KJHM Vol. 16, No. 1, 2003, 79-85

A note on the second Gaussian curvature of the helicoidal surfaces*

Division of Mathematics and Informational Statistics, Wonkwang University

Bong-Sik Cho

You-Mi Seo

Abstract

We define the second Gausssian curvature K_{II} by using Brioschi's formula and shall disscuss the helicoidal surfaces satisfying $K_{II} = H$.

0. History and Introduction

A regular surface $M \subset \boldsymbol{R}^3$ with Gaussian curvature $K > 0$ has a positive definite symmetric bilinear second fundamental form II. Viewing the second fundamental form as a new metric, we can obtain its Gaussian curvature K_{II}. It is well-known that the concept of the second Gaussian curvature K_{II} plays an important role in a regular surface.

A surface is called a Weingarten surface or W-space if there is a nontrivial relation between the mean curvature H and the second Gaussian curvature K_{II}. We first note that a minimal surface satisfies $K_{II} = 0$. In 1972, R. Schneider has shown that a surface with constant K_{II} is a sphere. In 1977, T. Koufogiorgos and T. Hasanis has given a proof that the sphere is the only closed ovaloid satisfying $K_{II} = H$. In 1981, W. Kuhnel

* This paper was supported by Wonkwang University in 2001.

is studied for surfaces of revolusion with $K_{II} = H$. In 1992, D. E. Blair and T. Koufogiorgos has shown that a ruled surface with $K_{II} = H$ yields to $K_{II} = 0$ and the sphere must be a piece of a helicoid.

A natural generalization of rotation surfaces are the helicoidal surface. Many authors have been concerned with the problem of characterization of the helicoidal surface by the curvature K_{II} of the second fundamental form.

The aim of this paper is to prove that the helicoidal surfaces satisfying $K_{II} = H$ are locally characterized by constancy of the ratio of the principal curvatures.

1. Second Gausssian curvature

In 1852, Brioschi expresses the Gaussian curvature of a patch $\boldsymbol{x} : U \to \boldsymbol{R}^3$ entirely in terms of the first fundamental form and obtains Brioschi's Formula which is

$$K = \frac{1}{(EG-F^2)^2}\left(\begin{vmatrix} -\frac{1}{2}E_{vv} + F_{uv} - \frac{1}{2}G_{uu} & \frac{1}{2}E_u & F_u - \frac{1}{2}E_v \\ F_v - \frac{1}{2}G_u & E & F \\ \frac{1}{2}G_v & F & G \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G \end{vmatrix} \right).$$

The Gaussian curvature K of M in $\boldsymbol{R}^3$ equipped with a metric I is given by Brioschi's formula. Since $E,\ G > 0$, $\begin{vmatrix} E & F \\ F & G \end{vmatrix} > 0$. A regular surface M in $\boldsymbol{R}^3$ with positive Gaussian curvature K has a positive definite second fundamental form II. Thus II is bilinear and symetric and positive definite.

Viewing the second fundamental form II as a new metric we can consider another Gaussian curvature , we derive the second Gaussian curvature K_{II} by applying $e,\ f,\ g$ instead of $E,\ F,\ G$ in Brioshi's formula. The second Gaussian curvature is define by

$$K_{II}=\frac{1}{(eg-f^{2})^{2}}\left\{\begin{vmatrix} -\frac{1}{2}e_{vv}+f_{uv}-\frac{1}{2}g_{uu} & \frac{1}{2}e_u & f_u-\frac{1}{2}e_v \\ f_v-\frac{1}{2}g_u & e & f \\ \frac{1}{2}g_v & f & g \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}e_v & \frac{1}{2}g_u \\ \frac{1}{2}e_v & e & f \\ \frac{1}{2}g_u & f & g \end{vmatrix}\right\}.$$

2. Helicoidal surfaces

A natural generalization of rotation surfaces are the helicoidal surfaces that can be defined as follows. Consider the one-parameter subgroup $g_t : \boldsymbol{R}^3 \rightarrow \boldsymbol{R}^3$ of the group of rigid motions of $\boldsymbol{R}^3$ given by

$$g_t(x,y,z) = (x\cos t + y\sin t,\ -x\sin t + y\cos t,\ z+ht),\quad t \in \boldsymbol{R}^3.$$

The motion g_t is called a helicoidal motion with axis O_z and pitch h. A helicoidal surface M with axis O_z and pitch h is a surface that is invariant by g_t for all t. A surface M has the form

$$\boldsymbol{x}(\phi,\ \rho) = (\ \rho\cos\phi,\ \rho\sin\phi,\ \lambda(\phi)+h\phi\ ),$$

where ρ and ϕ are polar coordinates in the xy-plane and the xy-plane has been rotated so that O_x is the origin of ϕ. If $h=0$, the helicoidal surface is a surface of revolution. If $\lambda' = \frac{d\lambda}{d\rho} = 0$, M is a helicoidal.

Lemma 1. (Bour) For a helicoidal surface M there exists a two-parameter family of helicoidal surfaces isometric to M.

Proof. The first fundamental form of (4-1) can be written

$$\begin{aligned} ds^2 &= Ed\rho^2 + 2Fd\rho d\phi + Gd\phi^2 \\ &= (1+\lambda')d\rho^2 + 2h\lambda' d\rho d\phi + (\rho^2+h^2)d\phi^2 \end{aligned}$$

where $\lambda' = \frac{d\lambda}{d\rho}$.

We introduce new parameter (u, v) by functions $u = u(\rho, \phi)$ and $v = v(\rho, \phi)$ that satisfy

$$\begin{aligned} \frac{\partial u}{\partial \rho} &= (\rho^2+h^2)^{-\frac{1}{2}}[1+\rho^2 \lambda'^2(\rho^2+h^2)^{-1}]^{\frac{1}{2}}, & \frac{\partial u}{\partial \phi} &= 0, \\ \frac{\partial v}{\partial \rho} &= (\rho^2+h^2)^{-1}h\lambda', & \frac{\partial v}{\partial \phi} &= 1. \end{aligned}$$

Then $\frac{\partial(u,v)}{\partial(\rho,\phi)} = \frac{\partial u}{\partial \rho} > 0$. We can write in the natural parametrization

$$ds^2 = (\rho^2+h^2)(du^2+dv^2).$$

We are now reduced to showing that given a function $x = x(u)$, we can find function ρ, λ and ϕ of u and v satisfying

$$\begin{aligned} x^2du^2 &= d\rho^2 + \rho^2(\rho^2+h^2)^{-1}d\lambda^2 \\ xdv &= \pm(\rho^2+h^2)^{\frac{1}{2}}(d\phi + h(\rho^2+h^2)^{-1}d\lambda) \end{aligned}$$

for an arbitrary constant h. From the above, ρ and λ do not depend on v. Since

$$x = \pm(\rho^2+h^2)^{\frac{1}{2}}\left\{\frac{\partial \phi}{\partial v} + \frac{h}{\rho^2+h^2}\frac{d\lambda}{dv}\right\}$$

and

$$x\frac{dv}{du} = \pm(\rho^2+h^2)^{\frac{1}{2}}\left\{\frac{\partial \phi}{\partial u} + \frac{h}{\rho^2+h^2}\frac{d\lambda}{du}\right\},$$

$$\frac{\partial \phi}{\partial u} = -\frac{h}{\rho^2+h^2}\dot{\lambda}, \quad \frac{\partial \phi}{\partial v} = \pm\frac{x}{(\rho^2+h^2)^{\frac{1}{2}}}$$

where dot denote the derivative in u.

Hence $\frac{\partial}{\partial u}\left(\frac{\partial \phi}{\partial v}\right) = 0$ and so $\frac{x}{(\rho^2+h^2)^{\frac{1}{2}}}$ does not depend on u. Therefore we can set

$$\frac{x}{(\rho^2+h^2)^{\frac{1}{2}}}=\frac{1}{m}\neq 0,\qquad m=\text{constant}.$$

If follows that

$$\dot{\rho}^2=m^4x^2\,\dot{x}^2(m^2x^2-h^2)^{-1}.$$

Since

$$\begin{aligned} x^2 &= \dot{\rho}^2+\rho^2(\rho^2+h^2)^{-1}\dot{\lambda}^2 \\ &= m^4x^2\,\dot{x}^2(m^2x^2-h^2)^{-1}+(m^2x^2-h^2)m^{-2}x^{-2}\,\dot{\lambda}^2, \end{aligned}$$

$$\dot{\lambda}^2=(m^2x^2-h^2)^{-2}m^2x^4y^2 \quad \text{where}\quad y=(m^2x^2-m^4\,\dot{x}^2-h^2)^{\frac{1}{2}}.$$

Since

$$\begin{aligned} x\,dv &= \pm mx(d\phi+h(m^2x^2)^{-1}d\lambda) \\ d\phi &= \pm m^{-1}dv-hm^{-2}x^{-2}d\lambda \\ &= \pm m^{-1}dv-\frac{y}{m(m^2x^2-h^2)}\,du \end{aligned}$$

Therefore the helicoidal surfaces

$$\mathbf{x}(u,v)=(\rho(u)\cos\phi(u,v),\ \rho(u)\sin\phi(u,v),\ \lambda(u)+h\phi(u,v)).$$

are all isometric with first fundamental form given by $s^2=x^2(du^2+dv^2)$, where

$$\begin{aligned} \rho &= \int\frac{m^2x}{\sqrt{m^2x^2-h^2}}\,dx=(m^2x^2-h^2)^{\frac{1}{2}}, \\ \phi &= \pm m^{-1}\left(\int dv-h\int\frac{y}{m^2x^2-h^2}\,du\right), \\ \lambda &= \pm m\int\frac{x^2y}{m^2x^2-h^2}\,du. \end{aligned}$$

When $m=1$ and $h=0$, Bour's lemma asserts the existence of a two-parameter family of helicoidal surfaces isometric to a given rotation surface.

Since the Gauss map of M is given by

$$U=\frac{1}{x^2}(\,\dot{\rho}h\sin\phi-\rho\dot{\lambda}\cos\phi,\ -\dot{\rho}h\cos\phi-\rho\dot{\lambda}\sin\phi,\ \rho\dot{\rho}),$$

we get

$$K=\frac{\dot{x}^2-x\ddot{x}}{x^4},\quad H=\frac{\dot{y}}{2x\dot{x}}$$

and

$$K_{II}=\frac{1}{(eg-f^2)^2}\left\{-\frac{1}{2}\ddot{g}(eg-f^2)+\begin{vmatrix}0 & -\frac{1}{2}\dot{g} & 0\\ \frac{1}{2}\dot{g} & e & f\\ 0 & f & g\end{vmatrix}-\begin{vmatrix}0 & 0 & \frac{1}{2}\dot{g}\\ 0 & e & f\\ \frac{1}{2}\dot{g} & f & g\end{vmatrix}\right\}$$

$$=-\frac{\ddot{y}}{4(\dot{x}^2-x\ddot{x})}-\frac{1}{4}\left\{\frac{\ddot{y}}{(\dot{x}^2-x\ddot{x})}-\frac{\dot{y}(\dot{x}^2-x\ddot{x})'}{(\dot{x}^2-x\ddot{x})^2}\right\}$$

$$=-\frac{\ddot{y}}{4(\dot{x}^2-x\ddot{x})}-\frac{1}{4}\left(\frac{\dot{y}}{(\dot{x}^2-x\ddot{x})}\right)'.$$

Let $c=\frac{k_1}{k_2}=\frac{H+\sqrt{H^2-K}}{H-\sqrt{H^2-K}}$ and $w=\frac{4H^2}{K}=\frac{(k_1+k_2)^2}{k_1k_2}=\frac{(1+c)^2}{c}$. (*)

Then we have $\dot{x}^2(\dot{x}^2-x\ddot{x})w=x^2\dot{y}^2$.

Theorem 2. Let M be a helicoidal surface with $K\neq 0$. Then $K_{II}=H$ if and only if $\frac{k_1}{k_2}$ is constant.

Proof. If the surface M is minimal, then $H=0$. Thus $\dot{y}=0$. Hence we have $K_{II}=0$. If M is not minimal,

$$4(K_{II}-H)H=\left\{-\frac{\ddot{y}}{(\dot{x}^2-x\ddot{x})}-\left(\frac{\dot{y}}{(\dot{x}^2-x\ddot{x})}\right)'-\frac{2\dot{y}}{x\dot{x}}\right\}\frac{\dot{y}}{2x\dot{x}}$$

$$=\left\{-\frac{\ddot{y}}{(\dot{x}^2-x\ddot{x})}-\left(\frac{\dot{x}^2}{x^2\dot{y}}\right)'w-\frac{2\dot{y}}{x\dot{x}}-\frac{\dot{x}^2}{x^2\dot{y}}\dot{w}\right\}\frac{\dot{y}}{2x\dot{x}}$$

$$=-\frac{\dot{x}\dot{w}}{2x^3}=(x^{-2})\cdot\left(\frac{H^2}{K}\right)^{\cdot}.$$

Using the above equation and (*) we have the proof.

References

1. Baikoussis, C. and Koufogiorgos, T. "On the inner curvature of the second fundamental form of helicoidal surfaces," *Arch. Math.* 68(1997), 169-176.
2. Baikoussis, C. and Koufogiorgos, T. "Helicoidal surfaces with prescribed mean Gaussian curvature," *Geom. J.* 63(1998), 25-29.
3. Blair, D. E. and Koufogiorgos, T. "Ruled surfaces with vanishing second Gaussian curvature," *Monatoh. Math.* 113(1992), 177-181.
4. Do Carmo, M. and Dajczer, M. "Helicoidal surfaces with constant mean curvature," *Tohoku Math. J.* 34(1982), 425-435.
5. Gray, A. *Modern Differential Deometry of Curves and Surfaces*, CRC Press, Inc., 1993.
6. Hasanis, T. and Koufogiorgos, T. "A characteristic property of the sphere," *Proc. Amer. Math. Soc.* 67(1977), 303-305.