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An Economic Design of a *k*-out-of-*n* System

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Abstract. A k-out-of-n system with n identical and independent components is considered in which the components takes two types of function: 0 (open-circuit) or 1 (closed) on command (e.g. electromagnetic relays and solid state switches). Components are subject to two types of failure on command: failure to close or failure to open. In our k-out-of-n system, failure of $(n-k)+1$ or more components to close causes to the close failure of the system, or failure of k or more components to open causes the open failure of the system. The long-run average cost rate is obtained. We find the optimal k minimizing the long run average cost rate for given n. A numerical example is presented.

Key Words : *two kinds of failure, economic design, long-run average cost rate*

1. INTRODUCTION

Many types of systems consist of components that can fail in two mutually exclusive ways, and where the system likewise can fail in either if two mutually exclusive ways. For example, a resistor may open-circuit, close-circuit, or function properly. A diode may fail in two ways: it may open-circuit (its resistance is infinite in both directions) or it may close-circuit (resistance is zero in both directions).

Consider now a model in which a k-out-of-n system is composed of n identical and statistically independent components that can assume either of two states, 0 (open) 1 (close) on command. The components are subject to two kinds of failure on command: failure to close and failure to open with their respective conditional probabilities of a component given that a failure has occurred $p>=0$ and $q=1-p$. Failure of $(n-k)+1$ or more components to close causes the system to experience failure to close (since at least k

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closed components we needed for the system to be closed). Failure of k or more components to open causes the system to experience failure to open (since at least $(n-k)+1$ open components are needed for the system to be open).

Barlow, Hunter and Proschan (1965), Rau (1970), and Ben-Dov (1980) considered series-parallel structures or k-out-of-n structures with two types of failures. They obtained system reliabilities and optimal n or k maximizing system reliabilities. Pirie and Bendell (1984) proposed cost model in which component failures and switching are considered. They determined optimal structure and switching policy for minimizing total cost of series-parallel system with 3 identical units. Phillips (1980) proved that k-out-of-n structure is optimal coherent system maximizing system reliability.

Ansell and Bendell (1982) proved that k-out-of-n structure is also optimal for dependent components. Pham and Pham (1991), Sah and Stiglitz (1988) proposed the mean system profit model and considered the optimization problem of obtaining the optimal decision for n and k.

For given number of components in the system, n, the problem is to find an optimal k, such that the corresponding k-out-of-n system minimizes a mean cost rate. The effect of the system parameters on the optimal k, then, is studied both analytically and numerically.

Notation

n : number of components in the system
k : minimal number of closed components for system closure
p : conditional probability of a failure to open of a component given that a failure has occurred
q : conditional probability of a failure to close of a component given that a failure has occurred, $q=1-p$
λ : failure rate of a component
C_1 : fixed cost for system replacement
C_2 : unit replacement cost of failure to open
C_3 : unit replacement cost of failure to close
θ : C_3/C_2
$F_c(t)$: probability distribution function of failure time to close
$F_o(t)$: probability distribution function of failure time to open
$R(t)$: system reliability
$S_d(k)$: expected length to system failure
$S_c(k)$: expected cost to system failure
C_k : average long run cost rate

Assumptions

1. System consists of n independent and identical components.
2. Each component has three states, operating, close and open failures.
3. If $(n-k)+1$ or more components fails to close, then it fails to close system failure and if k or more components fail to open, it fails to open system failure.
4. Failure of each component follows exponential distribution. At component failure, the probability of open failure is p and the probability of close failure is $q=1-p$
5. Repair of failed components cannot be done during system operation.

2. MODEL

Under assumptions, marginal distribution functions of open and close failures are given by:

$$F_o(t) = p(1-\exp(-\lambda t))$$
$$F_c(t) = q(1-\exp(-\lambda t))$$

Then system reliability is as follows (Ben-Dov (1980)):

$$\begin{aligned} R(t) = & \sum_{i=n-k+1}^{n} \binom{n}{i} (1-p(1-\exp(-\lambda t)))^i (p(1-\exp(-\lambda t)))^{n-i} \\ & - \sum_{i=n-k+1}^{n} \binom{n}{i} (q(1-\exp(-\lambda t)))^i (1-q(1-\exp(-\lambda t)))^{n-i} \\ = & \sum_{i=n-k+1}^{n} \binom{n}{i} \left[\sum_{j=0}^{i} \sum_{l=1}^{n+j-i} (-1)^{j+l} \binom{i}{j} \binom{n+j-i}{l} p^{n+j-i} \exp(-\lambda l t) \right. \\ & \left. - \sum_{j=0}^{n-i} \sum_{l=1}^{i+j} (-1)^{j+l} \binom{n-i}{j} \binom{i+j}{l} q^{i+j} \exp(-\lambda l t) \right] \end{aligned} \tag{1}$$

At a system failure, we replace the system immediately and the expected length of a system failure is given by:

$$S_d(k) = \int_0^\infty R(t)dt$$

$$= 1/\lambda \sum_{i=n-k+1}^{n} \binom{n}{i} \left[\sum_{j=0}^{i} \sum_{l=1}^{n+j-i} (-1)^{j+l} \binom{i}{j} \binom{n+j-i}{l} p^{n+j-i}/l - \sum_{j=0}^{n-i} \sum_{l=1}^{i+j} (-1)^{j+l} \binom{n-i}{j} \binom{i+j}{l} q^{i+j}/l \right]$$

$$= 1/\lambda \left[\sum_{i=0}^{n-k} \binom{k+i-1}{i} p^k q^i \left(\sum_{j=n-k+1-i}^{n} 1/j \right) + \sum_{i=0}^{k-1} \binom{n-k+i}{i} p^i q^{n-k+1} \left(\sum_{j=k-i}^{n} 1/j \right) \right] \tag{2}$$

At a system failure, we replace the failed components and the replacement cost of component is based on the failure type. The numbers of open and close failures is important to calculate the total cost. The expected cost to system failure is given by:

$$S_c(k) = C_1 + \sum_{i=0}^{n-k} \binom{k+i-1}{i} p^k q^i (kC_2 + iC_3) + \sum_{i=0}^{k-1} \binom{n-k+i}{i} p^i q^{n-k+1} (iC_2 + (n-k+1)C_3)$$

$$= C_1 + C_2\left[\sum_{i=0}^{n-k}\binom{k+i-1}{i}p^k q^i (k + i\cdot\theta) + \sum_{i=0}^{k-1}\binom{n-k+i}{i}p^i q^{n-k+1}(i + (n-k+1)\theta)\right] \tag{3}$$

Thus, a mean cost rate is obtained as follows:

$$C_k = S_c(k)/S_d(k)$$

$$= \lambda\left[C_1 + C_1\left(\sum_{i=0}^{n-k}\binom{k+i-1}{i}p^k q^i (k+i\cdot\theta) + \sum_{i=0}^{k-1}\binom{n-k+i}{i}p^i q^{n-k+1}(i+(n-k+1)\theta)\right)\right]$$
$$\Bigg/\left[\sum_{i=0}^{n-k}\binom{k+i-1}{i}p^i q^i\left(\sum_{j=n-k+1-i}^{n}1/j\right) + \sum_{i=0}^{k-1}\binom{n-k+i}{i}p^i q^{n-k+1}\left(\sum_{j=k-i}^{n}1/j\right)\right] \tag{4}$$

We consider a problem for obtaining the optimal k to minimize the mean cost rate. It is difficult to obtain the optimal k minimizing the mean cost rate analytically. But we cases where the mean cost rate is unimodal function.

As a typical characteristic, the optimal value of k is independent of replacement costs θ because the expected cost of a renewal is a linear function of θ. The first differential equation of the expected cost of a renewal by θ is given by:

$$\frac{dSc(k)}{d\theta} = \lambda C_2\left[\sum_{i=0}^{n-k}\binom{k+i-1}{i}p^k q^i i + \sum_{i=0}^{k-1}\binom{n-k+i}{i}p^i q^{n-k+1}(n-k+1)\right] \tag{5}$$

From Figure 1, we see that the expected cost depends on n and p.

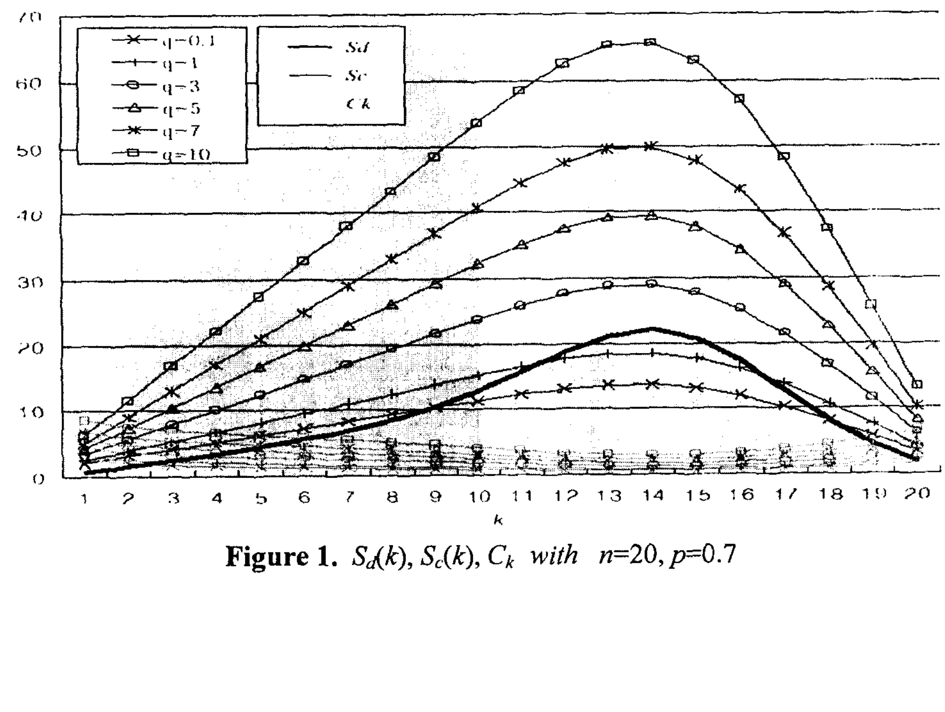

Figure 1. $S_d(k)$, $S_c(k)$, C_k *with* n=20, p=0.7

3. NUMERICAL EXAMPLE

Let us consider a case where n=20, p=0.7. The expected cost and the expected length for a renewal, and the mean cost rate are given in Table 1. Then the optimal k is 14.

We consider some cases in which n=10, 20, 45, 100, 200 and p=0.01, 0.1,0.3,0.5,0.7,0.9,0.99 and the results are given in Table 2.

We find some trends between the optimal k and model parameters. If p increases, optimal k also increases. For large n, the optimal k increases.

Table 1. $S_d(k)$, $S_c(k)$, C_k with n=20, p=0.7.

k	$S_d(k)$	$S_c(k)$	C_k	k	$S_d(k)$	$S_c(k)$	C_k
1	0.731	5.386	7.366	11	15.745	48.6835	3.092
2	1.522	9.771	6.420	12	18.716	52.0814	2.783
3	2.383	14.157	5.940	13	21.137	54.3086	2.569
4	3.330	18.543	5.569	14	22.054	54.6447	2.478
5	4.381	22.929	5.233	15	20.740	52.4594	2.530
6	5.569	27.314	4.906	16	17.291	47.5286	2.749
7	6.935	31.698	4.571	17	12.693	40.1954	3.167
8	8.551	36.076	4.219	18	8.208	31.2136	3.803
9	10.514	40.426	3.845	19	4.612	21.3768	4.635
10	12.915	44.687	3.460	20	1.974	11.2252	5.685

Table 2. $S_d(k)$, C_k and optimal k.

	P													
n	0.01		0.1		0.3		0.5		0.7		0.9		0.99	
	S_d	C_k	S_d	C_k	S_d	C_k	S_d	C_k	S_d	C_k	S_d	C_k	S_d	C_k
10	1	1	2	2	4	4	5,6	5,6	7	7	9	9	10	10
20	1	1	3	3	7	7	10,11	10,11	14	14	18	18	20	20
45	2	1	6	6	15	15	23	23	31	31	40	40	44	45
100	3	2	13	12	32	32	50,51	50,51	69	69	88	89	98	99
200	4	4	24	23	63	62	100,101	100,101	138	139	177	178	197	197

5. CONCLUSION

In this paper, we considered an optimization problem to obtain k to minimize the mean cost rate for the system with two types of failures. Numerical examples are studied for various values of model parameters. In this paper, just k is a decision variable but as a further study we can consider n another decision variable. Topics for further research consider preventive maintenance and dependency between component failures.

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