

# Finite Alphabet Control and Estimation

Graham C. Goodwin and Daniel E. Quevedo

**Abstract:** In many practical problems in signal processing and control, the signal values are often restricted to belong to a finite number of levels. These questions are generally referred to as “finite alphabet” problems. There are many applications of this class of problems including: on-off control, optimal audio quantization, design of finite impulse response filters having quantized coefficients, equalization of digital communication channels subject to intersymbol interference, and control over networked communication channels. This paper will explain how this diverse class of problems can be formulated as optimization problems having finite alphabet constraints. Methods for solving these problems will be described and it will be shown that a semi-closed form solution exists. Special cases of the result include well known practical algorithms such as optimal noise shaping quantizers in audio signal processing and decision feedback equalizers in digital communication. Associated stability questions will also be addressed and several real world applications will be presented.

**Keywords:** Constraints, finite sets, predictive control, binary control, quantization, networked control systems, sigma-delta modulation, noise shaping, equalizers, discrete coefficient filters, switch-mode power supplies.

## 1. INTRODUCTION

Many design problems in signal processing and control can be formulated as optimization problems in which the set of decision variables takes only a finite set of possible values. Typical examples include:

- (i) on-off control problems such as those typically encountered in air-conditioning systems,
- (ii) multilevel control problems such as those found in pumping of water in water distribution networks (Usually, in these problems, one can only choose from a finite set of pumping levels, see e.g. [1].),
- (iii) power electronics problems which invariably involve the use of switches [2-4],
- (iv) audio quantization based on psycho-acoustic considerations [5-8],
- (v) design of class D amplifiers based on digitized inputs [9],
- (vi) design of FIR and IIR filters having quantized coefficients to facilitate implementation in DSP hardware [10],
- (vii) equalization of band-limited digital communication channels subject to intersymbol interference [11-14],

(viii) networked control systems [15-21].

These seemingly different problems can be formulated within the common framework of optimization over a finite set of decision values. This paper will review this class of problems and discuss the associated design issues.

A straightforward approach to solving these problems would be to initially ignore input quantization (leading to a standard linear design) and subsequently to retro-fit the quantization aspect. This will give satisfactory results for simple problems, e.g. when the number of quantization levels is large. However, in general, it turns out to be preferable to include the quantization aspect in the design from the beginning. We describe a general framework for achieving this based upon a constrained finite horizon optimization setting. We also show that special cases of the methodology turn out to be well-known standard solutions in different areas, like for example:

- “optimal noise shaping” quantizers in the field of audio processing,
- “decision feedback equalizes” in digital communications, and
- “sigma delta modulations” in signal quantization.

We will show that these well-known solutions are equivalent to the optimization based solution described here when the optimization horizon is restricted to unity. Of course, improved performance is usually associated with the use of more general design choices. Indeed, we will show that significant improvements can often result by simply increasing

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the horizon to two or three. Fortunately, it also often turns out, that one does not need to consider very large horizons to obtain near optimal performance. (In the examples presented later we will see that horizons of the order of 3 to 10 normally suffice).

Of course, in other cases, larger horizons may be necessary to achieve the desired level of performance. In these cases, a difficulty arises since the required computations turn out to be exponential in the dimension of the problem. (It is for this reason that it is fortuitous that near optimal performance often occurs for relatively small horizons). If one must use a longer horizon, then the problem can become intractable if it is approached in a "brute force" fashion. To deal with these more difficult cases, we outline recent work on semi-definite programming relaxations which offer the potential to yield approximate solutions.

An outline of the remainder of this paper is as follows: In Section 2 we review selected application areas of finite alphabet control and estimation. Section 3 describes the main ideas which underly receding horizon optimization. These ideas are then applied, in Section 4, to problems of the control-type. Section 5 presents a semi-closed form solution to the receding optimization problem. This result gives rise to a partition of the state space, which is characterized in Section 6. Stability analysis and results are included in Section 7. Section 8 deals with estimation problems. Section 9 describes a unified view of finite alphabet control and estimation problems. Results of applying receding horizon optimization to diverse application areas are included in Section 10. Section 11 outlines some aspects related to semi-definite programming. Section 12 draws conclusions.

## 2. TYPICAL APPLICATION AREAS

As a motivation for the finite alphabet constrained receding horizon approximation methods under study in this paper, we will first present some applications where finite set constraints dominate performance.

### 2.1. Power conversion problems

Most electronics based power conversion problems depend on the use of switches, i.e., the "input" is restricted to a finite alphabet. As an illustration we refer to switched mode power supplies.

Switch-mode power supplies (SMPSs) are widely used in electronics equipment, such as computers, which use DC-voltages that need to be provided by the AC mains network. SMPSs utilize power semiconductor switches, such as MOSFETs, to synthesize the desired DC voltage levels. (See [22-24].)

A drawback of using SMPSs resides in the fact that, due to their switching nature, harmonic currents are

injected into the AC mains and ground connection. Also, electromagnetic noise is radiated, [25-29]. This issue becomes especially relevant at high switching frequencies (which are necessary in order to achieve low ripple in the output voltage) and is magnified if many SMPSs are connected to the same supply network.

In order to deal with the electromagnetic pollution problem, regulations, such as those of the FCC (in the USA) or the VDE (in the EU), have been elaborated. These specifications put limits on the peak values of EMI spectra, see e.g. [27].

One way to mitigate the EMI problem resides in the utilization of improved shielding, input filters and isolation of signal coupling paths, see e.g. [23, 24, 26, 29]. These hardware-based methods add to the complexity, size, weight and cost of the power supply. Thus, it may be more convenient to reduce EMI emissions directly at the source. This can be accomplished by careful design of the switching strategy of the power devices, where one can make use of ever increasing capabilities of DSP hardware. It turns out that EMI emission levels of an SMPS can be predicted by inspecting the spectrum of the switching signal. As a consequence, the indirect approach of mitigating EMI via careful design of the switching strategy has attracted significant research, see e.g. [30-34].

Conventional switching strategies for SMPS power utilize periodic pulse-width-modulation (PWM) and yield purely discrete spectra, see e.g. [35]. Thus, this strategy is generally not a good choice if EMI specifications are to be met. In order to broaden signal spectra and reduce harmonic peaks, modified PWM schemes, such as programmed PWM [30] and frequency modulated PWM [31] have been proposed. Another possibility lies in utilizing randomized PWM schemes, where a nominal PWM switching function is dithered in various ways, see e.g. [32, 33, 36]. Unfortunately, in these methodologies the mitigation of harmonic peaks is usually achieved only at the expense of increased output voltage ripple and this precludes the use of these strategies in SMPSs where tight voltage regulation is sought [35]. By realizing that switching in an SMPS can be regarded as a particular analog-to-digital conversion problem with a 1-bit output, in [34] strategies based upon  $\Sigma\Delta$ -Modulation, see e.g. [37], have been developed.

The EMI mitigation problem can readily be cast as a finite alphabet constrained receding horizon optimization problem, see [38]. Results are included in Section 10.1.

### 2.2. Audio quantization with psycho-acoustical considerations

An important problem in audio engineering corresponds to the mastering process of compact

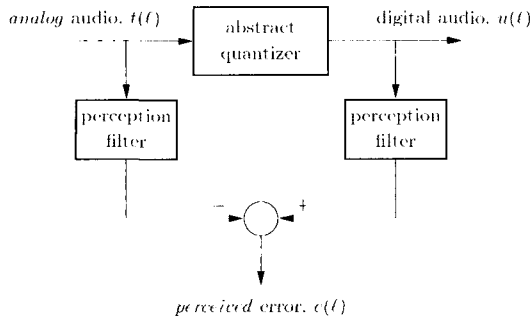


Fig. 1. Psycho-acoustical audio quantizer.

discs. Master recordings made in recording studios are usually very finely quantized (such as e.g. 24-bit) or, in case of older recordings, of analog nature. In order to reformat into a compact disc, signals need to be converted to 16-bit resolution. This means, that they need to be either quantized or re-quantized to lower wordlength.

The goal is to satisfy the listener. Thus, the difference between the original (analog) and the low-bit quantized audio signal, should be made, as far as possible, inaudible. Therefore, efforts should concentrate on sound quality as perceived by the human ear. Fig. 1 illustrates how psycho-acoustical aspects can be introduced into the conversion process. In this scheme a perception filter, which models the ear sensitivity to low level noise power, is utilized in order to give rise to the perceived error,  $e(\ell)$ .

We model the perception filter via a linear time-invariant linear filter  $W(z)$ . This filter can be fitted to equal-loudness curves, which are available in the literature, see e.g. [39]. The conversion problem can be interpreted as follows:

Given a sequence of analog audio  $\{t(\ell)\}$  and a stable perception filter

$$W(z) = D + C(zI - A)^{-1}B, \quad (1)$$

obtain a sequence  $\{u(\ell)\}$  which minimizes a measure of the perceived error

$$e(\ell) \triangleq W(z)(u(\ell) - t(\ell)).$$

Each of the values  $u(\ell)$  is restricted to belong to a finite alphabet  $U$ , such as the 16-bit set

$$U = \{-2^{15}, -2^{15} + 1, \dots, -1, 0, 1, \dots, 2^{15} - 1\}.$$

By concentrating on the perceived noise power, we obtain the cost function:

$$V = \sum_{\ell=0}^{\infty} (e(\ell))^2.$$

Note that the perceived error  $\{e(\ell)\}$  is the output to

the dynamical system:

$$\begin{aligned} x(\ell + 1) &= Ax(\ell) + B(u(\ell) - t(\ell)), \\ e(\ell) &= Cx(\ell) + D(u(\ell) - t(\ell)), \end{aligned} \quad (2)$$

where  $x(\ell) \in R^n$  is the system state. In Section 10.2 we will show that the receding horizon optimization approach gives a useful solution to this problem. Further details can also be found in [40, 41].

### 2.3. Design of FIR filters with quantized coefficients

Standard filter design techniques, such as the Parks-McClellan algorithm, give rise to filters whose coefficients are specified with infinite wordlength. However, in many hard-ware critical applications, such as those involving fixed-point application-specific integrated circuits, only a finite wordlength representation is allowed. In these cases, filter coefficients are restricted to belong to a finite alphabet, e.g. a linear combination of power-of-two terms. Thus, the problem of approximating an infinite precision target filter  $T(z)$ , with a discrete coefficient one,  $H(z)$ , arises.

A common situation is where  $H(z)$  is of finite impulse response (FIR) of length  $M$ , i.e.,

$$H(z) = \sum_{j=0}^{M-1} h_j z^{-j},$$

and is to be implemented in direct form. In this case, each of the terms of its impulse response is restricted to belong to a finite set  $U$ , i.e.,

$$h_j \in U, \quad \forall j = 0, \dots, M-1. \quad (3)$$

The problem now consists in choosing the coefficients in (3), such that the resultant filter approximates  $T(z)$  well, in some sense. Since filters are usually utilized because of their frequency filtering characteristics, it makes sense to state this approximation problem in the frequency domain and to pursue a frequency selective approximation. This can be accomplished by introducing a frequency dependent weighting function  $W(e^{j\omega})$  and the following (frequency domain)  $L_2$  performance measure:

$$V \triangleq \frac{1}{2\pi} \int_0^{2\pi} |W(e^{j\omega}) (H(e^{j\omega}) - T(e^{j\omega}))|^2 d\omega, \quad (4)$$

where  $H(e^{j\omega})$  and  $T(e^{j\omega})$  are the frequency responses of  $H(z)$  and  $T(z)$ , respectively. In this cost function, we have included frequency weighting by means of the term  $W(e^{j\omega})$ . This filter weights the

relative importance of the approximation error (ripple) in different frequency bands. Thus, the finite wordlength approximation effect can be concentrated in tolerant bands and reduced in more critical bands.

By means of Parseval's Theorem, we can equivalently evaluate  $V$  defined in (4) in the time domain, as

$$V = \sum_{i=0}^{\infty} (e(i))^2.$$

In this expression, the terms  $\{e(i)\}$  are defined implicitly via

$$\sum_{i=0}^{\infty} e(i)z^{-i} = E(z),$$

where

$$E(z) \triangleq W(z)(H(z) - T(z))$$

is the filtered error and  $W(z)$  denotes the  $Z$ -transform of  $W$ . Note that, if we describe  $W(z)$  as in Eq. (1), then it follows that  $e(i)$  is characterized via (2).

It can thus be seen that the quantized coefficient FIR filter design can be stated as a constrained quadratic optimization over an infinite horizon. Minimization of  $V$  yields the  $M$  finite set constrained filter coefficients,  $h_j, j = 0, 1, \dots, M - 1$ .

In practice, one would replace the infinite horizon by a suitable finite horizon  $N$ , see [42]. As shown in Section 10.3, the application of the receding horizon paradigm to the optimization problem may lead to good designs at only modest computational burden.

#### 2.4. Equalization of band-limited communication channels

When a digital signal drawn from a finite alphabet  $U$  is transmitted over a band-limited communication channel, the received signal is affected by intersymbol interference (ISI). This means, that, besides the unavoidable noise contamination, at each sampling instant, the received quantity contains contributions of several adjacent symbols. As a consequence, the problem of recovering the input sequence to the channel from the received output arises. Such an ISI compensator is commonly termed an equalizer. Fig. 2 depicts this situation. In it, the input signal sequence  $\{u(\ell)\}$  is passed through the channel in order to give rise to the output signal  $\{y(\ell)\}$ . The equalizer provides an estimate of the channel input, which we will denote as  $\{\hat{u}(\ell)\}$ .

In most applications the equalizer needs to work on-line. Thus, at time  $\ell = k$ , the equalization problem can be stated as that of obtaining an estimate of the sequence

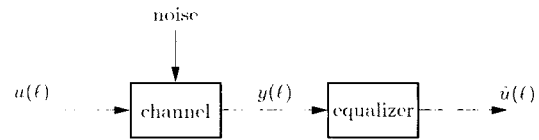


Fig. 2. Digital communication system.

$$\{u(0), u(1), \dots, u(k)\}$$

from the output sequence observed so far, i.e., from

$$\{y(0), y(1), \dots, y(k)\}.$$

In particular, the maximum likelihood sequence estimator (MLSE) yields the estimate

$$\{\hat{u}(0), \hat{u}(1), \dots, \hat{u}(k)\}$$

which maximizes the likelihood of the received data, i.e., the conditional probability density. In this estimate, each element is restricted to belong to the finite alphabet  $U$ .

It is common to model the dispersive nature of the channel as a linear discrete time system

$$y(\ell) = W(z)u(\ell) + n(\ell),$$

or, equivalently, in state space form as

$$x(\ell + 1) = Ax(\ell) + Bu(\ell),$$

$$y(\ell) = Cx(\ell) + Du(\ell) + n(\ell).$$

In both models,  $n(\ell)$ , is white and independent zero-mean Gaussian noise of variance  $r$ . These characterizations may include the effect of a whitening matched filter and any other filters.

With a zero input to the channel for  $\ell < 0$  (so that  $x(0) = 0$ ), it can easily be verified that the likelihood of the output sequence received is given by

$$P(\{y(0), y(1), \dots, y(k)\} | \{u(0), u(1), \dots, u(k)\}) = \alpha e^{-V/2},$$

where

$$V \triangleq r^{-1} \sum_{\ell=0}^k (W(z)u(\ell) - y(\ell))^2 \tag{5}$$

and  $\alpha$  is a constant.

By applying the natural logarithm, we obtain that the MLSE needs to minimize  $V$ , that is, it requires the solution of a finite set constrained quadratic optimization problem. Since the upper limit of the summation in (5) increases with time, the MLSE relies on solving an optimization problem which is intractable. In order to overcome this problem and to obtain an equalizer of limited complexity, one can deploy the receding horizon approach presented in Section 8. Further details are given in [43]. Results are also discussed in Section 10.4.

#### 2.5. Networked control systems

Networked control systems (NCS) are control

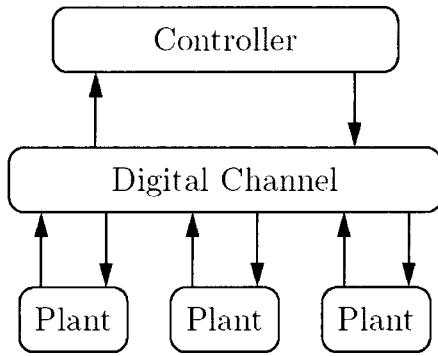


Fig. 3. Networked control system.

systems in which controller and plants are connected via a (digital) communication channel. One such configuration is depicted in Fig. 3. In this scheme, a centralized controller is connected to several sensors and actuators in order to control a collection of plants.

Practical applications of NCS abound. They have been made possible by technological developments, including the development of MEMS arrays, and may deploy wireless links (e.g. Bluetooth or IEEE 802.11), Ethernet (e.g. IEEE 802.3) or specialized protocols such as CAN.

While the use of digital communication channels enables novel teleoperating applications, new and interesting challenges also arise. The network itself is a dynamical system that exhibits characteristics which traditionally have not been taken into account in control system design. These special characteristics include quantization and time-delays and are a consequence of the fact that practical channels have only a limited bandwidth. As a consequence, a networked controller should be designed to take into account the communication channel.

Since practical digital communication channels can only carry a limited amount of bits per unit time, every signal transmitted needs to be expressed via a finite number of bits. Thus, quantization forms a central issue in control over digital networks. It is easy to see that the resulting control law problem can be formulated by means of the finite set constrained control framework developed in [44]. This idea has been further investigated in [45].

If the network traffic is to be kept at a minimum, one can add other restrictions. In particular, besides transmitting quantized signals, the down-link traffic can be further reduced by restricting the controller such that only one actuator can be addressed at any given time, see also related work in [19, 46]. We can then optimize performance subject to these constraints.

For that purpose, consider a set of linear time-invariant noiseless MIMO plants with a total number of  $m$  inputs and a given reference trajectory  $r^*(\ell)$ .

Furthermore, assume that the link between the controller and actuators is characterized by a known and fixed time-delay and that data is sent at a bounded rate. This is achieved by imposing the following two communication constraints on the design:

**Restriction 1:** The data sent from the controller to each actuator is restricted to belong to a (small and fixed) finite set of scalars,  $U$ .

**Restriction 2:** Only data corresponding to one input of the plants can be transmitted at a time. Between updates, (which may be separated by several sampling periods) all plant inputs are held at their previous values.

The delay between controller and plants can be incorporated into the model:

$$\begin{aligned} x(\ell+1) &= Ax(\ell) + Bu(\ell), \\ y(\ell) &= Cx(\ell), \end{aligned} \quad (6)$$

where

$$u(\ell) = [u_1(\ell), u_2(\ell), \dots, u_m(\ell)],$$

and  $x(\ell) \in R^n$  is the system state. Note that this description encompasses the entire set of plants.

The design problem can thus be stated as that of developing a control strategy, which drives the model (6) to the reference  $r^*(\ell)$ , while not violating Restrictions 1 or 2. Thus, the control strategy for the networked system is characterized by choosing, at each time step, which of the  $m$  inputs to access and what to send. The controller needs to divide its attention between all plant inputs.

Rather than sending the control signals directly, it is preferable to send their increments:

$$\Delta u_i(\ell) \triangleq u_i(\ell) - u_i(\ell-1), \quad (7)$$

when nonzero. This choice generally requires less bits to specify the control signal. The pair  $(\Delta u_i(\ell), i)$  is received at the actuator node specified by the index  $i$ . The actual signal  $u_i(\ell)$  is reconstructed at the plant side by discrete time integration as shown in Fig. 4.

Restrictions 1 and 2 can be summarized by means of a simple finite-alphabet constraint on the increments (7). More precisely, at every time instant  $\ell$ , the vector

$$\Delta u(\ell) \triangleq u(\ell) - u(\ell-1)$$

is restricted to belong to the finite alphabet  $\mathbb{V}$ , defined as:

$$\mathbb{V} \triangleq \{V \in R^m \text{ such that } \exists v \in U :$$

$$\mathbb{V} = [0 \ \dots \ 0 \ v \ 0 \ \dots \ 0]^T\}.$$

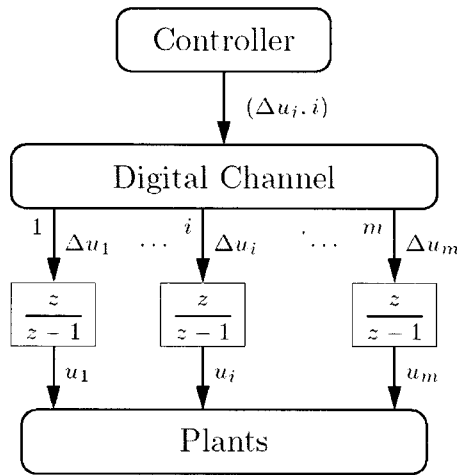


Fig. 4. Control architecture for an NCS.

As illustrated in Fig. 5 for the case  $m = 2$ , this set contains all column vectors formed by one element of  $U$ , whilst all its other components are zero.

The performance of the model (6) can be quantified by means of the quadratic cost:

$$V \triangleq \sum_{\ell=0}^{\infty} \|x(\ell) - x^*(\ell)\|_Q^2 + \sum_{\ell=0}^{\infty} \|u(\ell) - u^*(\ell)\|_R^2,$$

where  $Q$  and  $R$  are positive definite matrices. The signals  $x^*(\ell)$  and  $u^*(\ell)$  are target trajectories for the plant state and inputs, respectively and depend upon  $r^*(\ell)$ .

A key, and distinguishing feature of this strategy is that by minimizing  $V$  so that control increments are supplied only when they are required, bandwidth utilization is reduced. Clearly the minimization of  $V$  over a finite set constrained input cannot be achieved in practice. In contrast, the receding horizon approach presented in the present paper is computationally inexpensive and may yield good results, see Section 10.5 and also [20, 21].

### 3. RECEDING HORIZON OPTIMIZATION

In principle, all of the design problems referred to in Section 2 lead to infinite horizon minimization problems where decision variables are restricted to belong to a finite alphabet. These combinatorial optimization problems are computationally intractable. A useful practical approach is to restrict the optimization to a finite horizon and to solve this problem in a “receding horizon” fashion.

In order to apply the receding horizon optimization idea, it is convenient to categorize applications as Control Problems and Estimation Problems. The characteristic which we use to distinguish between those two classes depends on whether the system

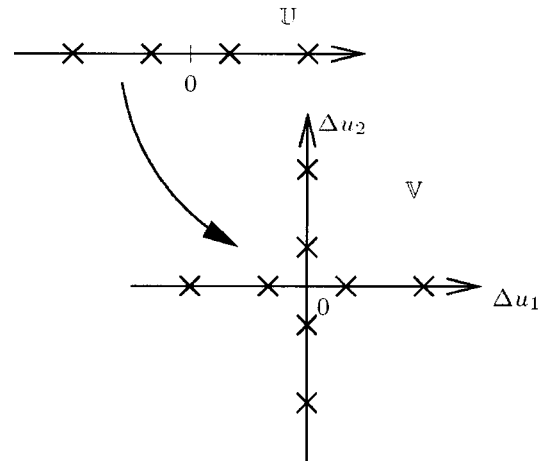


Fig. 5. Construction of the finite set  $V$ .

state is known or not. We use the term Control Problem in order to designate those applications, where the state is known. Thus, this category embraces binary- and multilevel- control problems, power electronics problems, signal quantization problems, quantized coefficient problems and the networked control system design method described above. On the other hand, the digital channel equalization problem falls into the Estimation Problem category. Although the channel model is assumed to be known, its input sequence is not. (Otherwise, the problem would be already solved.) As a consequence, the system state is unknown. Note, however, that some prior knowledge of the system state may exist and can be helpful in order to provide channel input estimates, see Section 8.

To highlight the main issues underlying receding horizon optimization, we temporarily restrict our attention to Control Problems. The core idea is that one considers a finite block (or horizon) of data stretching from time  $\ell$  to time  $\ell + N$ , say. Having carried out the associated optimization one “locks in” the decision variable at time  $\ell$ . One then moves onto time  $\ell + 1$  and considers a finite block of data stretching from time  $\ell + 1$  to time  $\ell + N + 1$  and “locks in” the decision variable at time  $\ell + 1$ . This procedure is repeated continuously leading to a “rolling horizon” optimization strategy. It is precisely this rolling horizon idea that allows us to simplify the associated computational requirements; i.e., one considers the finite alphabet restrictions in blocks of length  $N$  which roll-forward as time increases. A key observation is that the first decision variable in the block optimization is often insensitive to increasing the block size beyond modest numbers. The importance of this observation lies in the fact that the first decision variable is the only variable that is “locked in” at such time step; the remaining variables are simply used to evaluate the possible impact of future decisions. As a consequence, receding horizon optimization methods

may give excellent results even with modest horizons.

In what follows we address finite alphabet Control and Estimation problems separately. We will link them together in Section 9.

### 4. RECEDING HORIZON CONTROL PROBLEMS

For ease of exposition we will restrict our attention to the case where references are set equal to zero. The extension to more general reference trajectories presents no technical difficulties and is treated, e.g. in [40, 42, 47, 43].

In this framework, the Control Problem applications presented in Section 2 can be captured by the following fixed horizon cost function beginning at time  $k$ :

$$V_1 \triangleq \|x'(k+N)\|_P^2 + \sum_{\ell=k}^{k+N-1} \|x'(\ell)\|_Q^2 + \|u'(\ell)\|_R^2. \quad (8)$$

In this expression, we use  $x'(\ell)$  and  $y'(\ell)$  to denote predictions of the state and the output and  $u'(\ell)$  to denote the decision variables. More precisely, these quantities are related via:

$$\begin{aligned} x'(\ell+1) &= Ax'(\ell) + Bu'(\ell), \\ y'(\ell) &= Cx'(\ell) + Du'(\ell), \end{aligned} \quad (9)$$

with initial condition  $x'(k) = x(k)$ , the system state, which is assumed to be known. Each of the values  $u'(\ell)$ ,  $\ell = k, k+1, \dots, k+N-1$  is restricted to belong to the finite set:

$$U = \{s_1, s_2, \dots, s_{n_U}\}, \quad (10)$$

which contains  $n_U$  elements.

The value of the cost function (8) depends upon the values chosen for the sequence

$$\{u'(k), u'(k+1), \dots, u'(k+N-1)\}.$$

For notational convenience, we define the vector:

$$\bar{u}(k) \triangleq \begin{bmatrix} u'(k) \\ u'(k+1) \\ \vdots \\ u'(k+N-1) \end{bmatrix}$$

and write  $V_1(\bar{u}(k))$ .

The optimizers  $\bar{u}^{opt}(k)$  are obtained via minimization of (8), i.e., we seek:

$$\bar{u}^{opt}(k) \triangleq \arg \min_{\bar{u}(k) \in U^N} V_1(\bar{u}(k)). \quad (11)$$

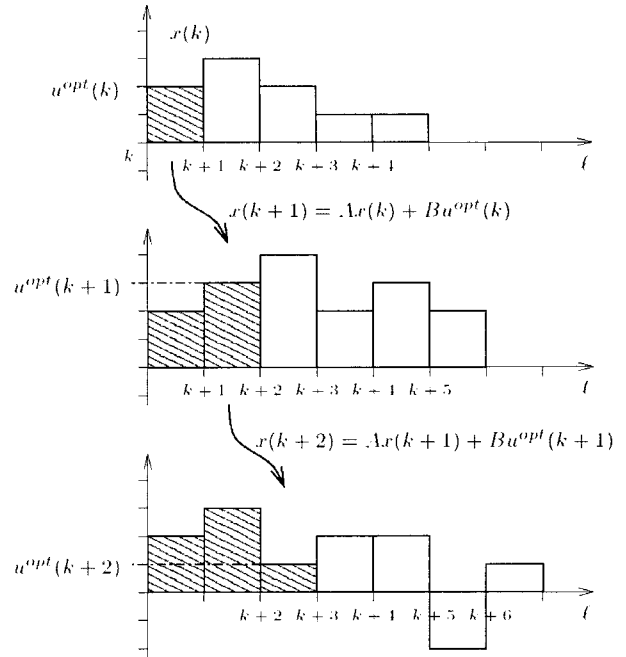


Fig. 6. Receding horizon principle.

Here, the finite set  $U^N$  is defined via the Cartesian product:

$$U^N \triangleq U \times \dots \times U,$$

in accordance with the restriction on each value,  $u(\ell) \in U$ .

Following the receding horizon idea described in Section 3, only the first element, namely:

$$u^{opt}(k) \triangleq [I \ 0 \ \dots \ 0] \bar{u}^{opt}(k)$$

is utilized. This value also yields the successor state:

$$x(k+1) = Ax(k) + Bu^{opt}(k).$$

Having fixed  $u^{opt}(k)$  and calculated  $x(k+1)$ , at the next step a new optimization is carried out on the interval  $[k+1, k+N]$  by using the updated state,  $x(k+1)$ . This gives rise to a new optimizer  $\bar{u}^{opt}(k+1)$  and the value  $u^{opt}(k+1)$ . The procedure is repeated ad-infinitum or, in the FIR filter design case, until all coefficients have been calculated. Fig. 6 illustrates this idea for the case  $N = 5$ . As can be seen, the window is of fixed size and moves (or slides) forward at each optimization step.

### 5. CHARACTERIZATION OF THE OPTIMAL SOLUTION

In this section we will provide a semi-closed form solution to the optimization problem (11). To proceed,

it is useful to vectorize the cost function (8) as follows:

Define:

$$\bar{x}(k) \triangleq \begin{bmatrix} x(k+1) \\ x(k+2) \\ \vdots \\ x(k+N) \end{bmatrix}, \quad \Lambda \triangleq \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix},$$

$$\Phi \triangleq \begin{bmatrix} B & 0 & & & \\ AB & B & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A^{N-1}B & A^{N-2}B & \dots & AB & B \end{bmatrix},$$

so that, given  $x(k)=x$  and by iterating (9), the predictor  $\bar{x}(k)$  satisfies:

$$\bar{x}(k) = \Phi \bar{u}(k) + \Lambda x.$$

Hence, the cost function (8) can be re-written as:

$$V_N(\bar{u}(k)) = \bar{V}_N + (\bar{u}(k))^T W \bar{u}(k) + 2(\bar{u}(k))^T Fx, \quad (12)$$

where

$$W \triangleq \Phi^T Q \Phi + R \in \mathbb{R}^{N \times N},$$

$$F \triangleq \Phi^T Q \Lambda \in \mathbb{R}^{N \times n},$$

$$Q \triangleq \text{diag}(Q, \dots, Q, P) \in \mathbb{R}^{Nn \times Nn},$$

$$R \triangleq \text{diag}(R, \dots, R) \in \mathbb{R}^{N \times N}$$

and  $\bar{V}_N$  does not depend upon  $\bar{u}(k)$ .

By direct calculation, it follows that the minimizer to (11), without taking into account any constraints on  $\bar{u}(k)$ , is:

$$\bar{u}_{uc}^{opt}(x) = -W^{-1} Fx. \quad (13)$$

Our subsequent development will utilize a nearest neighbour vector quantizer. It is defined as follows:

**Definition 1:** Vector Quantizer Given a countable (not necessarily finite) set of non-equal vectors  $B = \{b_1, b_2, \dots\} \subset \mathbb{R}^{n_B}$ , the nearest neighbour quantizer is defined as a mapping  $q_B: \mathbb{R}^{n_B} \rightarrow B$  which assigns to each vector  $c \in \mathbb{R}^{n_B}$  the closest element of  $B$  (as measured by the Euclidean norm), i.e.,  $q_B(c) = b_i \rightarrow B$  if and only if  $c$  belongs to the region:

$$\left\{ c \in \mathbb{R}^{n_B} : \|c - b_i\|^2 \leq \|c - b_j\|^2, \forall b_j \neq b_i, b_j \in B \right\}. \quad (14)$$

Note that in the special case, when  $n_B = 1$ , the quantizer defined above reduces to a standard scalar

quantizer.

Given Definition 1, we can now restate the solution to (11). This leads to:

**Theorem 1:** (Closed Form Solution) Suppose

$$U^N = \{v_1, v_2, \dots, v_p\},$$

where  $p = n_U^N$ , then the optimizer in (11) is given by:

$$\bar{u}^{opt}(x) = W^{-1/2} q_{\tilde{U}^N}(-W^{-T/2} Fx), \quad (15)$$

where the nearest neighbor quantizer  $q_{\tilde{U}^N}(\cdot)$  maps  $\mathbb{R}^N$  to the finite set  $\tilde{U}^N$ , defined as:

$$\tilde{U}^N = \{\tilde{v}_1, \tilde{v}_2, \dots, \tilde{v}_p\}, \quad \tilde{v}_i = W^{1/2} v_i, \quad v_i \in U^N. \quad (16)$$

**Proof:** For fixed  $x$ , the level sets of the cost (12) are ellipsoids in the input sequence space  $R^N$ . These are centered at the point  $\bar{u}_{uc}^{opt}(x)$  defined in (13). Thus, the optimization problem (11) can be geometrically interpreted as follows: Find the point  $\bar{u}(k) \in U^N$ , which belongs to the smallest ellipsoid defined by (12) (i.e., the point which provides the smallest cost whilst satisfying the constraints).

In order to simplify the problem, we introduce a change of variables:

$$\bar{\mu}(k) = W^{1/2} \bar{u}(k),$$

which transforms  $U^N$  into  $\tilde{U}^N$  defined in (16).

The optimizer  $\bar{u}^{opt}(x)$  can be defined in terms of this auxiliary variable as:

$$\bar{u}^{opt}(x) = W^{-1/2} \arg \min_{\bar{\mu}(k) \in \tilde{U}^N} J_N(\bar{\mu}(k)), \quad (17)$$

where:

$$J_N(\bar{\mu}(k)) \triangleq (\bar{\mu}(k))^T \bar{\mu}(k) + 2(\bar{\mu}(k))^T W^{-T/2} Fx. \quad (18)$$

The level sets of  $J_N$  are spheres in  $R^N$ , centred at

$$\bar{\mu}_{uc}^{opt}(x) \triangleq -W^{-T/2} Fx.$$

Hence, the constrained optimizer to (18), is given by the nearest neighbour to  $\bar{\mu}_{uc}^{opt}(x)$ , namely:

$$\arg \min_{\bar{\mu}(k) \in \tilde{U}^N} J_N(\bar{\mu}(k)) = q_{\tilde{U}^N}(-W^{-T/2} Fx). \quad (19)$$

The result (15) follows by substituting (19) into (17).  $\square$

It is worth noting, that the optimizer  $\bar{u}^{opt}(x)$  provided in Theorem 14 is, in general, different to the



sequence obtained by direct quantization of the unconstrained minimum (13), i.e.  $q_{\tilde{U}^N}(\bar{u}_{uc}^{opt}(x))$ .

As a consequence of Theorem 14, the receding horizon controller satisfies:

$$u^{opt}(x) = [1 \ 0 \ \dots \ 0] W^{-1/2} q_{\tilde{U}^N}(-W^{-T/2} Fx). \quad (20)$$

This solution can be illustrated as the composition of the following transformations:

$$x \in R^n \xrightarrow{-W^{-T/2} F} \bar{u}_{uc}^{opt} \xrightarrow{W^{-1/2} q_{\tilde{U}^N}(\cdot)} \bar{u}^{opt} \xrightarrow{[10 \dots 0]} u^{opt}. \quad (21)$$

It is worth noticing that  $q_{\tilde{U}^N}(\cdot)$  is a memoryless nonlinearity, so that (20) corresponds to a time-invariant nonlinear state feedback law. In a direct implementation, at each time step, the quantizer needs to perform  $r-1$  comparisons.

### 6. STATE SPACE PARTITION

Expression (14) partitions the domain of the quantizer into polyhedra, which are called Voronoi regions [48]. Since the constrained optimizer  $\bar{u}^{opt}(x)$  in (15) (see also (21)) is defined in terms of  $q_{\tilde{U}^N}(\cdot)$ , a partition of the state-space can be derived, as shown below:

**Theorem 2:** The constrained optimizing sequence  $\bar{u}^{opt}(x)$  in (15) can be characterized as:

$$\begin{aligned} \bar{u}^{opt}(x) = v_i &\Leftrightarrow x \in \mathfrak{R}_i, \\ \text{where} \\ \mathfrak{R}_i &\triangleq \{z \in R^n : 2(v_i - v_j)^T Fz \leq \|v_j\|_W^2 - \|v_i\|_W^2, \\ &\quad \forall v_j \neq v_i, v_j \in \tilde{U}^N\}. \end{aligned} \quad (22)$$

**Proof:** From Expressions (15) and (16) it follows that  $\bar{u}^{opt}(x) = v_i$  if and only if  $q_{\tilde{U}^N}(-W^{-T/2} Fx) = \tilde{v}_i$ . On the other hand,

$$\| -W^{-T/2} Fx - \tilde{v}_i \|^2 = \| W^{-T/2} Fx \|^2 + \| \tilde{v}_i \|^2 + 2\tilde{v}_i^T W^{-T/2} Fx$$

so that  $\| -W^{-T/2} Fx - \tilde{v}_i \|^2 = \| -W^{-T/2} Fx - \tilde{v}_j \|^2$  holds

if and only if  $2(\tilde{v}_i - \tilde{v}_j)^T W^{-T/2} Fx \leq \| \tilde{v}_j \|^2 - \| \tilde{v}_i \|^2$ .

This inequality together with expressions (16) and (14) shows that  $q_{\tilde{U}^N}(-W^{-T/2} Fx) = \tilde{v}_i$  if and only if  $x$  belongs to the set  $R_i$  defined in (22).

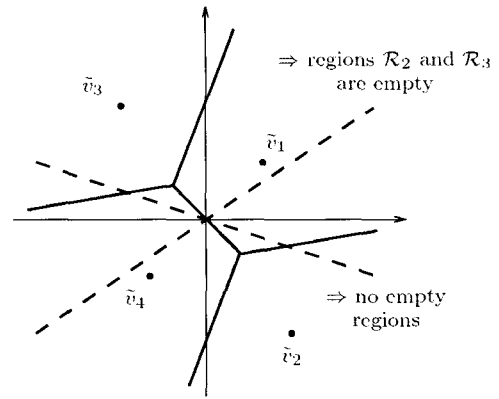


Fig. 7. Partition of the transformed input sequence space with  $N=2$  (solid lines) and two examples of  $-W^{-T/2} Fx, x \in R$  (dashed lines).

This fact completes the proof.  $\square$

The  $n_U^N$  regions  $R_i$  defined in (22) are polyhedra. They can be written in a compact form as:

$$R_i = \{x \in R^n : D_i x \leq H_i\},$$

where the rows of  $D_i$  are equal to all terms  $2(v_i - v_j)^T F$  as required, while the vector  $H_i$  contains the scalars  $\|v_j\|_W^2 - \|v_i\|_W^2$ .

**Remark 1:** Some of the inequalities in (22) may be redundant. In these cases, the corresponding regions do not share a common edge, i.e., are not adjacent. This phenomenon is illustrated in Fig. 7, where the regions  $R_2$  and  $R_3$  are not adjacent. The inequality separating them is redundant. Also, depending upon the matrix  $W^{-T/2} F$ , some of the regions  $R_i$  may be empty. This might happen, in particular, if  $N > n$ . In this case, the rank of  $F$  is equal to  $n$  and the transformation  $W^{-T/2} F$  does not span the entire space  $R^N$ . Fig. 7 illustrates this for the case  $n=1, n_U=2$  and  $N=2$ . As can be seen, depending on the unconstrained optimum locus given by the (dashed) line  $-W^{-T/2} Fx, x \in R$ , there exist situations in which some sequences  $\tilde{v}_j$  will never be optimal, yielding empty regions in the state space. On the other hand, if the pair  $(A, B)$  is completely controllable and  $A$  is invertible, then the rank of  $F$  is equal to  $\min(N, n)$ . In this case, if  $n \geq N$ , then  $W^{-T/2} F$  is onto, so that for every  $\tilde{v}_j \in \tilde{U}^N$  there exist, at least one,  $x$  such that  $q_{\tilde{U}^N}(-W^{-T/2} Fx) = \tilde{v}_j$  and none of the regions  $R_i$  are empty.

In the receding horizon law of (20), only  $n_U$  instead of (at most)  $n_U^N$  regions are needed to characterize the control law. Each of these regions is given by the union of all regions  $R_i$  corresponding to vertices  $v_i$ , having the same first element. The appropriate extension of Theorem 6 is presented below. This result follows directly from Theorem 6.

**Corollary 1: State Space Partition** Let the constraint alphabet  $U$  be as given in (10) and consider the following partition into equivalence classes:

$$U^N = \bigcup_{i=1, \dots, n_U} U_i^N,$$

where:

$$U_i^N \triangleq \{v \in U^N : [1 \ 0 \ \dots \ 0]v = s_i\}.$$

Then, the receding horizon control law (20) is equivalent to:

$$u^{opt}(x) = s_i, \quad \text{if } x \in X_i, \quad i = 1, 2, \dots, n_U. \quad (23)$$

Here, the polyhedra  $X_i$  are given by:

$$X_i \triangleq \bigcup_{j: v_j \in U_i^N} X_{ij}, \quad \text{where:}$$

$$X_{ij} \triangleq \{z \in R^n : 2(v_j - v_k)^T Fz \leq \|v_k\|_W^2 - \|v_j\|_W^2, \\ \|v_k\| \in U^N \setminus U_i^N\}.$$

It should be emphasized that this description requires less evaluations of inequalities than the direct calculation of the union of all  $\mathfrak{R}_j$  (as defined in (22)) with  $v_j \in U_i^N$ , since inequalities corresponding to internal borders are not evaluated. The state space partition obtained can be calculated off-line.

### 7. CLOSED LOOP STABILITY

We have seen above that the finite alphabet optimization problem leads to a semi-closed form solution, which can be utilized in the various Control Problems described in Section 2. However, optimality over finite horizons alone is usually an inadequate measure of performance. Ideally, one would like to establish other measures of the achieved performance, especially asymptotic stability. In this section we explore various stability issues associated with this finite alphabet Control problems under study.

The closed loop that results when joining the system of (6), or of (2) with zero reference, with the receding horizon law (21) can be described via the following piecewise-affine map:

$$\begin{aligned} x(k+1) &= g(x(k)), \\ g(x(k)) &\triangleq Ax(k) + Bs_i, \\ &\text{if } x(k) \in X_i, \quad i = 1, 2, \dots, n_U. \end{aligned} \quad (24)$$

This characterization follows directly from Corollary 1. Piecewise-affine maps are mixed mappings and also form a special class of hybrid systems with underlying discrete-time dynamics, see e.g. [49, 50] and the references therein. They also appear in connection with some signal processing problems, namely arithmetic overflow of digital filters [51] and  $\Sigma\Delta$ -Modulators [52, 37], and have also been studied in a more theoretical mathematical context, see e.g. [53, 54].

Since there exist fundamental differences in the dynamic behavior of (24), depending on whether the open loop systems of (6) and (2) are stable or unstable, i.e. on whether the matrix  $A$  is strictly Hurwitz or not, it is convenient to divide the discussion that follows accordingly.

#### 7.1. Stable open loop systems

If  $A$  is strictly Hurwitz, then the states in (24) (and thus in (6) and (2)) are always bounded, when its inputs belong to a finite, and hence bounded, constraint set law. Moreover, it can also be shown that essentially every state trajectory either converges towards a fixed point or towards a limit cycle, see e.g. [54, 55].

Whilst the above properties apply to general systems described by (24), where  $X_i$  defines any partition of the state space, the following result is more specific. It utilizes the fact that the law  $u^{opt}(x)$  is optimizing in a receding horizon sense in order to establish a stronger result.

**Theorem 3: (Asymptotic Stability)** If  $A$  is Hurwitz,  $0 \in U$  and  $P = P^T > 0$  satisfies the Lyapunov Equation  $A^T P A + Q = P$ , then the closed loop (24) is asymptotically stable.

**Proof:** The proof follows standard techniques used in the Model Predictive Control framework as summarized in [56]. In particular, using the notation of [56], we choose  $X_f = R^n$  and  $\kappa_f(x) = 0$ ,  $\forall x \in X_f$ . Clearly Assumptions A1–A3 hold and  $X_N = R^n$ .

Direct calculation yields, that  $\forall x \in X_f$ :

$$\begin{aligned} &F(f(x, \kappa_f(x))) - F(x) + \ell(x, \kappa_f(x)) \\ &= (Ax + B\kappa_f(x))^T P(Ax + B\kappa_f(x)) - x^T P x + x^T Q x \\ &\quad + (\kappa_f(x))^T R \kappa_f(x) = x^T (A^T P A + Q - P)x = 0, \end{aligned}$$

so that also A4 is satisfied. Global attractiveness of

the origin follows. □

As can be seen, the receding horizon law (23) ensures that the origin is not only a fixed point, but also that the entire space  $R^n$  is its basin of attraction. It should be emphasized that, if  $0 \notin U$ , then the origin is not a fixed point.

**Remark 2:** Note that the use of the final state weighting terms  $\|x'(k+N) - x^{opt}(k+N)\|_p^2$  in (8) effectively gives an infinite horizon cost save that the finite alphabet constraint is relaxed outside the interval  $(k, k+N-1)$ . Relaxations of this type and others are commonly deployed in finite alphabet problems, see Section 11.

7.2. Unstable open loop systems

In case of strictly unstable open loop systems (6) and (2), the situation becomes more involved. Although fixed points and periodic sequences may be admissible, they are basically non-attractive, see e.g. [54]. Moreover, with input signals which are limited in magnitude, there always exists an unbounded region, such that initial states contained in it lead to unbounded state trajectories. This does not mean that every state trajectory of (24) is unbounded. Despite the fact that the unstable open loop dynamics (as expressed in  $A$ ) makes neighboring trajectories diverge locally, under certain circumstances the receding horizon law may keep the state trajectory bounded.

As a consequence of the highly nonlinear (non-Lipschitz) dynamics resulting from the quantizer defining (15), in the bounded state case the resulting closed loop trajectories may be quite complex. In order to analyze them without exploring their fine geometrical structure, it is useful to relax the usual notion of asymptotic stability of the origin. A more useful characterization here is that of ultimate boundedness of state trajectories. This notion refers to convergence towards a bounded region of  $R^n$ , instead of to a point, see e.g. [57]. (Ultimate boundedness has also been considered in [58] and by several other authors in the context of practical stability.) We refer the reader to the literature, especially [44] where these more detailed issues are discussed and analyzed for the case of unstable open loop plants.

**8. RECEDING HORIZON ESTIMATION PROBLEMS**

Mirroring the development in Sections 4 to 7, we can develop a receding horizon estimation scheme for situations such as the channel equalization problem outlined in Section 2.4.

In order to obtain a scheme of fixed complexity, we fix two integers  $N_1 \geq 0$  and  $N_2 \geq 1$  and, at each

instant  $\ell = k$ , consider explicitly only those output samples contained in the set:

$$Y(k) \triangleq \{y(k-N_1), y(k-N_1+1), \dots, y(k+N_2-1)\}.$$

This set contains only  $N$  elements, where:

$$N = N_1 + N_2.$$

Note that, if  $N_2$  is chosen to be larger than unity, then the scheme includes previewing and thus decisions are delayed.

We will summarize the information prior to time  $(k-N_1)$ , i.e.,

$$\{y(0), y(1), \dots, y(k-N_1-2), y(k-N_1-1)\}$$

via an a priori state estimate  $z(k-N_1)$ . Thus, suppose that

$$x(k-N_1) \sim N(z(k-N_1), P_z), \tag{25}$$

where  $z(k-N_1)$  is an a priori estimate for  $x(k-N_1)$  which has a Gaussian distribution of covariance  $P_z$ . In this characterization, the matrix  $P_z^{-1}$  reflects the degree of belief in this a priori state estimate. Absence of prior knowledge of  $x(k-N_1)$  can be accommodated by using  $P_z^{-1} = 0$ .

The assumption of Gaussianity for the initial state and noise leads to the cost function

$$V_2 \triangleq \|x'(k-N_1) - z(k-N_1)\|_{P_z^{-1}}^2 + r_n^{-1} \sum_{\ell=k-N_1}^{k+N_2-1} (y(\ell) - Cx'(\ell) - Du'(\ell))^2. \tag{26}$$

In this expression,  $x'(k-N_1)$  and  $u'(\ell)$  are the decision variables. The other states follow the prediction model:

$$x'(\ell+1) = Ax'(\ell) + Bu'(\ell), \quad \ell = k-N_1, \dots, k+N_2-1.$$

Note that, as opposed to the predictors (9) utilized for Control Problems, here the initial state  $x'(k-N_1)$  is unknown and is thus another decision variable.

The cost  $V_2$  depends upon the values adopted by  $x'(k-N_1)$  and the vector

$$\underline{u}(k) \triangleq \begin{bmatrix} u'(k-N_1) \\ u'(k-N_1+1) \\ \vdots \\ u'(k) \\ \vdots \\ u'(k+N_2-1) \end{bmatrix}.$$

Hence, we write  $V_2(\underline{u}(k), x'(k-N_1))$ .

Minimization of  $V_2$  yields both a possible input sequence  $\underline{u}^{opt}(k)$  and an a posteriori state estimate  $x^{opt}(k - N_1)$ :

$$\begin{bmatrix} \underline{u}^{opt}(k) \\ x^{opt}(k - N_1) \end{bmatrix} = \arg \min_{\underline{u}(k) \in U^N} V_2(\underline{u}(k), x'(k - N_1)). \quad (27)$$

It should be emphasized here, that  $x^{opt}(k - N_1)$  constitutes a revised state estimate. It differs from the a priori estimate  $z(k - N_1)$  as permitted by the confidence matrix  $P^{-1}$ .

Following the receding horizon paradigm of Section 3, only the present value:

$$\hat{u}(k) \triangleq \begin{bmatrix} 0_{N_1} & 1 & 0_{N_2-1} \end{bmatrix} \underline{u}^{opt}(k)$$

is delivered at the output of the equalizer (estimator). At the next time instant, the shifted data window  $Y(k + 1)$  and an a priori estimate  $z(k - N_1 + 1)$  are used in an optimization which yields  $\hat{u}(k + 1)$  and so on.

**Remark 3:** The provision of an a priori estimate,  $z(k - N_1)$ , together with an associated degree of belief via the term  $\|x'(k - N_1) - z(k - N_1)\|_{P_z^{-1}}^2$  in the cost provides a means of propagating the information contained in the data received before  $\ell = k - N_1$ . Consequently, an information horizon of growing length is obtained. Notice, however, that we have again deployed a relaxation by only considering the finite alphabet constraints in blocks of size  $N$ . The initial state weighting term  $\|x'(k - N_1) - z(k - N_1)\|_{P_z^{-1}}^2$  effectively gives an infinite horizon problem where the constraints are relaxed outside the interval  $k - N_1, k + N_2 - 1$ . Thus, in contrast to considering the entire sequence  $\{y(0), \dots, y(k + N_2 - 1)\}$  explicitly, the computational effort is fixed by means of the window length  $N$ .

Various methods for information propagation via the a priori state estimate can be conceived. Each optimization step provides estimates for the state and input sequence. One can re-utilize these decisions in order to formulate a priori estimates for  $x(\ell)$ , via propagation in blocks according to:

$$z(\ell) = A^N x^{opt}(\ell - N) + M \underline{u}^{opt}(\ell - N_2), \quad (28)$$

where

$$M \triangleq \begin{bmatrix} A^{N-1}B & A^{N-2}B & \dots & AB & B \end{bmatrix}.$$

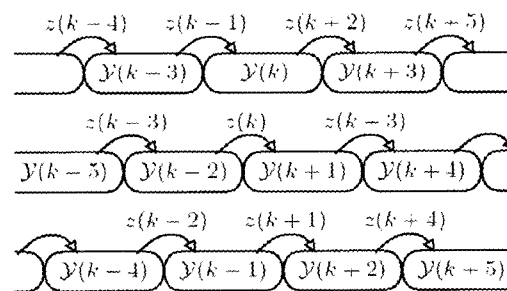


Fig. 8. Information propagation.

In this way, the estimate obtained in the previous block is rolled forward. Indeed, in order to operate in a receding horizon manner, it is necessary to store  $N$  a priori estimates. This is depicted graphically in Fig. 8 for the case  $N_1 = 1$  and  $N_2 = 2$ .

Since the states  $x(\ell)$  depend on the finite alphabet input, one may well question the assumption made in (25) that  $x(k - N_1)$  is Gaussian. However, we can always use this structure and simply interpret the matrix  $P_z$  in the cost (26) as a design parameter.

As a guide for tuning  $P_z$ , we recall that in the unconstrained case, where the input and initial state are Gaussian, i.e.,  $u(\ell) \sim N(0, Q_u)$  and  $x(0) \sim N(\bar{x}_0, P_0)$ , the Kalman Filter provides the minimum variance estimate for  $x(k - N_1)$ . Its covariance matrix  $P(k - N_1)$  obeys the Riccati difference equation:

$$\begin{aligned} P(\ell + 1) &= BQ_u B^T + AP(\ell)A^T \\ &\quad - K(\ell)(CP(\ell)C^T + r + DQ_u D^T)K^T(\ell) \quad (29) \\ P(0) &= P_0, \end{aligned}$$

where

$$K(\ell) \triangleq (AP(\ell)C^T + BQ_u D^T)(CP(\ell)C^T + r + DQ_u D^T)^{-1}.$$

A further simplification occurs if we replace the recursion (29) by its steady state equivalent. In particular, it is well known, see e.g. [59], that, under reasonable assumptions,  $P(\ell)$  converges to a steady state value,  $P$ , as  $\ell \rightarrow \infty$ . The matrix  $P$  satisfies the following algebraic Riccati equation:

$$\begin{aligned} P &= BQ_u B^T + APA^T - (APC^T + BQ_u D^T) \\ &\quad (CPC^T + r + DQ_u D^T)^{-1} (CPA^T + DQ_u B^T). \quad (30) \end{aligned}$$

This would be a possible choice for the matrix  $P_z$  in the cost function.

The receding horizon estimator results from bringing together the a priori estimate of (28) with the

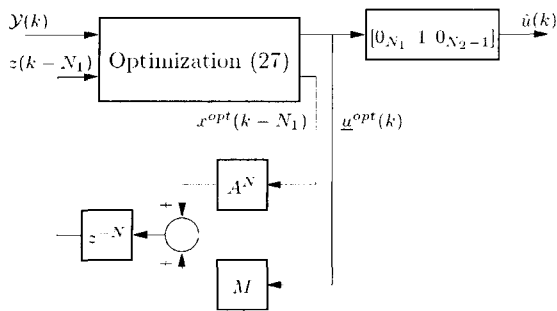


Fig. 9. Receding horizon estimation architecture.

optimization (27). It is schematically depicted in Fig. 9. As can be seen in this figure, the estimator includes a feedback path which provides the a priori estimate  $z(k - N_1)$  needed in the decision process (27).

By adapting the procedures outlined in Sections 4 to 6, we can develop a semi-closed form solution to the optimization problem, which yields an estimation scheme which is easy to implement. More details can be found in [43], see also Section 10.4 for results.

### 9. A UNIFIED VIEW

Inspection of the cost functions used for the Control Problems, see (8) and for the Estimation Problems given in (26) reveals similarities. In both formulations, decision variables are to be adjusted such that the states of a dynamic model behave in a certain manner.

The main difference lies in the boundary conditions. In the Control formulation, the final state weighting term

$$\|x'(k + N)\|_P^2$$

is introduced in order to summarize behaviour of the system over the semi-infinite future horizon  $\ell > k + N$  via state values at time  $\ell = k + N$ . On the other hand, in the Estimation Problem case, the term

$$\|x'(k - N_1) - z(k - N_1)\|_{P_z^{-1}}^2$$

is used to summarize the past system behaviour for  $\ell < k - N_1$ .

With this as a background, it is easy to conceive a formulation which includes both types of boundary conditions by penalizing both initial and terminal state deviations. The idea is to approximate the effect of future and past trajectories on decisions concerning the interval  $k - N_1 \leq \ell < k + N_2$ . This unifies the state estimation and control problems.

### 10. APPLICATIONS REVISITED

The general concepts introduced in the preceding sections can be applied to the applications discussed

in Section 2. A brief overview is given below:

#### 10.1. Power conversion problems

The problem is readily formulated as a receding horizon finite alphabet control problem of the type discussed in Section 4. It turns out that a unitary constraint horizon,  $N = 1$ , in the receding horizon formulation gives rise to a switching strategy which is equivalent to that provided by a general  $\Sigma\Delta$ -Modulator.  $\Sigma\Delta$ -Modulators have been mainly developed in an analog-to-digital conversion context and have also found their way into Switched-Mode Power Supplies [34]. It has recently been shown [38] that horizons greater than 1 give improved harmonic suppression. In particular, choosing a horizon  $N = 3$  gives about 10 dB improvement in the peak spectral distortion as can be appreciated in Fig. 10. As a consequence, an SMPS whose switching signal is provided by the receding horizon optimization scheme can be expected to have lower EMI emissions than an SMPS which uses a  $\Sigma\Delta$ -Modulator based approach.

#### 10.2. Audio quantization

Again this problem is readily converted to a receding horizon finite alphabet control problem as in

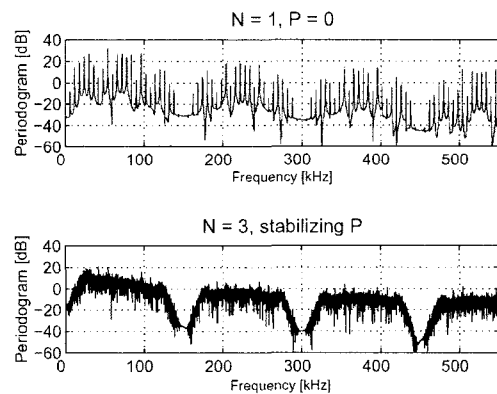


Fig. 10. Periodogram of the switching signal in an SMPS.

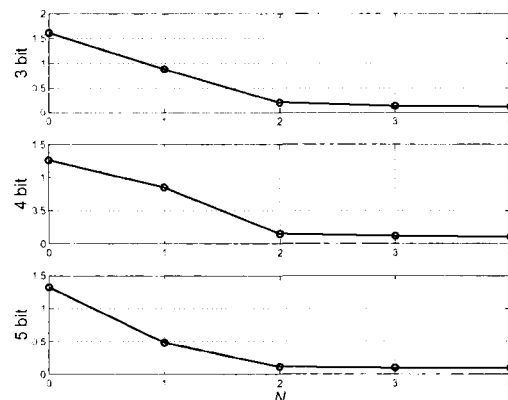


Fig. 11. Distortion introduced by receding horizon audio quantization.

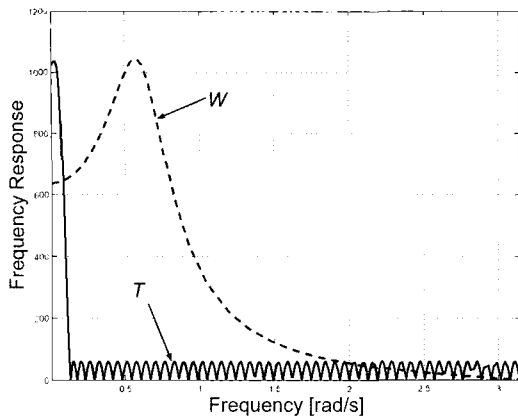


Fig. 12. Frequency response of the target filter  $T$  and of the frequency weighting filter.

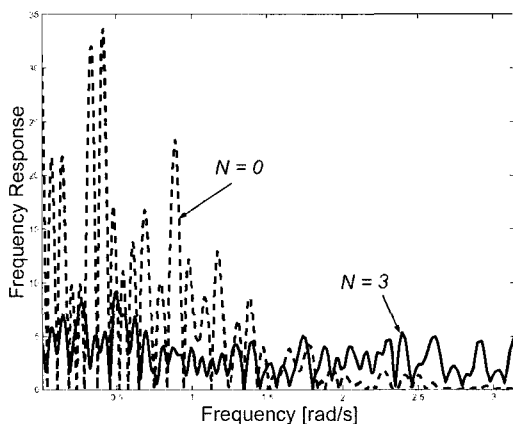


Fig. 13. Frequency response of the filtered approximation error for  $N=3$  (solid) and for direct quantization (dashed).

Section 4. In this case, the horizon-one solution turns out to be equivalent to the Noise Shaping Quantizer, which has been widely studied in the audio literature [5, 6, 7, 60] and has also been implemented in many consumer audio products. Again, using constraint horizons greater than one is generally beneficial. Fig 11, which is taken from [40], illustrates the effect of the horizon on the distortion introduced in the quantization process. In this figure,  $N=0$  denotes straight quantization. As can be appreciated, performance can be significantly improved by choosing a horizon  $N=2$ , see also [41]. Note, however, that the distortion levels are not reduced significantly with horizons larger than  $N=3$ . The performance is asymptotic in the horizon length. Thus, it is recommendable to use small, though non-unitary, horizons.

### 10.3. Design of FIR filters

A similar situation arises in the FIR filter design problem outlined in Section 2.3. The horizon-one solution is well-known in the signal-processing literature, see [61, 62, 63, 64]. Using larger horizons

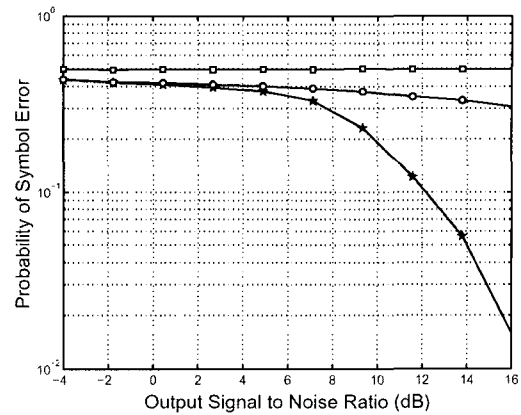


Fig. 14. Bit-error rate achieved with the receding horizon estimator of Fig. 9 with  $L_1=1$ ,  $L_2=2$  and stabilizing  $P$  as given in (30) (stars), with  $P=0$  (circles) and with a standard DFE (squares).

typically gives better performance [42].

As an example, taken from [42], consider an equiripple target filter  $T(z)$  and a frequency weighting filter  $W(z)$ , whose frequency response is included in Fig. 12. Fig. 13 contains the frequency response of the filtered approximation error for two designs. In both cases, the FIR filters designed are of length 100 and each of the coefficients is restricted to belong to the 6 bit set  $U = \{-32, -31, \dots, 31\}$ . The first filter is obtained via direct quantization of the impulse response of  $T$  and is denoted as  $N=0$ . The other design is provided by a receding horizon procedure with horizon  $N=3$ .

As is apparent from these figures, the filter obtained via receding horizon optimization is closer to  $T(z)$  than the design obtained by direct quantization. This performance increase is accomplished by concentrating the approximation error mostly in the less-important frequencies, as dictated by the weighting function  $W(z)$ .

### 10.4. Equalization of band-limited communication channels

Not surprisingly, the simplest case of  $L_1=0$ ,  $L_2=1$  and  $P=0$  corresponds to a well-known scheme, namely the Decision Feedback Equalizer, described e.g. in the textbook [11]. Horizons greater than one give improved performance, compare also to [65, 66]. It is also known that including degrees of belief in past estimates via  $P \neq 0$  improves performance even further [43]. Fig. 14, which is adapted from [67], illustrates these aspects for the case of a binary channel with impulse response given by:

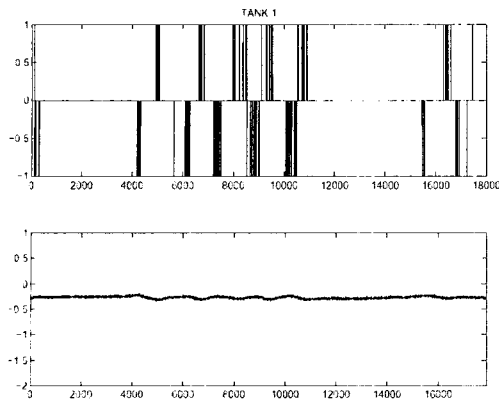


Fig. 15. Top: Control action sent by the Networked Controller, Bottom: Level Measurement.

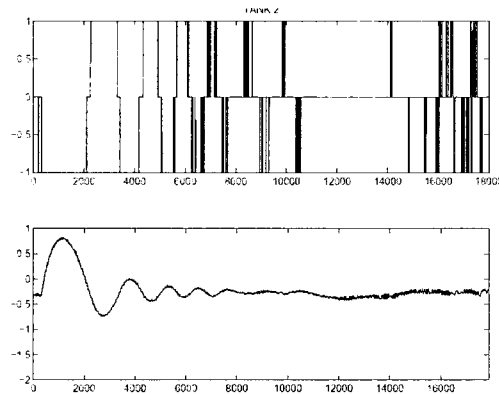


Fig. 16. Top: Control action sent by the Networked Controller, Bottom: Level Measurement.

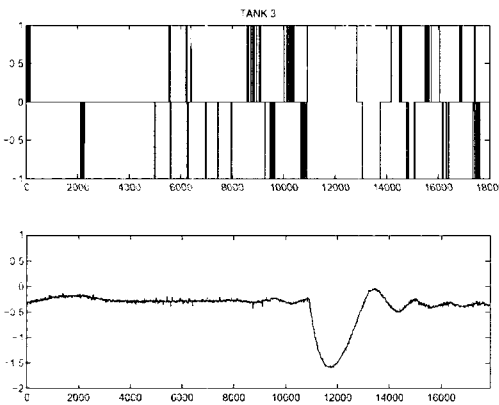


Fig. 17. Top: Control action sent by the Networked Controller, Bottom: Level Measurement.

$$H(z) = 1 - 12.8z^{-1} + 20.6z^{-2} + 17.9z^{-3} - 18.2z^{-4} + 4.5z^{-5}$$

It is apparent from this figure that using larger horizons and the incorporation of a finite degree of belief in past estimates is beneficial. Choosing  $P > 0$  leads to faster error recovery and thus prevents instability related phenomena. Further details can be

found in [43].

10.5. Networked control systems

This application is discussed in detail in [20, 21]. We have found that relatively small optimization horizons are effective when dealing with simple problems. A full scale experiment is described in [20], where we have applied the methodology to level control of five tanks. In the experiment, an unmeasured inflow disturbance was introduced into Tank 2 at 35 seconds and an extra outflow valve was opened on Tank 3 at 1090 seconds. Figs. 15–17, show the control increments sent by the controller to each tank together level measurements.

It can be seen from these figures that, when the disturbance occurs in Tank 2, most down-link bandwidth is dedicated to the control of its level. Also, it is easily observed that the controller pays attention to the other tanks when the measured level in Tank 2 approaches the desired level. A similar effect happens when Tank 3 is disturbed.

11. SEMI-DEFINITE PROGRAMMING RELAXATIONS

The above problems can be seen to be special cases of the following optimization problem:

$$\min \begin{bmatrix} u \\ 1 \end{bmatrix}^T \begin{bmatrix} H & r \\ r^r & s \end{bmatrix} \begin{bmatrix} u \\ 1 \end{bmatrix},$$

where  $u_i \in \{+1, -1\}$  for all  $i$ . This is a quadratic boolean optimization problem. The constraints can be converted to quadratic equality constraints of the form:

$$u_i^2 - 1 = 0.$$

This class of problems are non-convex and are known to be NP-hard. We have avoided this difficulty in Sections 7 to 9 by suggesting the use of a relaxed strategy in which the finite alphabet constraint is only applied over a fixed receding horizon window. Outside that window, the constraints are ignored leading to a relaxed infinite horizon problem. This yields computational feasible algorithms for simple problems (e.g., scalar input and constraint horizon 10 yields approximately 1,000 alternatives for the decision variables at each time step). However, it is clear that for larger problems, the required computations can become intractable (e.g., for 10 inputs and horizon 10, one has approximately  $10^{30}$  alternatives for the decision variables at each time step). Thus, some form of relaxation is necessary to obtain computationally tractable solutions. As an illustration, one can obtain bounds via semi-definite programming relaxations:

Consider the (non-convex) problem:

$$\begin{aligned} \min \quad & U^T Q U; \quad U = [u_1, \dots, u_N]^T, \\ \text{subject to} \quad & u_i^2 - 1 = 0. \end{aligned}$$

The associated Lagrangian function is

$$\begin{aligned} L(U, \Lambda) &= U^T Q U - \sum_{i=1}^m \Lambda_i (u_i^2 - 1) \\ &= U^T (Q - \Lambda) U + \text{trace } \Lambda. \end{aligned}$$

For the Lagrangian to be bounded from below, we require  $Q - \Lambda \succeq 0$  (i.e.,  $Q - \Lambda$  needs to be positive semi-definite). The dual problem is therefore a (convex) semi-definite program:

$$\begin{aligned} \max \quad & \text{trace } \Lambda, \\ \text{subject to} \quad & Q - \Lambda \succeq 0. \quad \Lambda \text{ diagonal} \end{aligned}$$

Note that, because the original (primal) problem is non-convex, there will, in general, be a duality gap. However, the solution to the convex dual problem gives a lower bound on the cost of the primal problem. This follows since for any feasible  $U, \Lambda$  we have

$$\underline{U}^T Q \underline{U} \succeq \underline{U}^T \Lambda \underline{U} = \sum_{i=1}^m \Lambda_{ii} u_i^2 = \text{trace } \Lambda. \quad (31)$$

This bound can be used in a variety of ways. For example, given a solution obtained by relaxing the primal problem (say by using finite constraint horizon methods) we can evaluate the associated cost and compare it with the lower bound in (31). Other relaxations are possible. Indeed, the problem of relaxing the primal problem has a long history in general optimization theory. A relaxation technique commonly deployed is as follows:

Let  $\Gamma = \underline{U} \underline{U}^T$ , then  $\underline{U}^T Q \underline{U} = \text{trace } Q \underline{U} \underline{U}^T = \text{trace } Q \Gamma$ . The matrix  $\Gamma$  has rank 1, is positive definite and satisfies  $\Gamma_{ii} = 1$ . Hence, the original problem can be expressed as

$$\begin{aligned} \min \quad & \text{trace } Q \Gamma \\ \text{subject to} \quad & \Gamma \succeq 0, \\ & \Gamma_{ii} = 1, \\ & \text{rank } \Gamma = 1. \end{aligned}$$

We can obtain a relaxation to the problem by dropping the (non-convex) rank constraint. The relaxed problem so obtained can be solved by semi-definite programming methods. Of course, if the resulting solution has rank 1, we are done. Otherwise, we can project the solution to obtain sub-optimal approximations and the associated cost compared with the lower bound (31) arising from the (convex) dual problem. These kinds of strategies are described, for example, in [68] and are capable of dealing with

problems having hundreds of decision variables at each step.

## 12. CONCLUSION

This paper has discussed the problem of finite alphabet control and estimation. This problem occurs in a broad spectrum of applications. It is encouraging (for control engineers) to see that their "tools" have a valuable role to play in many diverse fields including audio signal processing, digital communications and power electronics. We have discussed "closed form" solutions to the associated finite alphabet optimization problems, which are applicable to small constraint horizons. Moreover, we have seen that small constraint horizons can lead to near optimal performance when a receding horizon optimization strategy is used. For more complex problems, semi-definite programming relaxations can be used to obtain bounds on solutions.

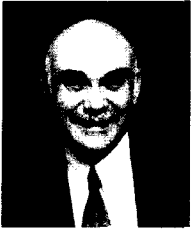
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