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Preservation Property of NBU_{Mg} under Shock Models

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Abstract. We propose, in this paper, the discrete version of NBU_{Mg} and show that the NBU_{Mg} class is preserved under both the non-homo geneous poisson shock model and the general shock model.

Key Words : *NBU, NBU_{Mg}, shock model, star-shaped.*

1. INTRODUCTION

In reliability, shock models have special importance. In this aspect, the closure properties of classes of life distribution under shock models are of great interests. Till now, many works have dealt with these problems. For example, *A-Hameed and Proschan* (1975) proved the closure property of NBU distribution under shock models. *Block and Savits* (1978) obtained similar result for the $NBUE$ class, *Klefsjö* (1981) for $HNBUE$ class and *Yue and Cao* (2001) for $NBUL$ class. The purpose of this paper is to obtain the corresponding properties of NBU_{Mg} class of life distributions under those shock models.

Let X be a non-negative random variables with absolutely continuous distribution functions F, and $\bar{F} = 1 - F$ be its survival functions. Let Z be a strictly positive discrete random variable. For all $k \in N = \{0, 1, \cdots\}$, denote

$$p_k = P(Z = k), \quad k = 0, 1, 2, \cdots,$$

and

$$\bar{P}_k = P(Z > k) = \sum_{j=k+1}^{\infty} p_j.$$

Before proposing the main results, we present some classes of life distributions and definition which will be used in following the context.

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1. X is NBU_{Mg} (*new better than used in moment generating function*) if $X_t \leq_{Mg} X$, or equivalently, for some $t \geq 0$,

$$\bar{F}(t)\int_0^{\infty} e^{\lambda u}\bar{F}(u)du \geq \int_0^{\infty} e^{\lambda u}\bar{F}(t+u)du.$$

2. Z is said to be discrete NBU (*new better than used*) if $\bar{P}_{j+k} \leq \bar{P}_j\bar{P}_k$, for all $k, j \in N$.

3. A function $g(x)$ defined on $[0,\infty)$ is called star-shaped if $g(t)/t$ is increasing in t, (see *Barlow and Proschan, 1981*).

NBU_{Mg} is recently introduced by *Li and Cheng* (2003), for more details about NBU please see *Barlow and Proschan* (1981).

In this paper, we will investigate a device subjected to a sequence of shocks occurring randomly in time according to a counting process $N(t), t \geq 0$. Let $\bar{P}_k$ be the probability that the device survives the first k shocks, where $1 = \bar{P}_0 \geq \bar{P}_1 \geq \cdots$, then the survival function of the device is given by

$$\bar{H}(t) = \sum_{k=0}^{\infty} \bar{P}_k P\left(N(t) = k\right).$$

2. MAIN RESULTS

Yue and Cao (2001) proved that $NBUL$ was preserved under the nonhomogeneous poisson shock model and general shock model. Our purpose here is to propose the corresponding closure properties of NBU_{Mg}.

2.1 Nonhomogeneous Poisson Shock Model

Suppose that a device is subjected to a sequence of shocks occurring randomly in the time according to a nonhomogeneous poisson process with intensity $\lambda(t)$. Suppose further that the device has probability $\bar{P}_k$ of surviving the first k shocks, where $1 = \bar{P}_0 \geq \bar{P}_1 \geq \cdots$. Denote $p_j = \bar{P}_{j-1} - \bar{P}_j, j \geq 1$. Then the survival function of the device is given by

$$\bar{H}(t) = \sum_{k=0}^{\infty} \bar{P}_k \frac{[\Lambda(t)]^k}{k!} e^{-\Lambda(t)}, \quad t \geq 0, \tag{2.1}$$

where $\Lambda(t) = \int_0^t \lambda(u)du$. Let

$$\bar{S}(t) = \sum_{k=0}^{\infty} \bar{P}_k \frac{t^k}{k!} e^{-t}, \quad t \geq 0, \tag{2.2}$$

then

$$\bar{H}(t) = \bar{S}(\Lambda(t)), \quad t \geq 0. \tag{2.3}$$

If $\lambda(t) = \lambda$ is a positive constant, then

$$\bar{H}(t) = \sum_{k=0}^{\infty} \bar{P}_k \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \quad t \geq 0. \tag{2.4}$$

Before developing our main results, we firstly propose the discrete version of the NBU_{Mg}.

Definition 2.1. Z is said to be discrete NBU_{Mg} (*new better than used in moment generating function*) if

$$\sum_{k=0}^{\infty} \bar{P}_{k+i} z^{-k} \leq \bar{P}_i \sum_{k=0}^{\infty} \bar{P}_k z^{-k}, \quad 0 \leq z \leq 1, \quad i = 0, 1, \cdots.$$

Now, let us introduce the following theorem.

Theorem 2.2. If p_k is discrete NBU_{Mg}, then $H(t)$ given by (2.4) is NBU_{Mg}.

Proof. p_k is discrete NBU_{Mg} implies that

$$\sum_{k=0}^{\infty} \bar{P}_{k+i} z^{-k} \leq \bar{P}_i \sum_{k=0}^{\infty} \bar{P}_k z^{-k}, \tag{2.5}$$

or equivalently,

$$\sum_{k=i}^{\infty} \bar{P}_k z^{-k} \leq \bar{P}_i z^{-i} \sum_{k=0}^{\infty} \bar{P}_k z^{-k}, \quad 0 < z < 1. \tag{2.6}$$

It suffices to show

$$e^{st} \bar{H}(t) \int_0^{\infty} e^{st} \bar{H}(x) dx \geq \int_t^{\infty} e^{sx} \bar{H}(x) dx. \tag{2.7}$$

Since

$$\begin{aligned}
\int_0^{\infty} e^{sx} \bar{H}(x) dx &= \int_0^{\infty} e^{sx} \sum_{k=0}^{\infty} \frac{(\lambda x)^k}{k!} e^{-\lambda x} \bar{P}_k dx \\
&= \sum_{k=0}^{\infty} \frac{\bar{P}_k}{k!} \int_0^{\infty} e^{-(\lambda - s)x} (\lambda x)^k dx \\
&= \frac{1}{\lambda - s} \sum_{k=0}^{\infty} \left(\frac{\lambda}{\lambda - s} \right)^k \bar{P}_k \\
&= \frac{1}{\lambda - s} \sum_{k=0}^{\infty} \left(\frac{\lambda - s}{\lambda} \right)^{-k} \bar{P}_k,
\end{aligned}$$

we have

$$
\begin{aligned}
& \int_t^{\infty} e^{sx}\bar{H}(x)dx \\
= & \sum_{k=0}^{\infty}\frac{\bar{P}_k}{k!}\int_t^{\infty} e^{-(\lambda-s)x}(\lambda x)^k dx \\
= & \frac{1}{\lambda-s}e^{-(\lambda-s)t}\sum_{k=0}^{\infty}(\frac{\lambda}{\lambda-s})^k\bar{P}_k\sum_{j=0}^{k}\frac{[(\lambda-s)t]^j}{j!} \\
= & \frac{1}{\lambda-s}e^{-(\lambda-s)t}\sum_{j=0}^{\infty}\frac{[(\lambda-s)t]^j}{j!}\sum_{k=j}^{\infty}\left(\frac{\lambda}{\lambda-s}\right)^k\bar{P}_k \\
= & \frac{1}{\lambda-s}e^{-(\lambda-s)t}\sum_{j=0}^{\infty}\frac{[(\lambda-s)t]^j}{j!}\sum_{k=j}^{\infty}\left(\frac{\lambda-s}{\lambda}\right)^{-k}\bar{P}_k \\
\leq & \frac{1}{\lambda-s}e^{-(\lambda-s)t}\sum_{j=0}^{\infty}\frac{[(\lambda-s)t]^j}{j!}\left(\frac{\lambda-s}{\lambda}\right)^{-j}\bar{P}_j\sum_{k=0}^{\infty}\left(\frac{\lambda-s}{\lambda}\right)^{-k}\bar{P}_k \\
= & e^{-(\lambda-s)t}\sum_{j=0}^{\infty}\frac{(\lambda t)^j}{j!}\bar{P}_j\frac{1}{\lambda-s}\sum_{k=0}^{\infty}\left(\frac{\lambda-s}{\lambda}\right)^{-k}\bar{P}_k \\
= & e^{st}\bar{H}(t)\int_0^{\infty} e^{sx}\bar{H}(x)dx.
\end{aligned}
$$

Thus (2.7) holds, $H(t)$ is NBU_{Mg}.

In the following theorem, we propose the closure property of NBU_{Mg} under nonhomogeneous poisson shock model.

Theorem 2.3. If p_k is discrete NBU_{Mg} and $\Lambda(t)$ is star-shaped in t, then $H(t)$ given by (2.1) is also NBU_{Mg}.

Proof. It is sufficient to show that

$$\int_t^{\infty} e^{sx}\bar{H}(x)dx \leq e^{st}\bar{H}(t)\int_0^{\infty} e^{sx}\bar{H}(x)dx. \tag{2.8}$$

Let

$$\Delta = e^{st}\bar{H}(t)\int_0^{\infty} e^{sx}\bar{H}(x)dx - \int_t^{\infty} e^{sx}\bar{H}(x)dx,$$

in view of the fact that $\Lambda(t)$ is star-shaped, it follows from Theorem 2.2 that

$$
\begin{aligned}
\Delta &= e^{st}\bar{S}(\Lambda(t))\int_0^{\infty} e^{sx}\bar{S}(\Lambda(x))\,dx - \int_t^{\infty} e^{sx}\bar{S}(\Lambda(x))\,dx \\
&= e^{st}\bar{S}(\Lambda(t))\int_0^{t} e^{sx}\bar{S}(\Lambda(x))\,dx - \left(1 - e^{st}\bar{S}(\Lambda(t))\right)\int_t^{\infty} e^{sx}\bar{S}(\Lambda(x))\,dx \\
&\geq e^{st}\bar{S}(\Lambda(t))\int_0^{t} e^{sx}\bar{S}\left(\frac{\Lambda(t)}{t}x\right)dx - \left(1 - e^{st}\bar{S}(\Lambda(t))\right)\int_t^{\infty} e^{sx}\bar{S}\left(\frac{\Lambda(t)}{t}x\right)dx
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{t}{\Lambda(t)}\left[e^{st}\bar{S}\left(\Lambda(t)\right)\int_0^{\Lambda(t)} e^{\frac{st}{\Lambda(t)}y}\bar{S}(y)dy\right] \\
&\quad -\frac{t}{\Lambda(t)}\left[\left(1-e^{st}\bar{S}\left(\Lambda(t)\right)\right)\int_{\Lambda(t)}^{\infty} e^{\frac{st}{\Lambda(t)}y}\bar{S}(y)dy\right] \\
&= \frac{t}{\Lambda(t)}\left[e^{st}\bar{S}\left(\Lambda(t)\right)\int_0^{\infty} e^{\frac{st}{\Lambda(t)}y}\bar{S}(y)dy - \int_{\Lambda(t)}^{\infty} e^{\frac{st}{\Lambda(t)}y}\bar{S}(y)dy\right] \\
&= \frac{t}{\Lambda(t)}\left[e^{s^*\Lambda(t)}\bar{S}\left(\Lambda(t)\right)\int_0^{\infty} e^{s^*y}\bar{S}(y)dy - \int_{\Lambda(t)}^{\infty} e^{s^*y}\bar{S}(y)dy\right] \\
&\geq 0.
\end{aligned}
$$

So, (2.8) holds.

2.2 General Shock Model

In this subsection, we deal with a device subjected to a sequence of shocks occurring randomly in time according to a counting process $N(t), t \geq 0$. Let $\bar{P}_k$ be the probability that the device survives the first k shocks, where $1 = \bar{P}_0 \geq \bar{P}_1 \geq \cdots$. Denote $p_j = \bar{P}_{j-1} - \bar{P}_j, j \geq 1$. Let U_k be the inter-arrival time between the $(k-1)$-th and k-th shocks, then the survival function of the device is given by

$$\bar{H}(t) = \sum_{k=0}^{\infty} \bar{P}_k P\left(N(t) = k\right). \tag{2.9}$$

For ease of reference, the following lemma is listed here. The proof can be completed in a similar manner to that of Lemma 5.4 in *Pellery* (1994).

Lemma 2.4. Let U_k be the inter-arrival time between the $(k-1)$-th and the k-th shocks, if variable U_k is NBU_{Mg}, for $k = 1, 2\cdots$, and stochastically decreasing in k, then for all $\lambda, t \geq 0$, and $j, k = 0, 1, 2\cdots$, we have

$$
\begin{aligned}
&\int_0^{\infty} e^{\lambda u} P\left(N(t+u) < j+k, N(u) = k\right) du \\
\leq\ & P\left(N(t) < j\right)\int_0^{\infty} e^{\lambda u} P\left(N(u) = k\right) du.
\end{aligned}\tag{2.10}
$$

In the next theorem, we show that NBU_{Mg} is preserved under the general shock model.

Theorem 2.5. Let the inter-arrival time U_k be NBU_{Mg}, and stochastically decreasing in k, and p_k be a discrete NBU, then $H(t)$ given by (2.9) is NBU_{Mg}.

Proof. It suffices to show that for all $\lambda, t \geq 0$,

$$\bar{H}(t)\int_0^{\infty} e^{\lambda u}\bar{H}(u)du \geq \int_0^{\infty} e^{\lambda u}\bar{H}(t+u)du, \tag{2.11}$$

since $p_j = \bar{P}_{j-1} - \bar{P}_j$, rewrite $\bar{H}(t)$ as follows

$$\bar{H}(t) = \sum_{i=1}^{\infty} p_j P\left(N(t) < j\right), \tag{2.12}$$

Notice that

$$\bar{H}(t+u) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \bar{P}_{j+k} P\left(N(t+u) = j+k, N(u) = k\right), \tag{2.13}$$

(2.11) is equivalent to

$$\begin{aligned} & \sum_{j=1}^{\infty} p_j P\left(N(t) < j\right) \sum_{k=0}^{\infty} \bar{P}_k \int_0^{\infty} e^{\lambda u} P\left(N(t) = k\right) du \\ \geq & \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \bar{P}_{j+k} \int_0^{\infty} e^{\lambda u} P\left(N(t+u) = j+k, N(u) = k\right) du, \end{aligned} \tag{2.14}$$

for $p_j = \bar{P}_{j-1} - \bar{P}_j$. Hence, we have

$$\begin{aligned} & \sum_{j=1}^{\infty} p_j P\left(N(t) < j\right) \sum_{k=0}^{\infty} \bar{P}_k \int_0^{\infty} e^{\lambda u} P\left(N(u) = k\right) du \\ = & \sum_{k=0}^{\infty} \bar{P}_k \sum_{j=1}^{\infty} (\bar{P}_{j-1} - \bar{P}_j) P\left(N(t) < j\right) \int_0^{\infty} e^{\lambda u} P\left(N(u) = k\right) du \\ \geq & \sum_{k=0}^{\infty} \bar{P}_k \sum_{j=1}^{\infty} (\bar{P}_{j-1} - \bar{P}_j) \int_0^{\infty} e^{\lambda u} P\left(N(u+t) < j+k, N(u) = k\right) du \\ = & \sum_{k=0}^{\infty} \bar{P}_k \sum_{j=1}^{\infty} (\bar{P}_{j-1} - \bar{P}_j) \sum_{i=0}^{j-1} \int_0^{\infty} e^{\lambda u} P\left(N(t+u) = i+k, N(u) = k\right) du \\ = & \sum_{k=0}^{\infty} \bar{P}_k \sum_{i=0}^{\infty} \bar{P}_i \int_0^{\infty} e^{\lambda u} P\left(N(t+u) = i+k, N(u) = k\right) du \\ \geq & \sum_{k=0}^{\infty}\sum_{i=0}^{\infty} \bar{P}_{i+k} \int_0^{\infty} e^{\lambda u} P\left(N(t+u) = i+k, N(u) = k\right) du. \end{aligned}$$

Now the NBU_{Mg} property of $H(t)$ follows.

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