle|_{(t,0)}\\ &+\langle N,\nabla_T\nabla_T N\rangle|_{(t,0)}\end{aligned}$$

and $L=\sum_{i=1}^2 L_i$, one obtains

$$\begin{aligned}L''(0)=&\int_a^b\Big(\langle R(T,W)W,\ T\rangle+\langle N,N''\rangle\Big)_{(t,0)}dt\\ &-\sum_{i=1}^{2}\langle N,N'\rangle|_{t_{i-1}^+}^{t_i^-}-\langle\nabla_W W,T\rangle|_a^b+\Big[\langle\nabla_W W,T\rangle\Big]\\ =&\int_a^b\Big(\Big\langle R(W,T)T,\ N-\langle W,T\rangle T\Big\rangle+\langle N'',N\rangle\Big)_{(t,0)}dt\\ &+\sum_{i=0}^{2}\langle N,\triangle_{t_i}N'\rangle-\langle\nabla_W W,T\rangle|_a^b+\Big[\langle\nabla_W W,T\rangle\Big]\end{aligned}$$

using $W=N-\langle W,T\rangle T$. Thus,

$$\begin{aligned}L''(0)=&\int_a^b\langle N''+R(W,T)T,N\rangle dt+\sum_{i=0}^{2}\langle N(t_i),\triangle_{t_i}N'\rangle\\ &-\langle\nabla_W W,T\rangle|_a^b+\Big[\langle\nabla_W W,T\rangle\Big]_0.\end{aligned}$$

COROLLARY 2.3. *Let H be a spacelike hypersurface, and assume that $c : [a,b] \to (M,g)$ is a unit speed timelike geodesic which is orthogonal to H at point $p = c(a) \in H$. Suppose that $\alpha : [a,b] \times (-\epsilon,\epsilon) \to (M,g)$ be a variation of c for the partition $t_0 = a < t_1 = 0 < t_2 = b$ such that $\alpha(a,s) \in H$ and $\alpha(b,s) = q = c(b)$ for all s with $-\epsilon \le s \le \epsilon$. If $W = \alpha_* \frac{\partial}{\partial s}|_{(t,0)}$ and $N = W + \langle W, c'\rangle c'$, then*

$$\begin{aligned} L''(0) &= \int_a^b \langle N'' + R(W,c')c', N\rangle dt + \langle N(0), \triangle_0 N'\rangle \\ &\quad + \langle N, N'\rangle|_{c(a)} + \langle L_{c'}(N), N\rangle|_{c(a)} + \Big[\langle \nabla_W W, T\rangle\Big]_0 \\ &= \int_a^b \langle N'' + R(W,c')c', N\rangle|_t dt + \sum_{i=0}^{1}\langle N(t_i), \triangle_{t_i} N'\rangle \\ &\quad + \langle L_{c'}(N), N\rangle|_{c(a)} + \Big[\langle \nabla_W W, T\rangle\Big]_0. \end{aligned}$$

Proof. In view of Proposition 2.2 and equation

$$\sum_{i=0}^{2}\langle N(t_i), \triangle_{t_i}(N')\rangle = \langle N(t), \triangle_t(N')\rangle|_{t=0} + \langle N, N'\rangle|_{c(a)}$$

it is only necessary to show that

$$\begin{aligned} -\langle \nabla_W W, T\rangle|_a^b &= \langle \nabla_W W, T\rangle|_{t=a} - \langle \nabla_W W, T\rangle|_{t=b} \\ &= \langle \nabla_W W, T\rangle|_{t=a} \\ &= \langle L_{c'}(N), N\rangle|_{c(a)}. \end{aligned}$$

To this end, we first note that $\alpha(b,s) = q$ implies that $\alpha_* \frac{\partial}{\partial s}|_{(b,s)} = 0$ for all s which yields $\langle \nabla_W W, T\rangle = 0$. Also $\alpha(a,s) \in H$ for all s with $-\epsilon < s < \epsilon$ implies that $\alpha_* \frac{\partial}{\partial s}|_{(a,s)}$ is tangential to H for all s, and hence $N(a) = \alpha_* \frac{\partial}{\partial s}|_{(a,0)}$. Let $\gamma(s) = \alpha(a,s)$ for all s with $-\epsilon < s < \epsilon$. Extend the vector $N(a) \in T_pH$ to a local vector field X along H with $X \circ \gamma(s) = \alpha_* \frac{\partial}{\partial s}|_{(a,s)}$ for all $-\epsilon < s < \epsilon$. Then

$$\langle L_{c'}(N), N\rangle|_{c(a)} = \langle \nabla_{X(a)} X, c'(a)\rangle$$

by definition of second fundamental form. Also let η be a unit normal field to H near p with $\eta(p) = c'(a)$. Then we have

$$\begin{aligned} \langle \nabla_W W, T\rangle &= \langle \nabla_W W, \eta \circ \alpha\rangle|_{(a,0)}|_{(a,0)} \\ &= \frac{d}{ds}\langle W, \eta \circ \alpha\rangle|_{(a,0)} - \langle W, \nabla_W \eta \circ \alpha\rangle|_{(a,0)}. \end{aligned}$$

But since $\alpha_* \frac{\partial}{\partial s}|_{(a,s)}$ is tangential to H, thus we obtain

$$\begin{aligned}\langle \nabla_W W, T\rangle|_{t=a} &= -\langle W, \nabla_W \eta \circ \alpha\rangle|_{(a,0)} \\ &= -\langle N(a), \nabla_{N(a)}\eta\rangle \\ &= -\langle X, \nabla_X \eta\rangle|_{c(a)} \\ &= -X|_p\langle X, \eta\rangle + \langle \nabla_X X|_p, \eta(a)\rangle \\ &= \langle \nabla_X X|_p, c'(a)\rangle \\ &= \langle L_{c'(a)}N,\ N\rangle\end{aligned}$$

as required. Here we have used the fact that since $X|_p = \alpha_* \frac{\partial}{\partial s}|_{(a,0)}$,

$$X|_p\langle X,\ \eta\rangle = \frac{d}{ds}\langle X \circ \gamma(s),\ \eta(s)\rangle|_{s=0} = \frac{d}{ds}(0) = 0.$$

□

DEFINITION 2.4. (Spacelike Hypersurface Index Form) Let $c : [a,b] \to (M,g)$ be a unit speed timelike geodesic which is orthogonal to a spacelike hypersurface H at $c(a)$. Assume that Z is a piecewise smooth vector field along c which is orthogonal to c. If $Z(a) \neq 0$ and $Z(b) = 0$, then the *index of Z with respect to H* is given by

$$I_H(Z,Z) = I(Z,Z) + \langle L_{c'}(Z), Z\rangle|_{t=a}$$

where,

$$I(Z,Z) = \int_a^b \langle Z'' + R(Z,c')c', Z\rangle|_t dt + \sum_{i=0}^{1}\langle Z(t_i), \triangle_{t_i} Z'\rangle$$

where the partition $t_0 = a < t_1 = 0 < t_2 = b$ of $[a,b]$ is chosen such that Z is differentiable except at $t_1 = 0$.

PROPOSITION 2.5. *Let $c : [a,b] \to (M,g)$ be a unit speed timelike geodesic segment and $\alpha : [a,b] \times (-\epsilon,\epsilon) \to (M,g)$ is a variation of c for the partition $t_0 = a < t_1 = 0 < t_2 = b$. Assume that $Z = \alpha_* \frac{\partial}{\partial s}$ is a piecewise smooth vector field along c which is orthogonal to c. If $Z(a) \neq 0$ and $Z(b) = 0$, then we have*

$$\begin{aligned}L''(0) = &\int_a^b \langle Z'' + R(Z,c')c', Z\rangle|_t dt + \langle L_{c'}(Z), Z\rangle|_{c(a)} \\ &+ \langle Z(a), \triangle_a Z'\rangle + \Big[Z\langle T, Z\rangle\Big]_0\end{aligned}$$

Proof.

$$\begin{aligned}
L''(0) &= \int_a^b \langle Z'' + R(Z,c')c', Z\rangle|_t dt + \sum_{i=0}^{1} \langle Z(t_i), \triangle_{t_i} Z'\rangle \\
&\quad + \langle L_{c'}(Z), Z\rangle|_{c(a)} + \Big[\langle \nabla_Z Z, T\rangle\Big]_0 \\
&= \int_a^b \langle Z'' + R(Z,c')c', Z\rangle|_t dt + \langle L_{c'}(Z), Z\rangle|_{c(a)} \\
&\quad + \langle Z(a), \triangle_a Z'\rangle + \langle Z(0), \triangle_0 Z'\rangle + \Big[\langle \nabla_Z Z, T\rangle\Big]_0 \\
&= \int_a^b \langle Z'' + R(Z,c')c', Z\rangle|_t dt + \langle L_{c'}(Z), Z\rangle|_{c(a)} \\
&\quad + \langle Z(a), \triangle_a Z'\rangle + \Big[\langle Z, \nabla_T Z\rangle\Big]_0 + \Big[\langle \nabla_Z Z, T\rangle\Big]_0 \\
&= \int_a^b \langle Z'' + R(Z,c')c', Z\rangle|_t dt + \langle L_{c'}(Z), Z\rangle|_{c(a)} \\
&\quad + \langle Z(a), \triangle_a Z'\rangle + \Big[Z\langle T, Z\rangle\Big]_0.
\end{aligned}$$

Also we can rewrite $L''(0)$ as follows

$$\begin{aligned}
L''(0) &= \int_a^b \langle Z'' + R(Z,c')c', Z\rangle|_t dt + \langle L_{c'}(Z), Z\rangle|_{c(a)} \\
&\quad + \langle Z(a), \triangle_a Z'\rangle + \Big[Z\langle T, Z\rangle\Big]_0 \\
&= I_H(Z,Z) + \langle Z(a), \triangle_a Z'\rangle + \Big[Z\langle T, Z\rangle\Big]_0.
\end{aligned}$$

□

Two hypersurfaces H_1, H_2 are said to be weakly parallel if every geodesic γ which intersects both faces is also orthogonal to other.

THEOREM 2.6. *Let M be a Lorentzian manifold with a C^0-singular hypersurface S, and with a hypersurface H weakly parallel to S. Let $c:[a,b]\to M$ $a<0<b$ be a unit speed timelike geodesic on orthogonal to S with $c(0)\in S$. If there is a nontrivial Jacobi field Y orthogonal to c with $Y'(a) = -L_{c'(a)}Y$ at $c(a)$, and $Y(b)=0$ with $\Big[Y\langle Y,T\rangle\Big] > 0$, then $c(t)$ is not of maximal length from $c(b)$ to H.*

Proof. From

$$\begin{aligned}L''(0) =& \int_a^b \langle Y'' + R(Y,c')c', Y\rangle|_t dt + \langle L_{c'}(Y), Y\rangle|_{c(a)} \\ &+ \langle Y(a), \triangle_a Y'\rangle + \Big[Y\langle T, Y\rangle\Big]_0 \\ =& \langle -Y', Y\rangle|_{c(a)} + \langle Y(a), \triangle_a Y'\rangle + \Big[Y\langle T, Y\rangle\Big]_0 \\ =& \langle -Y'(a^+), Y(a)\rangle|_{c(a)} + \langle Y(a), Y'(a^+)\rangle + \Big[Y\langle T, Y\rangle\Big]_0 \\ =& \Big[Y\langle T, Y\rangle\Big]_0\end{aligned}$$

we have $L''(0) > 0$.

THEOREM 2.7. *Let M be a Lorentzian manifold with a C^0-singular hypersurface S, and with a hypersurface H weakly parallel to S. Let $c : [a,b] \to M$ $a < 0 < b$ be a unit speed timelike geodesic with $c(0) \in S$, if there is a focal point $b' \in [a,b]$ and $\Big[Y\langle T, Y\rangle\Big] = 0$, then $c(t)$ is not of maximal length from $c(b)$ to H.*

Proof. We assume $b' > 0$ for otherwise the geodesic $c|_{[0,b']}$ doesn't cross the C^0-singular hypersurface and the theorem is true by known results. By hypothesis there exist a nontrivial Jacobi field Y_1 along c with Y_1 is orthogonal to c, and $Y_1(b') = 0$ and $Y_1'(a) = -L_{c'(a)}Y_1(a)$.

Define a piecewise unit smooth Jacobi field

$$Y(t) = \begin{cases} \frac{Y_1(t)}{\|Y_1(t)\|}, & \text{if } a \le t \le b' \\ 0, & \text{if } b' \le t \le b. \end{cases}$$

Since $\triangle_{b'} Y_1' \neq 0$, we may construct smooth vector field W orthogonal to c such that $W'(a) = W(a) = W(0) = W(b) = 0$ and $\langle W(b'), \triangle_{b'} Y'\rangle = -1$. Define vector field Z in $W^{\perp}(c)$ by

$$Z = \frac{1}{r}Y - rW$$

for $r \in R - \{0\}$.

Then

$$\begin{aligned}
Z'(a) &= \frac{1}{r}Y'(a) - rW'(a) \\
&= \frac{1}{r}Y_1'(a) - 0 \\
&= \frac{1}{r}\Big(-L_{c'(a)}\frac{Y_1(a)}{||Y_1(a)||}\Big) \\
&= -\Big(L_{c'(a)}(\frac{1}{r})\frac{Y_1(a)}{||Y_1(a)||}\Big) \\
&= -\Big(L_{c'(a)}(\frac{1}{r})Y(a)\Big) \\
&= -L_{c'(a)}Z(a)
\end{aligned}$$

and we have

$$\begin{aligned}
\langle Z(a), \triangle_a Z'\rangle &= \langle Z(a), Z'(a^+)\rangle \\
&= \langle \frac{1}{r}Y(a) - rW(a), \frac{1}{r}Y'(a) - rW'(a)\rangle \\
&= \frac{1}{r^2}\langle Y(a), Y'(a)\rangle \\
&= \frac{1}{2r^2}T\langle Y, Y\rangle|_{t=a} \\
&= 0
\end{aligned}$$

and the Index $I_H(Z, Z)$ is given by

$$\begin{aligned}
&I_H(Z, Z) \\
&= I(Z, Z) + \langle L_{c'}(Z),\ Z\rangle|_{t=a} \\
&= I(\frac{1}{r}Y - rW, \frac{1}{r}Y - rW) + \langle L_{c'}(\frac{1}{r}Y - rW),\ \frac{1}{r}Y - rW\rangle|_{t=a} \\
&= I(\frac{1}{r}Y - rW, \frac{1}{r}Y - rW) + \frac{1}{r^2}\langle -Y', Y\rangle|_{t=a} \\
&= \frac{1}{r^2}I(Y, Y) - 2\langle Y, W\rangle + r^2\langle W, W\rangle + \frac{1}{r^2}\langle -Y', Y\rangle|_{t=a} \\
&= \frac{1}{r^2}\Big(\int_a^b \langle Z'' + R(Z, c')c', Z\rangle|_t dt + \sum_{i=0}^{1}\langle Z(t_i), \triangle_{t_i}Z'\rangle\Big) \\
&\quad - 2\langle Y, W\rangle + r^2\langle W, W\rangle + \frac{1}{r^2}\langle -Y', Y\rangle|_{t=a}
\end{aligned}$$

therefore

$$\begin{aligned} I_H(Z,Z) =& \frac{1}{r^2}\langle Z(a), Z'(a)\rangle - 2\langle Y, W\rangle + r^2\langle W, W\rangle \\ &+ \frac{1}{r^2}\langle -Y', Y\rangle|_{t=a} \\ =& \frac{1}{r^2}\langle Y(a), Y'(a)\rangle - 2\langle Y, W\rangle + r^2\langle W, W\rangle \\ &+ \frac{1}{r^2}\langle -Y', Y\rangle|_{t=a} \\ =& -2\langle Y, W\rangle + r^2\langle W, W\rangle \\ =& 2 + r^2\langle W, W\rangle. \end{aligned}$$

Also, we have $I_H(Z,Z) > 0$ and

$$\begin{aligned} L''(0) =& \int_a^b \langle Z'' + R(Z,c')c', Z\rangle|_t dt + \langle L_{c'}(Z), Z\rangle|_{c(a)} \\ &+ \langle Z(a), \triangle_a Z'\rangle + \Big[Z\langle T, Z\rangle\Big]_0 \\ =& I_H(Z,Z) + \Big[Z\langle T, Z\rangle\Big]_0 \\ =& I_H(Z,Z) > 0. \end{aligned}$$

So, there exist small variations of c with variation vector Z which join H to $c(b)$ and have length greater than c. □

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