

## ON THE GROUP $SL(2, \mathbb{R}) \ltimes \mathbb{R}^{(m,2)}$

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ABSTRACT. In this paper, we find the unitary dual of the group  $SL(2, \mathbb{R}) \ltimes \mathbb{R}^{(m,2)}$  and study automorphic forms related to the group  $SL(2, \mathbb{R}) \ltimes \mathbb{R}^{(m,2)}$ .

### 1. Introduction

For any positive integer  $m \in \mathbb{Z}^+$ , we let

$$SL_{2,m}(\mathbb{R}) := SL(2, \mathbb{R}) \ltimes \mathbb{R}^{(m,2)}$$

be the semidirect product of the special linear group  $SL(2, \mathbb{R})$  and the commutative additive group  $\mathbb{R}^{(m,2)}$  equipped with the following multiplication law

$$(1.1) \quad (g, a) * (h, b) = (gh, a {}^t h^{-1} + b),$$

where  $g, h \in SL(2, \mathbb{R})$  and  $a, b \in \mathbb{R}^{(m,2)}$ . Here  $\mathbb{R}^{(m,2)}$  denotes the set of all  $m \times 2$  real matrices.

In this paper, we investigate some properties of the group  $SL_{2,m}(\mathbb{R})$  and find its unitary dual explicitly. The reason why we study the group  $SL_{2,m}(\mathbb{R})$  is that this group plays an important role in the study of certain automorphic forms. This paper is organized as follows. In Section 2, we present the basic ingredients of the group  $SL_{2,m}(\mathbb{R})$ . In Section 3, we describe the roots of  $SL_{2,m}(\mathbb{R})$  explicitly. In Section 4, we compute the Lie derivatives explicitly for functions on  $SL_{2,1}(\mathbb{R})$ . Similarly we can compute the Lie derivatives of  $SL_{2,m}(\mathbb{R})$  ( $m \geq 2$ ) explicitly. We

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leave the detail to the reader. In Section 5, we discuss the actions of  $SL_{2,m}(\mathbb{R})$  on  $\mathbb{H} \times \mathbb{C}^m$  and  $SP_2 \times \mathbb{R}^{(m,2)}$ , where  $\mathbb{H}$  denotes the Poincaré upper-half plane and  $SP_2$  is the symmetric space consisting of all the  $2 \times 2$  positive symmetric real matrices  $Y$  with  $\det Y = 1$ . In Section 6, we make some remarks on the notion of Maass-Jacobi forms associated to the group  $SL_{2,1}(\mathbb{R})$ . In Section 7, we compute the unitary dual of  $SL_{2,m}(\mathbb{R})$  using the Mackey's method.

NOTATIONS. We denote by  $\mathbb{Z}$ ,  $\mathbb{R}$  and  $\mathbb{C}$  the ring of integers, the field of real numbers and the field of complex numbers respectively.  $\mathbb{Z}^+$  denotes the set of all positive integers. The symbol “:=” means that the expression on the right is the definition of that on the left.  $F^{(k,l)}$  denotes the set of all  $k \times l$  matrices with entries in a commutative ring  $F$ . For a square matrix  $A$ ,  $\sigma(A)$  denotes the trace of  $A$ . For any  $M \in F^{(k,l)}$ ,  ${}^tM$  denotes the transpose of  $M$ . For  $A \in F^{(k,l)}$  and  $B \in F^{(k,k)}$ , we set  $B[A] = {}^tABA$ .  $\mathbb{H}$  denotes the Poincaré upper-half plane. For a field  $F$ , we denote by  $F^\times$  the multiplicative group consisting of nonzero elements.  $I_n$  denotes the identity matrix of degree  $n$ .

## 2. Basic ingredients of $SL_{2,m}(\mathbb{R})$

From now on, we write  $G = SL_{2,m}(\mathbb{R})$  for brevity. We let

$$\mathfrak{sl}(2, \mathbb{R}) = \left\{ X \in \mathbb{R}^{(2,2)} \mid \sigma(X) = 0 \right\}$$

be the Lie algebra of  $SL(2, \mathbb{R})$ . Then it is easy to see that the Lie algebra  $\mathfrak{g}$  of  $G$  is given by

$$(2.1) \quad \mathfrak{g} = \left\{ (X, Z) \mid X \in \mathfrak{sl}(2, \mathbb{R}), Z \in \mathbb{R}^{(m,2)} \right\}$$

equipped with the following Lie bracket

$$(2.2) \quad [(X_1, Z_1), (X_2, Z_2)] = ([X_1, X_2]_0, Z_2 {}^tX_1 - Z_1 {}^tX_2),$$

where  $[X_1, X_2]_0 := X_1X_2 - X_2X_1$  denotes the usual matrix bracket and  $(X_1, Z_1), (X_2, Z_2) \in \mathfrak{g}$ . The adjoint representation  $\text{Ad}$  of  $G$  is given by

$$(2.3) \quad \text{Ad}((g, a))(X, Z) = (gXg^{-1}, (Z - a {}^tX) {}^tg),$$

where  $(g, a) \in SL_{2,m}(\mathbb{R})$  and  $(X, Z) \in \mathfrak{g}$ . And the adjoint representation  $\text{ad}$  of  $\mathfrak{g}$  on  $\text{End}(\mathfrak{g})$  is given by

$$(2.4) \quad \text{ad}((X, Z))((X_1, Z_1)) = [(X, Z), (X_1, Z_1)].$$

We easily see that the Killing form  $B$  of  $\mathfrak{g}$  is given by

$$(2.5) \quad B((X_1, Z_1), (X_2, Z_2)) = (m + 4) \sigma(X_1 X_2).$$

Therefore the Killing form  $B$  is highly degenerate.

Let

$$K = \{ (k, 0) \in G \mid k \in SO(2) \}$$

be the compact subgroup of  $G$ . Then the Lie algebra  $\mathfrak{k}$  of  $K$  is

$$\mathfrak{k} = \left\{ (X, 0) \in \mathfrak{g} \mid X = \left( \begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix}, 0 \right), x \in \mathbb{R} \right\}.$$

We let  $\mathfrak{p}$  be the subspace of  $\mathfrak{g}$  defined by

$$\mathfrak{p} = \left\{ (X, Z) \in \mathfrak{g} \mid X = {}^t X \in \mathbb{R}^{(2,2)}, \sigma(X) = 0, Z \in \mathbb{R}^{(m,2)} \right\}.$$

Then we have the following relation

$$(2.6) \quad [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k} \quad \text{and} \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}.$$

In addition, we have

$$(2.7) \quad \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \quad (\text{direct sum}).$$

We note that the restriction of the Killing form  $B$  to  $\mathfrak{k}$  is negative definite and the restriction of  $B$  to the abelian subalgebra  $\mathfrak{r} = \{ (0, Z) \in \mathfrak{g} \}$  is identically zero. Since  $\mathfrak{r}$  is the radical of  $B$ ,  $B$  is degenerate (see (2.5)).

An Iwasawa decomposition of the group  $SL_{2,m}(\mathbb{R})$  is given by

$$(2.7) \quad G = NAK,$$

where

$$N = \left\{ \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, a \right) \in G \mid x \in \mathbb{R}, a \in \mathbb{R}^{(m,2)} \right\}$$

and

$$A = \left\{ \left( \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, 0 \right) \in G \mid a > 0 \right\}.$$

An Iwasawa decomposition of the Lie algebra  $\mathfrak{g}$  of  $G$  is given by

$$(2.8) \quad \mathfrak{g} = \mathfrak{n} + \mathfrak{a} + \mathfrak{k},$$

where

$$\mathfrak{n} = \left\{ \left( \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}, Z \right) \in \mathfrak{g} \mid x \in \mathbb{R}, Z \in \mathbb{R}^{(m,2)} \right\}$$

and

$$\mathfrak{a} = \left\{ \left( \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix}, 0 \right) \in \mathfrak{g} \mid x \in \mathbb{R} \right\}.$$

In fact,  $\mathfrak{a}$  is the Lie algebra of  $A$  and  $\mathfrak{n}$  is the Lie algebra of  $N$ .

### 3. The roots of the Lie algebra of $G$

In this section, we will determine the roots of  $\mathfrak{g}$  explicitly.

We let  $F_{kl}$  ( $1 \leq k \leq m, l = 1, 2$ ) be the  $m \times 2$  matrix with 1 in the  $(k, l)$ -entry and zero elsewhere. We put

$$E_0 = \left( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, (0, 0) \right), \quad E_1 = \left( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, (0, 0) \right),$$

$$E_2 = \left( \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, (0, 0) \right)$$

and

$$F_{kl}^0 = (0, F_{kl}), \quad 1 \leq k \leq m, \quad l = 1, 2.$$

Then  $E_0, E_1, E_2, F_{kl}^0$  ( $1 \leq k \leq m, l = 1, 2$ ) form a basis of  $\mathfrak{g}$ . Clearly

$$\mathfrak{a} = \mathbb{R} E_0 \quad \text{and} \quad \mathfrak{n} = \mathbb{R} E_1 + \sum_{k=1}^m (\mathbb{R} F_{k1}^0 + \mathbb{R} F_{k2}^0).$$

By an easy computation, we get

$$[E_0, E_1] = 2E_1, \quad [E_0, E_2] = -2E_2,$$

$$[E_0, F_{k1}^0] = F_{k1}^0, \quad [E_0, F_{k2}^0] = -F_{k2}^0.$$

Thus the roots of  $\mathfrak{g}$  relative to  $\mathfrak{a}$  are given by  $\pm e, \pm 2e$ , where  $e$  is the linear functional  $e : \mathfrak{a} \rightarrow \mathbb{C}$  defined by  $e(E_0) := 1$ . The set  $\Sigma^+ = \{e, 2e\}$  is the set of positive roots of  $\mathfrak{g}$  relative to  $\mathfrak{a}$ . We recall that for a root  $\alpha$ , the root space  $\mathfrak{g}_\alpha$  is defined by

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{a}\}.$$

Clearly we have

$$\mathfrak{g}_e = \sum_{k=1}^m \mathbb{R} F_{k1}^0, \quad \mathfrak{g}_{-e} = \sum_{k=1}^m \mathbb{R} F_{k2}^0,$$

$$\mathfrak{g}_{2e} = \mathbb{R} E_1, \quad \mathfrak{g}_{-2e} = \mathbb{R} E_2$$

and

$$\mathfrak{g} = \mathfrak{g}_{-2e} \oplus \mathfrak{g}_{-e} \oplus \mathfrak{a} \oplus \mathfrak{g}_e \oplus \mathfrak{g}_{2e}.$$

So we obtain

$$\dim \mathfrak{g}_e = \dim \mathfrak{g}_{-e} = m \quad \text{and} \quad \dim \mathfrak{g}_{2e} = \dim \mathfrak{g}_{-2e} = 1.$$

If we define

$$\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} (\dim \mathfrak{g}_\alpha) \cdot \alpha,$$

we get  $\rho = \frac{m+2}{2} e$ .

#### 4. The Lie derivatives on $G$

In this section, we compute the Lie derivatives for functions on  $G$  explicitly. For a simplicity, we consider only the case  $m = 1$ . We leave the case  $m \geq 2$  to the reader.

We put

$$W_1 = \left( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, (0,0) \right), \quad W_2 = \left( \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, (0,0) \right),$$

$$W_3 = \left( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, (0,0) \right)$$

and

$$W_4 = \left( \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, (1,0) \right), \quad W_5 = \left( \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, (0,1) \right).$$

Clearly  $W_1, \dots, W_5$  form a basis of  $\mathfrak{g}$ .

We define the differential operators  $L_k, R_k$  ( $1 \leq k \leq 5$ ) on  $G$  by

$$(4.1) \quad L_k f(\tilde{g}) = \left. \frac{d}{dt} \right|_{t=0} f(\tilde{g} * \exp tW_k)$$

and

$$(4.2) \quad R_k f(\tilde{g}) = \left. \frac{d}{dt} \right|_{t=0} f(\exp tW_k * \tilde{g}),$$

where  $f \in C^\infty(G)$  and  $\tilde{g} \in G$ .

We also put

$$H = \left( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, (0,0) \right) \quad \text{and} \quad V = \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, (0,0) \right).$$

We define the differential operators  $L_H, L_V, R_H$  and  $R_V$  by

$$L_H f(\tilde{g}) = \left. \frac{d}{dt} \right|_{t=0} f(\tilde{g} * \exp tH),$$

$$L_V f(\tilde{g}) = \left. \frac{d}{dt} \right|_{t=0} f(\tilde{g} * \exp tV),$$

$$R_H f(\tilde{g}) = \left. \frac{d}{dt} \right|_{t=0} f(\exp tH * \tilde{g})$$

and

$$R_V f(\tilde{g}) = \left. \frac{d}{dt} \right|_{t=0} f(\exp tV * \tilde{g}).$$

By an easy calculation, we get

$$\begin{aligned} \exp tW_1 &= \left( \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, (0, 0) \right), \quad \exp tW_2 = \left( \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, (0, 0) \right) \\ \exp tW_3 &= \left( \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, (0, 0) \right), \quad \exp tW_4 = \left( \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, (t, 0) \right) \end{aligned}$$

and

$$\exp tW_5 = \left( \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, (0, t) \right).$$

Now we use the following coordinates  $(g, \alpha)$  in  $G$  given by

$$(4.3) \quad g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

and

$$(4.4) \quad \alpha = (\alpha_1, \alpha_2),$$

where  $x, \alpha_1, \alpha_2 \in \mathbb{R}$ ,  $y > 0$  and  $0 \leq \theta < 2\pi$ . By an easy computation we have

$$\begin{aligned} L_1 &= y \cos 2\theta \frac{\partial}{\partial x} + y \sin 2\theta \frac{\partial}{\partial y} + \sin^2 \theta \frac{\partial}{\partial \theta} - \alpha_2 \frac{\partial}{\partial \alpha_1}, \\ L_2 &= y \cos 2\theta \frac{\partial}{\partial x} + y \sin 2\theta \frac{\partial}{\partial y} - \cos^2 \theta \frac{\partial}{\partial \theta} - \alpha_1 \frac{\partial}{\partial \alpha_2}, \\ L_3 &= -2y \sin 2\theta \frac{\partial}{\partial x} + 2y \cos 2\theta \frac{\partial}{\partial y} + \sin 2\theta \frac{\partial}{\partial \theta} - \alpha_1 \frac{\partial}{\partial \alpha_1} + \alpha_2 \frac{\partial}{\partial \alpha_2}, \\ L_4 &= \frac{\partial}{\partial \alpha_1}, \\ L_5 &= \frac{\partial}{\partial \alpha_2}, \end{aligned}$$

$$\begin{aligned}
R_1 &= \frac{\partial}{\partial x}, \\
R_2 &= (y^2 - x^2) \frac{\partial}{\partial x} - 2xy \frac{\partial}{\partial y} - y \frac{\partial}{\partial \theta}, \\
R_3 &= 2x \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial y}, \\
R_4 &= y^{-1/2} \cos \theta \frac{\partial}{\partial \alpha_1} + y^{-1/2} \sin \theta \frac{\partial}{\partial \alpha_2}, \\
R_5 &= -y^{-1/2} (x \cos \theta + y \sin \theta) \frac{\partial}{\partial \alpha_1} + y^{-1/2} (y \cos \theta - x \sin \theta) \frac{\partial}{\partial \alpha_2}.
\end{aligned}$$

Since

$$H = W_1 - W_2 \quad \text{and} \quad V = W_1 + W_2,$$

we have

$$\begin{aligned}
L_H &= L_1 - L_2 = \frac{\partial}{\partial \theta} - \alpha_2 \frac{\partial}{\partial \alpha_1} + \alpha_1 \frac{\partial}{\partial \alpha_2}, \\
L_V &= L_1 + L_2 \\
&= 2y \cos 2\theta \frac{\partial}{\partial x} + 2y \sin 2\theta \frac{\partial}{\partial y} - \cos 2\theta \frac{\partial}{\partial \theta} - \alpha_2 \frac{\partial}{\partial \alpha_1} - \alpha_1 \frac{\partial}{\partial \alpha_2}, \\
R_H &= R_1 - R_2 \\
&= (1 - y^2 + x^2) \frac{\partial}{\partial x} + 2xy \frac{\partial}{\partial y} + y \frac{\partial}{\partial \theta}
\end{aligned}$$

and

$$\begin{aligned}
R_V &= R_1 + R_2 \\
&= (1 + y^2 - x^2) \frac{\partial}{\partial x} - 2xy \frac{\partial}{\partial y} - y \frac{\partial}{\partial \theta}.
\end{aligned}$$

For instance, we present a calculation for  $L_3$  and  $R_5$ . For a simplicity, we denote the form (4.3) by  $[x, y, \theta]$ . Let  $(g, \alpha)$  be an element of  $SL_{2,1}(\mathbb{R})$  with  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = [x, y, \theta]$  and  $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^{(1,2)}$ . Then by an easy computation, we have

$$(4.5) \quad d - ic = y^{-1/2} e^{i\theta} \quad \text{and} \quad d + ic = y^{-1/2} e^{-i\theta}.$$

So we get

$$|d + ic| = y^{-1/2} \quad \text{and} \quad e^{i\theta} = \frac{d - ic}{|d + ic|}.$$

We put

$$(g(t), \alpha(t)) = [x_*(t), y_*(t), \theta_*(t)] = (g, \alpha) * \exp tW_3, \quad t \in \mathbb{R}$$

and

$$\tau_*(t) = g(t) \langle i \rangle = x_*(t) + iy_*(t), \quad t \in \mathbb{R}.$$

Then

$$g(t) = \begin{pmatrix} ae^t & be^{-t} \\ ce^t & de^{-t} \end{pmatrix} \quad \text{and} \quad \alpha(t) = (\alpha_1 e^{-t}, \alpha_2 e^t).$$

Thus

$$\tau_*(t) = \frac{iae^t + be^{-t}}{ice^t + de^{-t}} = \frac{iae^{2t} + b}{ice^{2t} + d} = x_*(t) + iy_*(t).$$

Differentiating  $\tau_*(t)$  with respect to  $t$  and evaluating at  $t = 0$ , by (4.5), we have

$$\tau'_*(0) = 2iy e^{2i\theta}.$$

Thus

$$x'_*(0) = -2y \sin 2\theta \quad \text{and} \quad y'_*(0) = 2y \cos 2\theta.$$

And since

$$i\theta_*(t) = \log \frac{de^{-t} - ice^t}{(d^2 e^{-2t} + c^2 e^{2t})^{1/2}},$$

we obtain

$$\theta'_*(0) = 2 \sin \theta \cdot \cos \theta = \sin 2\theta.$$

We set

$$\alpha(t) = (\alpha_{1,*}(t), \alpha_{2,*}(t)), \quad t \in \mathbb{R}.$$

Since

$$\alpha_{1,*}(t) = \alpha_1 e^{-t} \quad \text{and} \quad \alpha_{2,*}(t) = \alpha_2 e^t,$$

we get

$$\alpha'_{1,*}(0) = -\alpha_1 \quad \text{and} \quad \alpha'_{2,*}(0) = \alpha_2.$$

For a smooth function  $f$  on  $SL_{2,1}(\mathbb{R})$ , using the chain rule, we get the desired formula for  $L_3$ .

Since

$$\exp tW_5 = (E_2, (0, t)),$$



we have

$$\exp tW_5 * (g, \alpha) = (g, (0, t) {}^t g^{-1} + \alpha).$$

We put

$$(g(t), \alpha(t)) = \exp tW_5 * (g, \alpha)$$

and

$$g(t) = [x_*(t), y_*(t), \theta_*(t)], \quad \alpha(t) = (\alpha_{1,*}(t), \alpha_{2,*}(t)).$$

Then we have

$$g(t) = g, \quad \alpha_{1,*}(t) = -bt + \alpha_1 \quad \text{and} \quad \alpha_{2,*}(t) = at + \alpha_2.$$

From (4.3), we get

$$a = y^{1/2} \cos \theta - x y^{-1/2} \sin \theta \quad \text{and} \quad b = y^{1/2} \sin \theta + x y^{-1/2} \cos \theta.$$

Finally we obtain the desired explicit formula for  $R_5$ . We leave the other formulas to the reader.

**5. The actions of  $G$  on  $\mathbb{H} \times \mathbb{C}^m$  and  $SP_2 \times \mathbb{R}^{(m,2)}$**

We recall that  $SP_2$  is the set of all  $2 \times 2$  positive symmetric real matrices  $Y$  with  $\det Y = 1$ . Then  $G$  acts on  $SP_2 \times \mathbb{R}^{(m,2)}$  transitively by

$$(5.1) \quad (g, \alpha) \cdot (Y, V) := (gY {}^t g, (V + \alpha) {}^t g),$$

where  $g \in SL(2, \mathbb{R})$ ,  $\alpha \in \mathbb{R}^{(m,2)}$ ,  $Y \in SP_2$  and  $V \in \mathbb{R}^{(m,2)}$ . It is easy to see that  $K$  is a maximal compact subgroup of  $G$  stabilizing the origin  $(I_2, 0)$ . Thus  $SP_2 \times \mathbb{R}^{(m,2)}$  may be identified with the homogeneous space  $G/K$  as follows:

$$(5.2) \quad G/K \ni (g, \alpha)K \longmapsto (g, \alpha) \cdot (I_2, 0) \in SP_2 \times \mathbb{R}^{(m,2)},$$

where  $g \in SL(2, \mathbb{R})$  and  $\alpha \in \mathbb{R}^{(m,2)}$ .

We recall that  $SL(2, \mathbb{R})$  acts on  $\mathbb{H}$  transitively by

$$g < \tau > := (a\tau + b)(c\tau + d)^{-1}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}), \quad \tau \in \mathbb{H}.$$

Now we define the action of  $G$  on  $\mathbb{H} \times \mathbb{C}^m$  by

$$(5.3) \quad (g, \alpha) \circ (\tau, z) := ({}^t g^{-1} \langle \tau \rangle, (z + \alpha_1 \tau + \alpha_2)(-b\tau + a)^{-1}),$$

where  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$  and  $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^{(m,2)}$ , and  $(\tau, z) \in \mathbb{H} \times \mathbb{C}^m$ . Since the action (5.3) is transitive and  $K$  is the stabilizer of this action at the origin  $(i, 0)$ ,  $\mathbb{H} \times \mathbb{C}^m$  can be identified with the homogeneous space  $G/K$  as follows:

$$(5.4) \quad G/K \ni (g, \alpha)K \longmapsto (g, \alpha) \circ (i, 0).$$

We observe that we can express an element  $Y$  of  $SP_2$  uniquely as

$$(5.5) \quad Y = \begin{pmatrix} y^{-1} & 0 \\ 0 & y \end{pmatrix} \left[ \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \right] = \begin{pmatrix} y^{-1} & -xy^{-1} \\ -xy^{-1} & x^2y^{-1} + y \end{pmatrix}$$

with  $x, y \in \mathbb{R}$  and  $y > 0$ .

PROPOSITION 5.1. We define the mapping  $T : SP_2 \times \mathbb{R}^{(m,2)} \longrightarrow \mathbb{H} \times \mathbb{C}^m$  by

$$(5.6) \quad T(Y, V) = (x + iy, v_1(x + iy) + v_2),$$

where  $Y$  is of the form (5.5) and  $V = (v_1, v_2) \in \mathbb{R}^{(m,2)}$ . Then the mapping  $T$  is a bijection which is compatible with the above two actions (5.1) and (5.3).

For any  $Y \in SP_2$  of the form (5.5), we put

$$(5.7) \quad g_Y = \begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix} \begin{pmatrix} y^{-1/2} & 0 \\ 0 & y^{1/2} \end{pmatrix} = \begin{pmatrix} y^{-1/2} & 0 \\ -xy^{-1/2} & y^{1/2} \end{pmatrix}.$$

and

$$(5.8) \quad \alpha_{Y,V} = V {}^t g_Y^{-1}.$$

Then we have

$$(5.9) \quad T(Y, V) = (g_Y, \alpha_{Y,V}) \circ (i, 0).$$

*Proof.* It is obvious that  $T$  is a bijection. Let  $T_0 : SP_2 \longrightarrow \mathbb{H}$  be the mapping defined by

$$T_0(Y) = x + iy,$$

where  $Y$  is of the form (5.5). We observe that

$$Y = g_Y {}^t g_Y \quad \text{and} \quad T_0(Y) = {}^t g_Y^{-1} \langle i \rangle,$$

where  $g_Y$  is an element of  $SL(2, \mathbb{R})$  defined by (5.7).

For  $g \in SL(2, \mathbb{R})$  and  $Y \in SP_2$ ,

$$Y_* = gY {}^t g = g_* {}^t g_* \quad \text{for some } g_* \in SL(2, \mathbb{R}).$$

Since  $Y_* = g_{Y_*} {}^t g_{Y_*}$ , there is an element  $k \in K$  such that  $g_{Y_*} = g_* k$ . So we have

$$\begin{aligned} T_0(Y_*) &= {}^t g_{Y_*}^{-1} \langle i \rangle = {}^t (g_* k)^{-1} \langle i \rangle \\ &= {}^t g_*^{-1} {}^t k^{-1} \langle i \rangle = {}^t g_*^{-1} \langle i \rangle. \end{aligned}$$

So far we showed that if  $Y_0 \in SP_2$  such that  $Y_0 = g_0 {}^t g_0$  with  $g_0 \in SL(2, \mathbb{R})$ , then  $T_0(Y_0) = g_0 \langle i \rangle$ . If  $Y$  is of the form (5.5), we take

$$g_* = g \cdot g_Y.$$

Then  $g_* {}^t g_* = gY {}^t g$ . Therefore we have

$$T_0(gY {}^t g) = {}^t g_*^{-1} \langle i \rangle = {}^t g^{-1} \cdot {}^t g_Y^{-1} \langle i \rangle = {}^t g^{-1} \langle \tau \rangle,$$

where we put  $\tau = x + iy$ .

Let  $(g, \alpha) \in G$  with  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$ ,  $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^{(m,2)}$  and  $(Y, V) \in SP_2 \times \mathbb{R}^{(m,2)}$  such that  $Y \in SP_2$  is of the form (5.5) and  $V = (v_1, v_2) \in \mathbb{R}^{(m,2)}$  with  $v_1, v_2 \in \mathbb{R}^{(m,1)}$ . Now we have, if  $\tau = x + iy \in \mathbb{H}$ ,

$$\begin{aligned} &(g, \alpha) \circ T(Y, V) \\ &= (g, \alpha) \circ (\tau, v_1 \tau + v_2) \\ &= ({}^t g^{-1} \langle \tau \rangle, \{(v_1 + \alpha_1) \tau + (v_2 + \alpha_2)\} (-b \tau + a)^{-1}). \end{aligned}$$

On the other hand.

$$\begin{aligned} T((g, \alpha) \cdot (Y, V)) &= T(gY {}^t g, (V + \alpha) {}^t g) \\ &= (T_0(gY {}^t g), \tilde{v}_1 \cdot T_0(gY {}^t g) + \tilde{v}_2) \\ &= ({}^t g^{-1} \langle \tau \rangle, \tilde{v}_1 \cdot {}^t g^{-1} \langle \tau \rangle + \tilde{v}_2), \end{aligned}$$

where  $(\tilde{v}_1, \tilde{v}_2) = (V + \alpha) {}^t g$ .

Since

$$\begin{aligned} \tilde{v}_1 &= (v_1 + \alpha_1) a + (v_2 + \alpha_2) b, \\ \tilde{v}_2 &= (v_1 + \alpha_1) c + (v_2 + \alpha_2) d, \end{aligned}$$

by a simple calculation, we get

$$\tilde{v}_1 {}^t g^{-1} \langle \tau \rangle + \tilde{v}_2 = \{ (v_1 + \alpha_1) \tau + (v_2 + \alpha_2) \} (-b\tau + a)^{-1}.$$

Hence we obtain

$$T((g, \alpha) \cdot (Y, V)) = (g, \alpha) \circ T(Y, V).$$

Consequently  $T$  is compatible with the actions (5.1) and (5.3). It is easily checked that (5.9) holds.  $\square$

DEFINITION 5.2. For any  $(\tau, z) \in \mathbb{H} \times \mathbb{C}^m$  with  $\tau \in \mathbb{H}$  and  $z \in \mathbb{C}^m$ , we define  $Y_{(\tau, z)} \in SP_2$  and  $V_{(\tau, z)} \in \mathbb{R}^{(m, 2)}$  by

$$(Y_{(\tau, z)}, V_{(\tau, z)}) = T^{-1}(\tau, z).$$

PROPOSITION 5.3. Let  $(g, \alpha)$  be an element of  $G$  such that  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$  and  $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^{(m, 2)}$  with  $\alpha_1, \alpha_2 \in \mathbb{R}^{(m, 1)}$ . For  $(\tau, z) \in \mathbb{H} \times \mathbb{C}^m$  with  $\tau \in \mathbb{H}$ ,  $z = u + iv$ ,  $u = \operatorname{Re} z$  and  $v = \operatorname{Im} z$ , we set

$$(\tau_*, z_*) = (g, \alpha) \circ (\tau, z)$$

and

$$\tau_* = x_* + i y_*, \quad z_* = u_* + i v_*$$

where  $x_* = \operatorname{Re} \tau_*$ ,  $y_* = \operatorname{Im} \tau_*$ ,  $u_* = \operatorname{Re} z_*$  and  $v_* = \operatorname{Im} z_*$ . Then we have

$$Y_{(\tau_*, z_*)} = \begin{pmatrix} y_*^{-1} & 0 \\ 0 & y_* \end{pmatrix} \left[ \begin{pmatrix} 1 & -x_* \\ 0 & 1 \end{pmatrix} \right]$$

and

$$V_{(\tau_*, z_*)} = (y_*^{-1} v_*, u_* - x_* y_*^{-1} v_*), \quad u_*, v_* \in \mathbb{R}^{(m, 1)},$$

where

$$\begin{aligned} x_* &= \{ -bd|\tau|^2 - ac + (ad + bc)x \} | -b\tau + a|^{-2}, \\ y_* &= y | -b\tau + a|^{-2}, \\ u_* &= \{ (a - bx)(u + x\alpha_1 + \alpha_2) - by(v + \alpha_1 y) \} | -b\tau + a|^{-2}, \\ v_* &= \{ (a - bx)(v + y\alpha_1) + by(u + x\alpha_1 + \alpha_2) \} | -b\tau + a|^{-2}. \end{aligned}$$

*Proof.* It is easy to prove the above proposition and so we omit it.  $\square$

Now we present a  $G$ -invariant metric on  $SP_2 \times \mathbb{R}^{(m,2)}$ .

**PROPOSITION 5.4.** *The following Riemannian metric  $ds^2$  on  $SP_2 \times \mathbb{R}^{(m,2)}$  defined by*

$$ds^2 = \frac{dx^2 + dy^2}{y} + \frac{1}{y} \sum_{k=1}^m \{ (x^2 + y^2) dv_{k1}^2 + 2x dv_{k1} dv_{k2} + dv_{k2}^2 \}$$

*is invariant under the action (5.1) of  $G$ . The Laplace-Beltrami operator  $\Delta_m$  of  $(SP_2 \times \mathbb{R}^{(m,2)}, ds^2)$  is given by*

$$\begin{aligned} \Delta_m = & y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \\ & + y^{-1} \sum_{k=1}^m \left\{ \frac{\partial^2}{\partial v_{k1}^2} - 2x \frac{\partial^2}{\partial v_{k1} \partial v_{k2}} + (x^2 + y^2) \frac{\partial^2}{\partial v_{k2}^2} \right\}. \end{aligned}$$

*Proof.* The proof of the case  $m = 1$  can be found in [9]. In a similar way, the proof of the above proposition can be done.  $\square$

### 6. Maass-Jacobi forms

In this section, we make some comments on the study of Maass-Jacobi forms related to  $G = SL_{2,1}(\mathbb{R})$ .

First we review the notion of Maass-Jacobi forms (cf. [10]). We let

$$\Gamma = SL(2, \mathbb{Z}) \ltimes \mathbb{Z}^{(1,2)}$$

be the discrete subgroup of  $G$ . Let  $\mathbb{D}(G)$  be the algebra of all differential operators on  $G$  invariant under all the left translations of  $G$ . For a diffeomorphism  $\phi$  of  $G$  and an element  $D \in \mathbb{D}(G)$ , we define  $D^\phi$  by

$$D^\phi(f) = (D(f \circ \phi)) \circ \phi^{-1}, \quad f \in C^\infty(G).$$

For  $g \in G$ , we denote by  $R_g$  the right translation of  $G$  by  $g$ . We define

$$\mathbb{D}_K(G) = \{ D \in \mathbb{D}(G) \mid D^{R_k} = D \text{ for all } k \in K \}.$$

In [10], the author defined the notion of Maass-Jacobi forms. We recall this concept here. A smooth bounded function  $\phi : G \rightarrow \mathbb{C}$  is called a *Maass-Jacobi form* if it satisfies the following conditions (MJ1)-(MJ3):

(MJ1)  $\phi(\gamma x k) = \phi(x)$  for all  $\gamma \in \Gamma$ ,  $x \in G$  and  $k \in K$ .

(MJ2)  $\phi$  is an eigenfunction for the Laplace-Beltrami operator in  $\mathbb{D}_K(G)$ .

(MJ3)  $\phi$  is slowly increasing in the sense of [1].

We observe that the notion of Maass-Jacobi forms generalizes that of Maass-wave forms. For the convenience of the reader, we review Maass wave forms. Let  $s \in \mathbb{C}$  and  $\Gamma_1 = SL(2, \mathbb{Z})$ . We denote by  $W_s(\Gamma_1)$  the vector space of all smooth bounded functions  $f : SL(2, \mathbb{R}) \rightarrow \mathbb{C}$  satisfying the following conditions (a) and (b):

(a)  $f(\gamma g k) = f(g)$  for all  $\gamma \in \Gamma_1$ ,  $g \in SL(2, \mathbb{R})$  and  $k \in K$ .

(b)  $\Delta_0 f = \frac{1-s^2}{4} f$ ,

where

$$\Delta_0 = y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + y \frac{\partial^2}{\partial x \partial \theta}$$

is the Laplace-Beltrami operator on  $SL(2, \mathbb{R})$  whose coordinates  $x, y, \theta$  ( $x \in \mathbb{R}$ ,  $y > 0$ ,  $0 \leq \theta < 2\pi$ ) are given by

$$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad g \in SL(2, \mathbb{R})$$

by means of the Iwasawa decomposition of  $SL(2, \mathbb{R})$ . The elements in  $W_s(\Gamma_1)$  are called *Maass wave forms*. We observe that the algebra  $\mathbb{D}_K(SL(2, \mathbb{R}))$  is generated by the Laplace-Beltrami operator  $\Delta_0$  on  $SL(2, \mathbb{R})$ . It is well known that  $W_s(\Gamma_1)$  is nontrivial for infinitely many values of  $s$ . For more detail, we refer to [2], [3], [5] and [8].

We can prove that Maass-Jacobi forms may be realized as functions on  $\mathbb{H} \times \mathbb{C}$  or  $SP_2 \times \mathbb{R}^{(1,2)}$  satisfying a certain invariance property. We explain these facts in some more detail.

**THEOREM 6.1.** *Let  $\mathbb{D}(\mathbb{H} \times \mathbb{C})$  be the algebra of all differential operators on  $\mathbb{H} \times \mathbb{C}$  invariant under the action (5.3). Then the algebra  $\mathbb{D}(\mathbb{H} \times \mathbb{C})$  is generated by the following differential operators*

$$(6.1) \quad D = y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + v^2 \left( \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) + 2yv \left( \frac{\partial^2}{\partial x \partial u} + \frac{\partial^2}{\partial y \partial v} \right),$$

$$(6.2) \quad \Psi = y \left( \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right),$$

$$(6.3) \quad D_1 = 2y^2 \frac{\partial^3}{\partial x \partial u \partial v} - y^2 \frac{\partial}{\partial y} \left( \frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2} \right) + \left( v \frac{\partial}{\partial v} + 1 \right) \Psi$$

and

$$(6.4) \quad D_2 = y^2 \frac{\partial}{\partial x} \left( \frac{\partial^2}{\partial v^2} - \frac{\partial^2}{\partial u^2} \right) - 2y^2 \frac{\partial^3}{\partial y \partial u \partial v} - v \frac{\partial}{\partial u} \Psi,$$

where  $\tau = x + iy$  and  $z = u + iv$  with  $x, u, v$  real and  $y > 0$ . Moreover, we have

$$[D, \Psi] := D\Psi - \Psi D = 2y^2 \frac{\partial}{\partial y} \left( \frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2} \right) - 4y^2 \frac{\partial^3}{\partial x \partial u \partial v} - 2 \left( v \frac{\partial}{\partial v} \Psi + \Psi \right).$$

In particular, the algebra  $\mathbb{D}(\mathbb{H} \times \mathbb{C})$  is not commutative. The homogeneous space  $\mathbb{H} \times \mathbb{C}$  is not weakly symmetric in the sense of A. Selberg [7].

*Proof.* The proof can be found in [10]. □

For any right  $K$ -invariant function  $\phi : G \rightarrow \mathbb{C}$  on  $G$ , we define the function  $f_\phi : \mathbb{H} \times \mathbb{C} \rightarrow \mathbb{C}$  by

$$(6.5) \quad f_\phi(\tau, z) = \phi(g, \alpha), \quad \tau \in \mathbb{H}, \quad z \in \mathbb{C},$$

where  $(g, \alpha)$  is an element of  $G$  with  $(g, \alpha) \circ (i, 0) = (\tau, z)$ . Clearly it is well defined because (6.5) is independent of the choice of  $(g, \alpha) \in G$  such that  $(g, \alpha) \circ (i, 0) = (\tau, z)$ .

**PROPOSITION 6.2.** *Let  $\phi : G \rightarrow \mathbb{C}$  be a nonzero Maass-Jacobi form. Then the function  $f_\phi$  defined by (6.5) satisfies the following conditions:*

(MJ1)<sup>0</sup>  $f_\phi(\tilde{\gamma} \circ (\tau, z)) = f_\phi(\tau, z)$  for all  $\tilde{\gamma} \in \Gamma$ .

(MJ2)<sup>0</sup>  $f_\phi$  is a joint eigenfunction for the differential operator  $\Delta = D + \Psi$ .

(MJ3)<sup>0</sup>  $f_\phi$  has a polynomial growth.

Conversely, if a smooth function  $f : \mathbb{H} \times \mathbb{C} \rightarrow \mathbb{C}$  satisfies the above conditions  $(MJ1)^0 - (MJ3)^0$ , then the function  $\phi_f : G \rightarrow \mathbb{C}$  defined by

$$(6.6) \quad \phi_f(g, \alpha) = f((g, \alpha) \circ (i, 0)), \quad (g, \alpha) \in G$$

is a Maass-Jacobi form.

We denote by  $\mathbb{D}(SP_2 \times \mathbb{R}^{(1,2)})$  the algebra of all differential operators on  $SP_2 \times \mathbb{R}^{(1,2)}$  invariant under the action (5.1).

**THEOREM 6.3.** *The algebra  $\mathbb{D}(SP_2 \times \mathbb{R}^{(1,2)})$  is generated by the following differential operators*

$$(6.7) \quad \tilde{D} = y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

$$(6.8) \quad \tilde{\Psi} = y^{-1} \left\{ \frac{\partial^2}{\partial v_1^2} - 2x \frac{\partial^2}{\partial v_1 \partial v_2} + (x^2 + y^2) \frac{\partial^2}{\partial v_2^2} \right\},$$

$$(6.9) \quad \begin{aligned} \tilde{D}_1 = & \left\{ \frac{\partial^2}{\partial v_1^2} - 2x \frac{\partial^2}{\partial v_1 \partial v_2} + (x^2 - y^2) \frac{\partial^2}{\partial v_2^2} \right\} \frac{\partial}{\partial y} \\ & + 2y \left( \frac{\partial^2}{\partial v_1 \partial v_2} - x \frac{\partial^2}{\partial v_2^2} \right) \frac{\partial}{\partial x} - \Psi \end{aligned}$$

and

$$(6.10) \quad \begin{aligned} \tilde{D}_2 = & \left\{ \frac{\partial^2}{\partial v_1^2} - 2x \frac{\partial^2}{\partial v_1 \partial v_2} + (x^2 - y^2) \frac{\partial^2}{\partial v_2^2} \right\} \frac{\partial}{\partial x} \\ & - 2y \left( \frac{\partial^2}{\partial v_1 \partial v_2} - x \frac{\partial^2}{\partial v_2^2} \right) \frac{\partial}{\partial y}, \end{aligned}$$

where we used the coordinates

$$Y = \begin{pmatrix} y^{-1} & 0 \\ 0 & y \end{pmatrix} \left[ \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \right] = \begin{pmatrix} y^{-1} & -xy^{-1} \\ -xy^{-1} & x^2y^{-1} + y \end{pmatrix}$$

with  $x, y \in \mathbb{R}$ ,  $y > 0$  and  $V = (v_1, v_2)$  with  $x, y \in \mathbb{R}$ .

*Proof.* The proof of the above theorem can be found in [9]. □



For a right  $K$ -invariant function  $\phi : G \rightarrow \mathbb{C}$ , we define another function  $h_\phi : SP_2 \times \mathbb{R}^{(1,2)} \rightarrow \mathbb{C}$  by

$$(6.11) \quad h_\phi(Y, V) = \phi(g, V {}^t g^{-1}), \quad Y \in SP_2, \quad V \in \mathbb{R}^{(1,2)},$$

where  $Y = g {}^t g$  with some  $g \in SL(2, \mathbb{R})$ . The definition (6.11) is well defined because it is independent of the choice of  $g$  with  $Y = g {}^t g$ . Indeed, such  $g$  is unique up to  $K$ .

Now we can realize Maass-Jacobi forms as functions on  $SP_2 \times \mathbb{R}^{(1,2)}$  satisfying a certain invariance property.

**PROPOSITION 6.4.** *Let  $\phi : G \rightarrow \mathbb{C}$  be a nonzero Maass-Jacobi form. Then the function  $h_\phi$  defined by (6.7) satisfies the following conditions:*

(MJ1)\*  $h_\phi(\gamma Y {}^t \gamma, (V + \delta) {}^t \gamma) = h_\phi(Y, V)$  for all  $(\gamma, \delta) \in \Gamma_{2,1}$  with  $\gamma \in SL(2, \mathbb{Z})$  and  $\delta \in \mathbb{Z}^{(1,2)}$ .

(MJ2)\*  $h_\phi$  is a joint eigenfunction for the differential operator  $\tilde{\Delta} = \tilde{D} + \tilde{\psi}$ .

(MJ3)\*  $h_\phi$  has a suitable polynomial growth.

Conversely, if a smooth function  $h : SP_2 \times \mathbb{R}^{(1,2)} \rightarrow \mathbb{C}$  satisfies the conditions (MJ1)\* – (MJ3)\*, the function  $\phi_h : G \rightarrow \mathbb{C}$  defined by

$$(6.12) \quad \phi_h(g, \alpha) = h(g {}^t g, \alpha {}^t g), \quad g \in SL(2, \mathbb{R}), \quad \alpha \in \mathbb{R}^{(m,2)}$$

is a Maass-Jacobi form.

The proof of Theorem 6.2 and 6.4. can be found in [10].

### 7. The unitary dual of $SL_{2,m}(\mathbb{R})$

For brevity, we set  $H = \mathbb{R}^{(m,2)}$ . Then we have the split exact sequence

$$0 \rightarrow H \rightarrow G \rightarrow SL(2, \mathbb{R}) \rightarrow 1.$$

We see that the unitary dual  $\hat{H}$  of  $H$  is isomorphic to  $\mathbb{R}^{(m,2)}$ . The unitary character  $\chi_{(\lambda, \mu)}$  of  $H$  corresponding to  $(\lambda, \mu) \in \mathbb{R}^{(m,2)}$  with  $\lambda, \mu \in \mathbb{R}^{(m,1)}$  is given by

$$\chi_{(\lambda, \mu)}(x, y) = e^{2\pi i({}^t \lambda x + {}^t \mu y)}, \quad (x, y) \in H.$$

$G$  acts on  $H$  by conjugation and hence this action induces the action of  $G$  on  $\hat{H}$  as follows.

$$(7.1) \quad G \times \hat{H} \rightarrow \hat{H}, \quad (g, \chi) \mapsto \chi^g, \quad g \in G, \quad \chi \in \hat{H},$$

where the character  $\chi^g$  is defined by

$$\chi^g(a) = \chi(g^{-1}ag), \quad a \in H.$$

If  $g = (g_0, \alpha) \in G$  with  $g_0 \in SL(2, \mathbb{R})$  and  $\alpha \in H$ , it is easy to check that for each  $(\lambda, \mu) \in \mathbb{R}^{(m,2)}$ ,

$$(7.2) \quad \chi_{(\lambda, \mu)}^g = \chi_{(\lambda, \mu)g_0^{-1}}.$$

Indeed, if  $a \in H$ , then

$$\begin{aligned} g^{-1}ag &= (g_0, \alpha)^{-1}(I_2, a)(g_0, \alpha) \\ &= (g_0^{-1}, -\alpha^t g_0)(g_0, a^t g_0^{-1} + \alpha) \\ &= (I_2, a^t g_0^{-1}) \end{aligned}$$

and therefore we have for each  $c \in \mathbb{R}^{(m,2)}$  and  $a \in H$ ,

$$\begin{aligned} \chi_c^g(a) &= \chi_c(g^{-1}ag) = \chi_c(a^t g_0^{-1}) \\ &= e^{2\pi i \sigma({}^t c a^t g_0^{-1})} \\ &= e^{2\pi i \sigma({}^t (c g_0^{-1}) a)} \\ &= \chi_{c g_0^{-1}}(a), \end{aligned}$$

where  $\sigma(A)$  denotes the trace of a square matrix  $A$ .

We know that  $H$  is of type I. We denote by  $\Omega_\chi$  the  $G$ -orbit of  $\chi \in \hat{H}$  and let

$$G_\chi = \{g \in G \mid \chi^g = \chi\}$$

be the stabilizer of  $G$  at  $\chi$ . The mapping

$$G/G_\chi \longrightarrow \Omega_\chi, \quad g \cdot G_\chi \mapsto \chi^g$$

is a homeomorphism, in other words,  $H$  is regularly embedded. Obviously  $H$  is a subgroup of  $G_\chi$ . We define the subset  $G_\chi^*$  of the unitary dual  $\hat{G}_\chi$  of  $G_\chi$  by

$$G_\chi^* = \left\{ \tau \in \hat{G}_\chi \mid \tau|_A \text{ is a multiple of } \chi \right\}.$$

According to Mackey [6], we obtain

**THEOREM 7.1 (Mackey).** *For any  $\tau \in G_\chi^*$ , the induced representation  $Ind_{G_\chi}^G \tau$  is an irreducible unitary representation of  $G$ . And the unitary dual  $\hat{G}$  of  $G$  is given by*

$$\hat{G} = \bigcup_{[\chi] \in G \backslash \hat{H}} \{Ind_{G_\chi}^G \tau \mid \tau \in G_\chi^*\}.$$

**Case I.**  $m = 1$ .

We see easily from (7.2) that the  $G$ -orbits in  $\hat{H} \cong \mathbb{R}^2$  consist of two orbits  $\Omega_0, \Omega_1$  given by

$$\Omega_0 = \{(0, 0)\}, \quad \Omega_1 = \mathbb{R}^2 - \{(0, 0)\}.$$

We observe that  $\Omega_0$  is the  $G$ -orbit of  $(0, 0)$  and  $\Omega_1$  is the  $G$ -orbit of any element  $(\lambda, \mu) \neq 0$ .

Now we choose the element  $\delta = \chi_{(1,0)}$  of  $\hat{H}$ . That is,  $\delta(x, y) = e^{2\pi i x}$  for all  $(x, y) \in \mathbb{R}^2$ . It is easy to check that the stabilizer of  $\chi_{(0,0)}$  is  $G$  and the stabilizer  $G_\delta$  of  $\delta$  is given by

$$G_\delta = \left\{ \left( \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \alpha \right) \mid c \in \mathbb{R}, \alpha \in \mathbb{R}^{(1,2)} \right\}.$$

In order to find the unitary dual  $\hat{G}$  of  $G$ , we need the following fact.

**LEMMA 7.2.** *Let  $(\tau, V)$  be an irreducible unitary representation of  $G_\delta$  whose restriction to  $H$  is a multiple of  $\delta$ . Then  $(\tau, V)$  is a one-dimensional representation of  $G_\delta$ .*

*Proof.* Let  $W$  be a nonzero subspace of  $V$  invariant under the subgroup

$$G_\delta^0 = \left\{ \left( \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, 0 \right) \in G_\delta \mid c \in \mathbb{R} \right\}$$

of  $G_\delta$ .

Since

$$\left( \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \alpha \right) = \left( \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, 0 \right) (I_2, \alpha),$$

we have

$$\begin{aligned} \tau \left( \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \alpha \right) W &= \tau \left( \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, 0 \right) \tau((I_2, \alpha))W \\ &\subseteq \tau \left( \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, 0 \right) W \subseteq W. \end{aligned}$$

Therefore  $W$  is a nonzero subspace of  $V$  invariant under  $G_\delta$ . By the assumption on the irreducibility of  $V$ ,  $W = V$ . Since  $(\tau, V)$  is an irreducible unitary representation of  $G_\delta^0$  and  $G_\delta^0$  is an abelian group,  $(\tau, V)$  must be a one-dimensional representation of  $G_\delta^0$  and hence  $G_\delta$ .  $\square$

From now on, we denote by  $\mathbb{R}^\times$  and  $\mathbb{C}_1^\times$  the set of all nonzero real numbers and the set of all complex numbers  $z$  with  $|z| = 1$  respectively.

We obtain the following

**THEOREM 7.3.** *Let  $m = 1$ . Then the irreducible unitary representations of  $G$  are the following:*

(a) *The irreducible unitary representations  $\pi$ , where the restriction of  $\pi$  to  $H$  is trivial and the restriction of  $\pi$  to  $SL(2, \mathbb{R})$  is an irreducible unitary representation of  $SL(2, \mathbb{R})$ . For the unitary dual of  $SL(2, \mathbb{R})$ , we refer to [4], p.123.*

(b) *The representations  $\pi_{(r)} := \text{Ind}_{G_\delta}^G \sigma_r$  ( $r \in \mathbb{R}$ ) induced from the unitary character  $\sigma_r$  of  $G_\delta$  defined by*

$$\sigma_r \left( \left( \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, (\lambda, \mu) \right) \right) = \delta(rc + \lambda) = e^{2\pi i(rc + \lambda)}, \quad c, \lambda, \mu \in \mathbb{R}.$$

*Proof.* The  $G$ -orbits in  $\hat{H}$  consist of two orbits  $\Omega_0 = \{\chi_{(0,0)}\}$  and  $\Omega_1 = \{\chi_{(\lambda,\mu)} \mid \lambda, \mu \in \mathbb{R}, (\lambda, \mu) \neq (0,0)\}$ . We choose a representative  $\delta := \chi_{(1,0)}$  of  $\Omega_1$ . Since  $G$  is the stabilizer of  $\chi_{(0,0)}$ , an element of  $G_{\chi_{(0,0)}}^* = G^*$  is of the form

$$\pi_\rho((g, \alpha)) = \rho \cdot \pi(g), \quad g \in SL(2, \mathbb{R}), \alpha \in \mathbb{R}^{(1,2)},$$

where  $\pi$  is an irreducible unitary representation of  $SL(2, \mathbb{R})$  and  $\rho \in \mathbb{C}_1^\times$ . We observe that  $\pi_\rho$  is unitarily equivalent to  $\pi = \pi_1$  for any  $\rho \in \mathbb{C}_1^\times$ . On the other hand, since the stabilizer  $G_\delta$  of  $\delta = \chi_{(1,0)}$  is given by

$$G_\delta = \left\{ \left( \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \alpha \right) \mid c \in \mathbb{R}, \alpha \in \mathbb{R}^{(1,2)} \right\},$$

according to Lemma 7.2, we see that an element of  $G_\delta^*$  is of the form

$$\sigma_{r,\rho} \left( \left( \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, (\lambda, \mu) \right) \right) = \rho \cdot e^{2\pi i(rc + \lambda)}, \quad c, \lambda, \mu \in \mathbb{R},$$

where  $r$  is a fixed real number and  $\rho \in \mathbb{C}_1^\times$ . We observe that  $\sigma_{r,\rho}$  is unitarily equivalent to  $\sigma_r := \sigma_{r,1}$  for any  $\rho \in \mathbb{C}_1^\times$ . By Theorem 7.1, we complete the proof.  $\square$

**Case II.**  $m = 2$ .

In this case,  $G = SL(2, \mathbb{R}) \times \mathbb{R}^{(2,2)}$  and  $\hat{H} \cong \mathbb{R}^{(2,2)}$ . From now on, we identify  $\hat{H}$  with  $\mathbb{R}^{(2,2)}$ .

LEMMA 7.4. *Let  $m = 2$ . Then the  $G$ -orbits in  $\hat{H}$  consist of the following orbits*

$$\begin{aligned} \Omega_0 &= \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}, \\ \Omega_1 &= \left\{ \begin{pmatrix} A \\ 0 \end{pmatrix} \in \mathbb{R}^{(2,2)} \mid A \in \mathbb{R}^{(1,2)}, A \neq 0 \right\}, \\ \Omega_2 &= \left\{ \begin{pmatrix} 0 \\ A \end{pmatrix} \in \mathbb{R}^{(2,2)} \mid A \in \mathbb{R}^{(1,2)}, A \neq 0 \right\}, \\ \Omega_3(\delta) &= \left\{ \begin{pmatrix} A \\ \delta A \end{pmatrix} \in \mathbb{R}^{(2,2)} \mid A \in \mathbb{R}^{(1,2)}, A \neq 0 \right\} \quad (\delta \in \mathbb{R}^\times) \end{aligned}$$

and

$$\Omega_4(\theta) = \left\{ M \in \mathbb{R}^{(2,2)} \mid \det M = \theta \right\} \quad (\theta \in \mathbb{R}^\times).$$

$\Omega_0$  is the  $G$ -orbit of  $\mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ ,  $\Omega_1$  is the  $G$ -orbit of  $\begin{pmatrix} \alpha \\ 0 \end{pmatrix}$  with  $0 \neq \alpha \in \mathbb{R}^{(1,2)}$ ,  $\Omega_2$  is the  $G$ -orbit of  $\begin{pmatrix} 0 \\ \beta \end{pmatrix}$  with  $0 \neq \beta \in \mathbb{R}^{(1,2)}$ ,  $\Omega_3(\delta)$  is the  $G$ -orbit of  $\begin{pmatrix} \alpha \\ \delta \alpha \end{pmatrix}$  with  $0 \neq \alpha \in \mathbb{R}^{(1,2)}$  and  $\Omega_4(\theta)$  is the  $G$ -orbit of  $M \in \mathbb{R}^{(2,2)}$  with  $\det M = \theta$ .

*Proof.* We recall that  $G$  acts on  $\hat{H} \cong \mathbb{R}^{(2,2)}$  via (7.2). If we identify  $\hat{H}$  with  $\mathbb{R}^{(2,2)}$ , we know that  $G$  acts on  $\mathbb{R}^{(2,2)}$  by

$$(g_0, \alpha) \cdot X := Xg_0^{-1}, \quad (g_0, \alpha) \in G, \quad g_0 \in SL(2, \mathbb{R}), \quad X \in \mathbb{R}^{(2,2)}.$$

Then we see without difficulty that the  $G$ -orbits  $\Omega_0, \Omega_1, \Omega_2, \Omega_3(\delta)$  ( $\delta \in \mathbb{R}^\times$ ) and  $\Omega_4(\theta)$  ( $\theta \in \mathbb{R}^\times$ ) form all the  $G$ -orbits in  $\hat{H} \cong \mathbb{R}^{(2,2)}$ . □

We put

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Obviously  $e \in \Omega_1$  and  $f \in \Omega_2$ .

Then we may prove the following.

LEMMA 7.5. (a) The stabilizer of  $\mathbf{0}$  is  $G$ .

(b) The stabilizer  $G_e$  of  $e$  is given by

$$G_e = \left\{ \left( \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \alpha \right) \mid c \in \mathbb{R}, \alpha \in \mathbb{R}^{(2,2)} \right\}.$$

For each  $x \in \Omega_1$ , the stabilizer  $G(x)$  of  $x$  is conjugate to  $G_e$ . Precisely if  $x = eg_0$  with  $g_0 \in SL(2, \mathbb{R})$ , then  $G(x) = (g_0, 0)^{-1}G_e(g_0, 0)$ .

(c) The stabilizer  $G_f$  of  $f$  is given by

$$G_f = \left\{ \left( \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \alpha \right) \mid c \in \mathbb{R}, \alpha \in \mathbb{R}^{(2,2)} \right\}.$$

For each  $y \in \Omega_2$ , the stabilizer  $G(y)$  of  $y$  is conjugate to  $G_f$ .

(d) The stabilizer  $G_\delta$  of  $\begin{pmatrix} 1 & 0 \\ \delta & 0 \end{pmatrix}$  ( $\delta \in \mathbb{R}^\times$ ) is given by

$$G_\delta = \left\{ \left( \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \alpha \right) \mid c \in \mathbb{R}, \alpha \in \mathbb{R}^{(2,2)} \right\}.$$

For each  $z \in \Omega_3(\delta)$ , the stabilizer  $G(z)$  of  $z$  is conjugate to  $G_\delta$ .

(e) The stabilizer  $G_\theta$  of  $M_\theta \in \Omega_4(\theta)$  ( $\theta \in \mathbb{R}^\times$ ) is given by

$$G_\theta = \left\{ (I_2, \alpha) \mid \alpha \in \mathbb{R}^{(2,2)} \right\} \cong \mathbb{R}^{(2,2)},$$

where  $M_\theta = \begin{pmatrix} 1 & 0 \\ 0 & \theta \end{pmatrix}$ . Therefore  $H$  is regularly embedded.

*Proof.* (a) is obvious. Since the stabilizer  $G_e$  of  $e$  is

$$G_e = \{ (g_0, \alpha) \in G \mid e = eg_0, g_0 \in SL(2, \mathbb{R}) \},$$

by an easy computation, we get

$$G_e = \left\{ \left( \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \alpha \right) \mid c \in \mathbb{R}, \alpha \in \mathbb{R}^{(2,2)} \right\}.$$

For  $x \in \Omega_1$ , we let  $x = eg_0$  with some  $g_0 \in SL(2, \mathbb{R})$ . Let  $G(x)$  be the stabilizer of  $x$ . If  $g = (g_1, \alpha) \in G(x)$  with  $g_1 \in SL(2, \mathbb{R})$ , then  $x = xg_1$  and so  $eg_0 = eg_0g_1$ . So  $(g_0, 0)(g_1, \alpha)(g_0, 0)^{-1} \in G_e$ , in other words,  $(g_1, \alpha) \in (g_0, 0)^{-1}G_e(g_0, 0)$ . Similarly we obtain (c) and (d). Let

$M_\theta \in \mathbb{R}^{(2,2)}$  such that  $\det M_\theta = \theta$ ,  $\theta \in \mathbb{R}^\times$ . Let  $G_\theta$  be the stabilizer of  $M_\theta$ . If  $(g_0, \alpha) \in G_\theta$  with  $g_0 \in SL(2, \mathbb{R})$ , then  $M_\theta g_0 = M_\theta$ . Since  $M_\theta$  is nonsingular,  $g_0 = I_2$ . Therefore we obtain (e).  $\square$

Now we obtain the following

**THEOREM 7.6.** *Let  $m = 2$ . Then the irreducible unitary representations of  $G$  are the following:*

(a) *The irreducible unitary representations  $\pi$ , where the restriction of  $\pi$  to  $H$  is trivial and the restriction of  $\pi$  to  $SL(2, \mathbb{R})$  is an irreducible unitary representation of  $SL(2, \mathbb{R})$ .*

(b) *The representations  $\pi_{e;r} := \text{Ind}_{G_e}^G \sigma_r$  ( $r \in \mathbb{R}$ ) induced from the unitary character  $\sigma_r$  of  $G_e$  defined by*

$$\sigma_r \left( \left( \left( \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \right) \right) \right) = e^{2\pi i(rc + \alpha_1)}, \quad c, \alpha_1, \dots, \alpha_4 \in \mathbb{R}.$$

(c) *The representations  $\pi_{f;r} := \text{Ind}_{G_f}^G \tau_r$  ( $r \in \mathbb{R}$ ) induced from the unitary character  $\tau_r$  of  $G_f$  defined by*

$$\tau_r \left( \left( \left( \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \right) \right) \right) = e^{2\pi i(rc + \alpha_3)}, \quad c, \alpha_1, \dots, \alpha_4 \in \mathbb{R}.$$

(d) *The representations  $\pi_{\delta;r} := \text{Ind}_{G_\delta}^G \theta_r$  ( $\delta \in \mathbb{R}^\times$ ,  $r \in \mathbb{R}$ ) induced from the unitary character  $\theta_r$  of  $G_\delta$  defined by*

$$\theta_r \left( \left( \left( \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \right) \right) \right) = e^{2\pi i(rc + \alpha_1 + \alpha_3 \delta)}, \quad c, \alpha_1, \dots, \alpha_4 \in \mathbb{R}.$$

(e) *The representations  $\pi_\theta := \text{Ind}_{G_\theta}^G \psi_\theta$  ( $\theta \in \mathbb{R}^\times$ ) induced from the unitary character  $\psi_\theta$  of  $G_\theta$  defined by*

$$\psi_\theta \left( \left( \left( I_2 \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \right) \right) \right) = e^{2\pi i(\alpha_1 + \alpha_4 \theta)}, \quad \alpha_1, \dots, \alpha_4 \in \mathbb{R}.$$

*Proof.* According to Lemma 7.4 and Lemma 7.5,  $\Omega_0$  is the  $G$ -orbit of  $\mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$  and  $G$  is the stabilizer of  $\mathbf{0}$ . We note that the representation  $\chi_0$  of  $H$  corresponding to  $\mathbf{0}$  is the trivial representation of  $H$ . Thus an element of  $G^*$  is of the form

$$\pi_\rho((g, \alpha)) = \rho \cdot \pi(g), \quad g \in SL(2, \mathbb{R}), \alpha \in \mathbb{R}^{(2,2)},$$

where  $\pi$  is an irreducible unitary representation of  $SL(2, \mathbb{R})$  and  $\rho \in \mathbb{C}_1^\times$ . We observe that  $\pi_\rho$  is unitarily equivalent to  $\pi_1$  for any  $\rho \in \mathbb{C}_1^\times$ .

From Lemma 7.4 and Lemma 7.5, we see that  $\Omega_1$  is the  $G$ -orbit of  $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$  and the stabilizer  $G_e$  of  $e$  is given by

$$G_e = \left\{ \left( \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \alpha \right) \mid c \in \mathbb{R}, \alpha \in \mathbb{R}^{(2,2)} \right\}.$$

We note that the representation  $\chi_e$  of  $H$  corresponding to  $e$  is given by  $\chi_e(\alpha) = e^{2\pi i \sigma({}^t e \alpha)}$ ,  $\alpha \in H$ , where  $\sigma({}^t e \alpha)$  denotes the trace of a  $2 \times 2$  matrix  ${}^t e \alpha$ . Precisely,

$$\chi_e \left( \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \right) = e^{2\pi i \alpha_1}, \quad \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \in H.$$

Since Lemma 7.2 holds in this case, we see that an element of  $G_e^*$  is of the form

$$\sigma_{r,\rho} \left( \left( \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \right) \right) = \rho \cdot e^{2\pi i (rc + \alpha_1)}, \quad c, \alpha_1, \dots, \alpha_4 \in \mathbb{R}$$

where  $r$  is a fixed real number and  $\rho \in \mathbb{C}_1^\times$ . We observe that  $\sigma_{r,\rho}$  is unitarily equivalent to  $\sigma_r := \sigma_{r,1}$  for any  $\rho \in \mathbb{C}_1^\times$ . Similarly  $\Omega_2$  is the  $G$ -orbit of  $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$  and the stabilizer  $G_f$  of  $f$  is given by

$$G_f = \left\{ \left( \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \alpha \right) \mid c \in \mathbb{R}, \alpha \in \mathbb{R}^{(2,2)} \right\}.$$

We see easily that the representation  $\chi_f$  of  $H$  corresponding to  $f$  is given by

$$\chi_f \left( \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \right) = e^{2\pi i \alpha_3}, \quad \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \in H.$$



Since Lemma 7.2 holds in this case, we see that an element of  $G_f^*$  is of the form

$$\tau_{r,\rho} \left( \left( \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \right) \right) = \rho \cdot e^{2\pi i(rc + \alpha_3)}, \quad c, \alpha_1, \dots, \alpha_4 \in \mathbb{R}$$

where  $r$  is a fixed real number and  $\rho \in \mathbb{C}_1^\times$ . We observe that  $\tau_{r,\rho}$  is unitarily equivalent to  $\tau_r := \tau_{r,1}$  for any  $\rho \in \mathbb{C}_1^\times$ .

According to Lemma 7.4 and Lemma 7.5,  $\Omega_3(\delta)$  ( $\delta \in \mathbb{R}^\times$ ) is the  $G$ -orbit of  $\begin{pmatrix} 1 & 0 \\ \delta & 0 \end{pmatrix}$  and the stabilizer  $G_\delta$  of  $\begin{pmatrix} 1 & 0 \\ \delta & 0 \end{pmatrix}$  is given by

$$G_\delta = \left\{ \left( \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \alpha \right) \mid c \in \mathbb{R}, \alpha \in \mathbb{R}^{(2,2)} \right\}.$$

We see easily that the representation  $\chi_\delta$  of  $H$  corresponding to  $\delta$  is given by

$$\chi_\delta \left( \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \right) = e^{2\pi i(\alpha_1 + \alpha_3 \delta)}, \quad \alpha_1, \dots, \alpha_4 \in \mathbb{R}.$$

Since Lemma 7.2 holds in this case, we see that an element of  $G_\delta^*$  is of the form

$$\theta_{r,\rho} \left( \left( \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \right) \right) = \rho \cdot e^{2\pi i(rc + \alpha_1 + \alpha_3 \delta)}, \quad c, \alpha_1, \dots, \alpha_4 \in \mathbb{R}$$

where  $r$  is a fixed real number and  $\rho \in \mathbb{C}_1^\times$ . We observe that  $\theta_{r,\rho}$  is unitarily equivalent to  $\theta_r := \theta_{r,1}$  for any  $\rho \in \mathbb{C}_1^\times$ .

It follows from Lemma 7.4 and Lemma 7.5 that  $\Omega_4(\theta)$  ( $\theta \in \mathbb{R}^\times$ ) is the  $G$ -orbit of  $M_\theta := \begin{pmatrix} 1 & 0 \\ 0 & \theta \end{pmatrix}$  and the stabilizer  $G_\theta$  of  $M_\theta$  is given by

$$G_\theta = \left\{ (I_2, \alpha) \in G \mid \alpha \in \mathbb{R}^{(2,2)} \right\}.$$

We see that the representation  $\chi_\theta$  of  $H$  corresponding to  $M_\theta$  is given by

$$\chi_\theta \left( \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \right) = e^{2\pi i(\alpha_1 + \alpha_4 \theta)}, \quad \alpha_1, \dots, \alpha_4 \in \mathbb{R}.$$

We see easily that an element of  $G_\theta^*$  is of the form

$$\psi_{\theta,\rho} \left( \left( I_2, \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \right) \right) = \rho \cdot e^{2\pi i(\alpha_1 + \alpha_4 \theta)}, \quad \alpha_1, \dots, \alpha_4 \in \mathbb{R},$$

where  $\theta$  is a fixed nonzero real number and  $\rho \in \mathbb{C}_1^\times$ . We observe that  $\psi_{\theta,\rho}$  is unitarily equivalent to  $\psi_\theta := \psi_{\theta,1}$  for any  $\rho \in \mathbb{C}_1^\times$ . By Theorem 7.1, the proof is done.  $\square$

**Case III.**  $m > 2$ .

This case is more complicated than the above cases. Here we consider only the case  $m = 3$ . The other case  $m \geq 4$  may be dealt similarly.

**LEMMA 7.7.** *Let  $m = 3$ , that is,  $G = SL_{2,3}(\mathbb{R})$  and  $H = \mathbb{R}^{(3,2)}$ . Then the  $G$ -orbits in  $\hat{H} \cong \mathbb{R}^{(3,2)}$  are given by*

$$\begin{aligned}\Omega_0 &= \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}, \\ \Omega_1 &= \left\{ \begin{pmatrix} A \\ 0 \\ 0 \end{pmatrix} \in \mathbb{R}^{(3,2)} \mid A \in \mathbb{R}^{(1,2)}, A \neq 0 \right\}, \\ \Omega_2 &= \left\{ \begin{pmatrix} 0 \\ A \\ 0 \end{pmatrix} \in \mathbb{R}^{(3,2)} \mid A \in \mathbb{R}^{(1,2)}, A \neq 0 \right\}, \\ \Omega_3 &= \left\{ \begin{pmatrix} 0 \\ 0 \\ A \end{pmatrix} \in \mathbb{R}^{(3,2)} \mid A \in \mathbb{R}^{(1,2)}, A \neq 0 \right\},\end{aligned}$$

$$\begin{aligned}\Omega(1; \delta) &= \left\{ \begin{pmatrix} 0 \\ A \\ \delta A \end{pmatrix} \in \mathbb{R}^{(3,2)} \mid A \in \mathbb{R}^{(1,2)}, A \neq 0 \right\} \quad (\delta \in \mathbb{R}^\times), \\ \Omega(2; \delta) &= \left\{ \begin{pmatrix} A \\ 0 \\ \delta A \end{pmatrix} \in \mathbb{R}^{(3,2)} \mid A \in \mathbb{R}^{(1,2)}, A \neq 0 \right\} \quad (\delta \in \mathbb{R}^\times), \\ \Omega(3; \delta) &= \left\{ \begin{pmatrix} A \\ \delta A \\ 0 \end{pmatrix} \in \mathbb{R}^{(3,2)} \mid A \in \mathbb{R}^{(1,2)}, A \neq 0 \right\} \quad (\delta \in \mathbb{R}^\times), \\ \Omega(\lambda, \mu) &= \left\{ \begin{pmatrix} A \\ \lambda A \\ \mu A \end{pmatrix} \in \mathbb{R}^{(3,2)} \mid A \in \mathbb{R}^{(1,2)}, A \neq 0 \right\} \quad (\lambda, \mu \in \mathbb{R}^\times)\end{aligned}$$

and

$$\begin{aligned} \Omega_{12}(\theta; \lambda, \mu) &= \left\{ \begin{pmatrix} A \\ B \\ \lambda A + \mu B \end{pmatrix} \in \mathbb{R}^{(3,2)} \mid \det \begin{pmatrix} A \\ B \end{pmatrix} = \theta \right\} \\ &\quad (\theta \in \mathbb{R}^\times, \lambda, \mu \in \mathbb{R}), \\ \Omega_{13}(\theta; \lambda, \mu) &= \left\{ \begin{pmatrix} A \\ \lambda A + \mu B \\ B \end{pmatrix} \in \mathbb{R}^{(3,2)} \mid \det \begin{pmatrix} A \\ B \end{pmatrix} = \theta \right\} \\ &\quad (\theta \in \mathbb{R}^\times, \lambda, \mu \in \mathbb{R}), \\ \Omega_{23}(\theta; \lambda, \mu) &= \left\{ \begin{pmatrix} \lambda A + \mu B \\ A \\ B \end{pmatrix} \in \mathbb{R}^{(3,2)} \mid \det \begin{pmatrix} A \\ B \end{pmatrix} = \theta \right\} \\ &\quad (\theta \in \mathbb{R}^\times, \lambda, \mu \in \mathbb{R}). \end{aligned}$$

*Proof.* We recall that  $G = SL(2, \mathbb{R}) \ltimes \mathbb{R}^{(3,2)}$  acts on  $\hat{H} \cong \mathbb{R}^{(3,2)}$  via (7.2). If we identify  $\hat{H}$  with  $\mathbb{R}^{(3,2)}$ , we know that  $G$  acts on  $\mathbb{R}^{(3,2)}$  by

$$(g_0, \alpha) \cdot X := Xg_0^{-1}, \quad (g_0, \alpha) \in G, \quad g_0 \in SL(2, \mathbb{R}), \quad X \in \mathbb{R}^{(3,2)}.$$

Considering the rank of a  $3 \times 2$  real matrix, we see without difficulty that all the  $G$ -orbits in  $\mathbb{R}^{(3,2)}$  are given by  $\Omega_0, \Omega_1, \Omega_2, \Omega_3, \Omega(1; \delta), \Omega(2; \delta), \Omega(3; \delta), \Omega(\lambda_0, \mu_0)$ , where  $\delta, \lambda_0, \mu_0 \in \mathbb{R}^\times$  and  $\Omega_{12}(\theta; \lambda, \mu), \Omega_{13}(\theta; \lambda, \mu), \Omega_{23}(\theta; \lambda, \mu)$ , where  $\theta \in \mathbb{R}^\times, \lambda, \mu \in \mathbb{R}$ .  $\square$

We put

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and for each  $\delta \in \mathbb{R}^\times$

$$\begin{aligned} f_{1,\delta} &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ \delta & 0 \end{pmatrix}, \quad f_{2,\delta} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ \delta & 0 \end{pmatrix}, \quad f_{3,\delta} = \begin{pmatrix} 1 & 0 \\ \delta & 0 \\ 0 & 0 \end{pmatrix}, \\ f_{\lambda,\mu} &= \begin{pmatrix} 1 & 0 \\ \lambda & 0 \\ \mu & 0 \end{pmatrix} \quad (\lambda, \mu \in \mathbb{R}^\times). \end{aligned}$$

We also set for each  $\theta \in \mathbb{R}^\times$ ,  $\lambda, \mu \in \mathbb{R}$ ,

$$h_{12}(\theta; \lambda, \mu) = \begin{pmatrix} 1 & 0 \\ 0 & \theta \\ \lambda & \mu\theta \end{pmatrix}, \quad h_{13}(\theta; \lambda, \mu) = \begin{pmatrix} 1 & 0 \\ \lambda & \mu\theta \\ 0 & \theta \end{pmatrix}$$

and

$$h_{23}(\theta; \lambda, \mu) = \begin{pmatrix} \lambda & \mu\theta \\ 1 & 0 \\ 0 & \theta \end{pmatrix}.$$

We note that  $\mathbf{0} \in \Omega_0$ ,  $e_i \in \Omega_i$  ( $i = 1, 2, 3$ ),  $f_{j,\delta} \in \Omega(j; \delta)$  ( $j = 1, 2, 3$ ),  $f_{\lambda,\mu} \in \Omega(\lambda, \mu)$ ,  $h_{12}(\theta; \lambda, \mu) \in \Omega_{12}(\theta; \lambda, \mu)$ ,  $h_{13}(\theta; \lambda, \mu) \in \Omega_{13}(\theta; \lambda, \mu)$ ,  $h_{23}(\theta; \lambda, \mu) \in \Omega_{23}(\theta; \lambda, \mu)$ .

Then we may prove the following lemma without difficulty.

LEMMA 7.8. (a) *The stabilizer of  $\mathbf{0}$  is  $G$ .*

(b) *Let  $G_i$  be the stabilizer of  $e_i$  ( $i = 1, 2, 3$ ). Then*

$$G_i = \left\{ \left( \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \alpha \right) \mid c \in \mathbb{R}, \alpha \in \mathbb{R}^{(3,2)} \right\}, \quad i = 1, 2, 3.$$

*For each  $x_i \in \Omega_i$  ( $i = 1, 2, 3$ ), the stabilizer  $G(x_i)$  of  $x_i$  is conjugate to  $G_i$ . Precisely if  $x_i = e_i g_i$  with  $g_i \in SL(2, \mathbb{R})$ , then  $G(x_i) = (g_i, 0)^{-1} G_i (g_i, 0)$ .*

(c) *For  $i = 1, 2, 3$  and  $\delta \in \mathbb{R}^\times$ , we let  $G_{i,\delta}$  be the stabilizer of  $f_{i,\delta}$  ( $i = 1, 2, 3$ ). Then*

$$G_{i,\delta} = \left\{ \left( \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \alpha \right) \mid c \in \mathbb{R}, \alpha \in \mathbb{R}^{(3,2)} \right\}, \quad i = 1, 2, 3, \delta \in \mathbb{R}^\times.$$

*For each  $y_i \in \Omega(i; \delta)$  ( $i = 1, 2, 3$ ), the stabilizer  $G(y_i)$  of  $y_i$  is conjugate to  $G_{i,\delta}$ .*

(d) *For any  $\lambda, \mu \in \mathbb{R}^\times$ , we let  $G_{\lambda,\mu}$  be the stabilizer of  $f_{\lambda,\mu}$ . Then*

$$G_{\lambda,\mu} = \left\{ \left( \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \alpha \right) \mid c \in \mathbb{R}, \alpha \in \mathbb{R}^{(3,2)} \right\} \quad (\lambda, \mu \in \mathbb{R}^\times).$$

(e) *For any  $\theta \in \mathbb{R}^\times$ , we let  $G_{12}(\theta; \lambda, \mu)$ ,  $G_{13}(\theta; \lambda, \mu)$ ,  $G_{23}(\theta; \lambda, \mu)$  be the stabilizers of  $h_{12}(\theta; \lambda, \mu)$ ,  $h_{13}(\theta; \lambda, \mu)$ ,  $h_{23}(\theta; \lambda, \mu)$  respectively. Then*

$$G_{12}(\theta; \lambda, \mu) = G_{13}(\theta; \lambda, \mu) = G_{23}(\theta; \lambda, \mu) = \left\{ (I_2, \alpha) \mid \alpha \in \mathbb{R}^{(3,2)} \right\}.$$

Therefore we see easily that  $H$  is regularly embedded.

*Proof.* (a) is obvious. Since the stabilizer  $G_i$  of  $e_i$  ( $i = 1, 2, 3$ ) is given by

$$G_i = \left\{ (g_0, \alpha) \in G \mid e_i = e_i g_0, g_0 \in SL(2, \mathbb{R}), \alpha \in \mathbb{R}^{(3,2)} \right\},$$

by an easy computation, we get

$$G_i = \left\{ \left( \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \alpha \right) \mid c \in \mathbb{R}, \alpha \in \mathbb{R}^{(3,2)} \right\}, \quad i = 1, 2, 3.$$

For  $x_i \in \Omega_i$ , we let  $x_i = e_i g_i$  with some  $g_i \in SL(2, \mathbb{R})$  ( $i = 1, 2, 3$ ). Let  $G(x_i)$  be the stabilizer of  $x_i$ . if  $g = (g_0, \alpha) \in G(x_i)$  with  $g_0 \in SL(2, \mathbb{R})$  and  $\alpha \in \mathbb{R}^{(3,2)}$ , then  $x_i = x_i g_0$  and so  $e_i g_i = e_i g_i g_0$ . Thus  $(g_0, \alpha) \in (g_i, 0)^{-1} G_i (g_i, 0)$ . In a similar way, we obtain (c) and (d). Let  $G_{12}(\theta; \lambda, \mu)$  be the stabilizer  $h_{12}(\theta; \lambda, \mu)$  ( $\theta \in \mathbb{R}^\times, \lambda, \mu \in \mathbb{R}$ ). Then if  $g = (g_0, \alpha) \in G_{12}(\theta; \lambda, \mu)$  with  $g_0 \in SL(2, \mathbb{R})$ , then  $h_{12}(\theta; \lambda, \mu) = h_{12}(\theta; \lambda, \mu) g_0$ . By an easy computation,  $g_0 = I_2$ . Therefore we get

$$G_{12}(\theta; \lambda, \mu) = \left\{ (I_2, \alpha) \mid \alpha \in \mathbb{R}^{(3,2)} \right\}.$$

Similarly we see that the stabilizers  $G_{13}(\theta; \lambda, \mu)$  and  $G_{23}(\theta; \lambda, \mu)$  of  $h_{13}(\theta; \lambda, \mu)$  and  $h_{23}(\theta; \lambda, \mu)$  respectively are given by

$$G_{13}(\theta; \lambda, \mu) = G_{23}(\theta; \lambda, \mu) = \left\{ (I_2, \alpha) \mid \alpha \in \mathbb{R}^{(3,2)} \right\}.$$

□

Finally we obtain the following theorem.

**THEOREM 7.9.** *Let  $m = 3$ . Then the irreducible unitary representations of  $G$  are the following:*

(a) *The irreducible unitary representations  $\pi$ , where the restriction of  $\pi$  to  $H$  is trivial and the restriction of  $\pi$  to  $SL(2, \mathbb{R})$  is an irreducible unitary representation of  $SL(2, \mathbb{R})$ .*

(b) *The representations  $\pi_{1,r} := \text{Ind}_{G_1}^G \sigma_{1,r}$  ( $r \in \mathbb{R}$ ) induced from the unitary character  $\sigma_{1,r}$  of  $G_1$  defined by*

$$\sigma_{1,r} \left( \left( \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \\ \alpha_5 & \alpha_6 \end{pmatrix} \right) \right) = e^{2\pi i (rc + \alpha_1)}, \quad c, \alpha_1, \dots, \alpha_6 \in \mathbb{R}.$$

(c) The representations  $\pi_{2,r} := \text{Ind}_{G_2}^G \sigma_{2,r}$  ( $r \in \mathbb{R}$ ) induced from the unitary character  $\sigma_{2,r}$  of  $G_2$  defined by

$$\sigma_{2,r} \left( \left( \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \\ \alpha_5 & \alpha_6 \end{pmatrix} \right) \right) = e^{2\pi i(rc + \alpha_3)}, \quad c, \alpha_1, \dots, \alpha_6 \in \mathbb{R}.$$

(d) The representations  $\pi_{3,r} := \text{Ind}_{G_3}^G \sigma_{3,r}$  ( $r \in \mathbb{R}$ ) induced from the unitary character  $\sigma_{3,r}$  of  $G_1$  defined by

$$\sigma_{3,r} \left( \left( \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \\ \alpha_5 & \alpha_6 \end{pmatrix} \right) \right) = e^{2\pi i(rc + \alpha_5)}, \quad c, \alpha_1, \dots, \alpha_6 \in \mathbb{R}.$$

(e) The representations  $\pi_{(1,\delta),r} := \text{Ind}_{G_{1,\delta}}^G \tau_{(1,\delta),r}$  ( $r \in \mathbb{R}, \delta \in \mathbb{R}^\times$ ) induced from the unitary character  $\tau_{(1,\delta),r}$  of  $G_{1,\delta}$  defined by

$$\tau_{(1,\delta),r} \left( \left( \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \\ \alpha_5 & \alpha_6 \end{pmatrix} \right) \right) = e^{2\pi i(rc + \alpha_3 + \alpha_5 \delta)},$$

$$c, \alpha_1, \dots, \alpha_6 \in \mathbb{R}.$$

(f) The representations  $\pi_{(2,\delta),r} := \text{Ind}_{G_{2,\delta}}^G \tau_{(2,\delta),r}$  ( $r \in \mathbb{R}, \delta \in \mathbb{R}^\times$ ) induced from the unitary character  $\tau_{(2,\delta),r}$  of  $G_{2,\delta}$  defined by

$$\tau_{(2,\delta),r} \left( \left( \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \\ \alpha_5 & \alpha_6 \end{pmatrix} \right) \right) = e^{2\pi i(rc + \alpha_1 + \alpha_5 \delta)},$$

$$c, \alpha_1, \dots, \alpha_6 \in \mathbb{R}.$$

(g) The representations  $\pi_{(3,\delta),r} := \text{Ind}_{G_{3,\delta}}^G \tau_{(3,\delta),r}$  ( $r \in \mathbb{R}, \delta \in \mathbb{R}^\times$ ) induced from the unitary character  $\tau_{(3,\delta),r}$  of  $G_{3,\delta}$  defined by

$$\tau_{(3,\delta),r} \left( \left( \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \\ \alpha_5 & \alpha_6 \end{pmatrix} \right) \right) = e^{2\pi i(rc + \alpha_1 + \alpha_3 \delta)},$$

$$c, \alpha_1, \dots, \alpha_6 \in \mathbb{R}.$$

(h) The representations  $\pi_{(r;\lambda,\mu)} := \text{Ind}_{G_{\lambda,\mu}}^G \varphi_{(\lambda,\mu),r}$  ( $r \in \mathbb{R}$ ,  $\lambda, \mu \in \mathbb{R}^\times$ ) induced from the unitary character  $\varphi_{(\lambda,\mu),r}$  of  $G_{\lambda,\mu}$  defined by

$$\varphi_{(\lambda,\mu),r} \left( \left( \left( \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \\ \alpha_5 & \alpha_6 \end{pmatrix} \right) \right) \right) = e^{2\pi i(r c + \alpha_1 + \alpha_3 \lambda + \alpha_5 \mu)},$$

$c, \alpha_1, \dots, \alpha_6 \in \mathbb{R}.$

(i) The representations  $\pi_{(12;\theta,\lambda,\mu)} := \text{Ind}_{G_{12}(\theta;\lambda,\mu)}^G \psi_{(12;\theta,\lambda,\mu)}$  ( $\theta \in \mathbb{R}^\times$ ,  $\lambda, \mu \in \mathbb{R}$ ) induced from the unitary character  $\psi_{(12;\theta,\lambda,\mu)}$  of  $G_{12}(\theta; \lambda, \mu)$  defined by

$$\psi_{(12;\theta,\lambda,\mu)} \left( \left( \left( I_2, \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \\ \alpha_5 & \alpha_6 \end{pmatrix} \right) \right) \right) = e^{2\pi i(\alpha_1 + \alpha_5 \lambda + (\alpha_4 + \alpha_6 \mu)\theta)},$$

$\alpha_1, \dots, \alpha_6 \in \mathbb{R}.$

(j) The representations  $\pi_{(13;\theta,\lambda,\mu)} := \text{Ind}_{G_{13}(\theta;\lambda,\mu)}^G \psi_{(13;\theta,\lambda,\mu)}$  ( $\theta \in \mathbb{R}^\times$ ,  $\lambda, \mu \in \mathbb{R}$ ) induced from the unitary character  $\psi_{(13;\theta,\lambda,\mu)}$  of  $G_{13}(\theta; \lambda, \mu)$  defined by

$$\psi_{(13;\theta,\lambda,\mu)} \left( \left( \left( I_2, \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \\ \alpha_5 & \alpha_6 \end{pmatrix} \right) \right) \right) = e^{2\pi i(\alpha_1 + \alpha_3 \lambda + (\alpha_6 + \alpha_4 \mu)\theta)},$$

$\alpha_1, \dots, \alpha_6 \in \mathbb{R}.$

(k) The representations  $\pi_{(23;\theta,\lambda,\mu)} := \text{Ind}_{G_{23}(\theta;\lambda,\mu)}^G \psi_{(23;\theta,\lambda,\mu)}$  ( $\theta \in \mathbb{R}^\times$ ,  $\lambda, \mu \in \mathbb{R}$ ) induced from the unitary character  $\psi_{(23;\theta,\lambda,\mu)}$  of  $G_{23}(\theta; \lambda, \mu)$  defined by

$$\psi_{(23;\theta,\lambda,\mu)} \left( \left( \left( I_2, \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \\ \alpha_5 & \alpha_6 \end{pmatrix} \right) \right) \right) = e^{2\pi i(\alpha_3 + \alpha_1 \lambda + (\alpha_6 + \alpha_2 \mu)\theta)},$$

$\alpha_1, \dots, \alpha_6 \in \mathbb{R}.$

*Proof.* According to Lemma 7.7 and Lemma 7.8,  $\Omega_0$  is the  $G$ -orbit of  $\mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$  and  $G$  is the stabilizer of  $\mathbf{0}$ . We see that the representation  $\chi_0$  of  $H$  corresponding to  $\mathbf{0}$  is the trivial representation of  $H$ . Thus an element of  $G^* = G_0^*$  is of the form

$$\pi_\rho((g, \alpha)) = \rho \cdot \pi(g), \quad g \in SL(2, \mathbb{R}), \alpha \in \mathbb{R}^{(3,2)},$$

where  $\pi$  is an irreducible unitary representation of  $SL(2, \mathbb{R})$  and  $\rho \in \mathbb{C}_1^\times$ . We observe that  $\pi_\rho$  is unitarily equivalent to  $\pi_1 = \pi$  for any  $\rho \in \mathbb{C}_1^\times$ . According to Lemma 7.7 and Lemma 7.8, we see that  $\Omega_1, \Omega_2, \Omega_3$  are the  $G$ -orbits of  $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ ,  $e_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}$ ,  $e_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}$  respectively and the stabilizers  $G_i$  of  $e_i$  ( $i = 1, 2, 3$ ) are given by

$$G_i = \left\{ \left( \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \alpha \right) \mid c \in \mathbb{R}, \alpha \in \mathbb{R}^{(3,2)} \right\}, \quad i = 1, 2, 3.$$

We note that the representations  $\chi_{e_i}$  of  $H$  corresponding to  $e_i$  ( $i = 1, 2, 3$ ) are given respectively by

$$\begin{aligned} \chi_{e_1} \left( \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \\ \alpha_5 & \alpha_6 \end{pmatrix} \right) &= e^{2\pi i \alpha_1}, \\ \chi_{e_2} \left( \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \\ \alpha_5 & \alpha_6 \end{pmatrix} \right) &= e^{2\pi i \alpha_3}, \\ \chi_{e_3} \left( \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \\ \alpha_5 & \alpha_6 \end{pmatrix} \right) &= e^{2\pi i \alpha_5}, \end{aligned}$$

where  $\alpha_1, \dots, \alpha_6 \in \mathbb{R}$ . Since Lemma 7.2 holds in these cases, we see that an element of  $G_1^*$  is of the form

$$\sigma_{1,r,\rho} \left( \left( \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \\ \alpha_5 & \alpha_6 \end{pmatrix} \right) \right) = \rho \cdot e^{2\pi i (rc + \alpha_1)}, \quad c, \alpha_1, \dots, \alpha_6 \in \mathbb{R},$$



and an element of  $G_2^*$  is of the form

$$\sigma_{2,r,\rho} \left( \left( \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \\ \alpha_5 & \alpha_6 \end{pmatrix} \right) \right) = \rho \cdot e^{2\pi i(r c + \alpha_3)}, \quad c, \alpha_1, \dots, \alpha_6 \in \mathbb{R},$$

and an element of  $G_3^*$  is of the form

$$\sigma_{3,r,\rho} \left( \left( \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \\ \alpha_5 & \alpha_6 \end{pmatrix} \right) \right) = \rho \cdot e^{2\pi i(r c + \alpha_3)}, \quad c, \alpha_1, \dots, \alpha_6 \in \mathbb{R},$$

where  $r$  is a real number and  $\rho \in \mathbb{C}_1^\times$ . We observe that  $\sigma_{i,r,\rho}$  is unitarily equivalent to  $\sigma_{i,r} := \sigma_{i,r,1}$  ( $i = 1, 2, 3$ ) for any  $\rho \in \mathbb{C}_1^\times$ .

It follows from Lemma 7.7 and Lemma 7.8 that  $\Omega(1; \delta)$ ,  $\Omega(2; \delta)$ ,  $\Omega(3; \delta)$  ( $\delta \in \mathbb{R}^\times$ ) are the  $G$ -orbits of

$$f_{1,\delta} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ \delta & 0 \end{pmatrix}, \quad f_{2,\delta} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ \delta & 0 \end{pmatrix}, \quad f_{3,\delta} = \begin{pmatrix} 1 & 0 \\ \delta & 0 \\ 0 & 0 \end{pmatrix}$$

respectively and the stabilizers  $G_{i,\delta}$  of  $f_{i,\delta}$  ( $i = 1, 2, 3$ ) are given by

$$G_{i,\delta} = \left\{ \left( \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \alpha \right) \mid c \in \mathbb{R}, \alpha \in \mathbb{R}^{(3,2)} \right\}, \quad i = 1, 2, 3, \delta \in \mathbb{R}^\times.$$

We see easily that the representations  $\chi_{f_{i,\delta}}$  of  $H$  corresponding to  $f_{i,\delta}$  ( $i = 1, 2, 3$ ) are given respectively by

$$\chi_{f_{1,\delta}} \left( \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \\ \alpha_5 & \alpha_6 \end{pmatrix} \right) = e^{2\pi i(\alpha_3 + \alpha_5 \delta)}, \quad \alpha_1, \dots, \alpha_6 \in \mathbb{R},$$

$$\chi_{f_{2,\delta}} \left( \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \\ \alpha_5 & \alpha_6 \end{pmatrix} \right) = e^{2\pi i(\alpha_1 + \alpha_5 \delta)}, \quad \alpha_1, \dots, \alpha_6 \in \mathbb{R}$$

and

$$\chi_{f_{3,\delta}} \left( \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \\ \alpha_5 & \alpha_6 \end{pmatrix} \right) = e^{2\pi i(\alpha_1 + \alpha_3 \delta)}, \quad \alpha_1, \dots, \alpha_6 \in \mathbb{R}.$$

Since Lemma 7.2 holds in these cases, we see that an element of  $G_{1,\delta}^*$  is of the form

$$\tau_{(1,\delta),r,\rho} \left( \left( \left( \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \\ \alpha_5 & \alpha_6 \end{pmatrix} \right) \right) \right) = \rho \cdot e^{2\pi i(r c + \alpha_3 + \alpha_5 \delta)}$$

and an element of  $G_{2,\delta}^*$  is of the form

$$\tau_{(2,\delta),r,\rho} \left( \left( \left( \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \\ \alpha_5 & \alpha_6 \end{pmatrix} \right) \right) \right) = \rho \cdot e^{2\pi i(r c + \alpha_1 + \alpha_5 \delta)}$$

and an element of  $G_{3,\delta}^*$  is of the form

$$\tau_{(3,\delta),r,\rho} \left( \left( \left( \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \\ \alpha_5 & \alpha_6 \end{pmatrix} \right) \right) \right) = \rho \cdot e^{2\pi i(r c + \alpha_1 + \alpha_3 \delta)},$$

where  $c, \alpha_1, \dots, \alpha_6 \in \mathbb{R}$ ,  $\rho \in \mathbb{C}_1^\times$  and  $r$  is a real number. We observe that  $\tau_{(i,\delta),r,\rho}$  is unitarily equivalent to  $\tau_{(i,\delta),r} := \tau_{(i,\delta),r,1}$  ( $i = 1, 2, 3$ ) for any  $\rho \in \mathbb{C}_1^\times$ . And according to Lemma 7.7 and Lemma 7.8 that

$\Omega(\lambda, \mu)$  ( $\lambda, \mu \in \mathbb{R}^\times$ ) is the  $G$ -orbit of  $f_{\lambda, \mu} = \begin{pmatrix} 1 & 0 \\ \lambda & 0 \\ \mu & 0 \end{pmatrix}$  and the stabilizer

$G_{\lambda, \mu}$  of  $f_{\lambda, \mu}$  is given by

$$G_{\lambda, \mu} = \left\{ \left( \left( \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \alpha \right) \mid c \in \mathbb{R}, \alpha \in \mathbb{R}^{(3,2)} \right\}.$$

We see easily that the representation  $\chi_{f_{\lambda, \mu}}$  of  $H$  corresponding to  $f_{\lambda, \mu}$  is given by

$$\chi_{f_{\lambda, \mu}} \left( \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \\ \alpha_5 & \alpha_6 \end{pmatrix} \right) = e^{2\pi i(\alpha_1 + \alpha_3 \lambda + \alpha_5 \mu)}, \quad \alpha_1, \dots, \alpha_6 \in \mathbb{R}.$$

Since Lemma 7.2 holds in these cases, we see that an element of  $G_{\lambda, \mu}^*$  is of the form

$$\varphi_{(\lambda, \mu), r, \rho} \left( \left( \left( \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \\ \alpha_5 & \alpha_6 \end{pmatrix} \right) \right) \right) = \rho \cdot e^{2\pi i(r c + \alpha_1 + \alpha_3 \lambda + \alpha_5 \mu)},$$

where  $c, \alpha_1, \dots, \alpha_6 \in \mathbb{R}$ ,  $\rho \in \mathbb{C}_1^\times$  and  $r$  is a real number. We observe that  $\varphi_{(\lambda, \mu), r, \rho}$  is unitarily equivalent to  $\varphi_{(\lambda, \mu), r} := \varphi_{(\lambda, \mu), r, 1}$  for any  $\rho \in \mathbb{C}_1^\times$ .

It follows from Lemma 7.7 and Lemma 7.8 that  $\Omega_{12}(\theta; \lambda, \mu), \Omega_{13}(\theta; \lambda, \mu), \Omega_{23}(\theta; \lambda, \mu)$  ( $\theta \in \mathbb{R}^\times, \lambda, \mu \in \mathbb{R}$ ) are the  $G$ -orbits of

$$h_{12}(\theta; \lambda, \mu) = \begin{pmatrix} 1 & 0 \\ 0 & \theta \\ \lambda & \mu\theta \end{pmatrix}, \quad h_{13}(\theta; \lambda, \mu) = \begin{pmatrix} 1 & 0 \\ \lambda & \mu\theta \\ 0 & \theta \end{pmatrix},$$

$$h_{23}(\theta; \lambda, \mu) = \begin{pmatrix} \lambda & \mu\theta \\ 1 & 0 \\ 0 & \theta \end{pmatrix}$$

respectively and the stabilizers  $G_{12}(\theta; \lambda, \mu), G_{13}(\theta; \lambda, \mu), G_{23}(\theta; \lambda, \mu)$  of  $h_{12}(\theta; \lambda, \mu), h_{13}(\theta; \lambda, \mu), h_{23}(\theta; \lambda, \mu)$  respectively are given by

$$G_{12}(\theta; \lambda, \mu) = G_{13}(\theta; \lambda, \mu) = G_{23}(\theta; \lambda, \mu) = \left\{ (I_2, \alpha) \mid \alpha \in \mathbb{R}^{(3,2)} \right\}.$$

We see easily that the representations  $\chi_{(12; \theta, \lambda, \mu)}, \chi_{(13; \theta, \lambda, \mu)}, \chi_{(23; \theta, \lambda, \mu)}$  of  $H$  corresponding to  $h_{12}(\theta; \lambda, \mu), h_{13}(\theta; \lambda, \mu), h_{23}(\theta; \lambda, \mu)$  respectively are given by

$$\chi_{(12; \theta, \lambda, \mu)} \left( \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \\ \alpha_5 & \alpha_6 \end{pmatrix} \right) = e^{2\pi i(\alpha_1 + \alpha_5\lambda + (\alpha_4 + \alpha_6\mu)\theta)}, \quad \alpha_1, \dots, \alpha_6 \in \mathbb{R},$$

$$\chi_{(13; \theta, \lambda, \mu)} \left( \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \\ \alpha_5 & \alpha_6 \end{pmatrix} \right) = e^{2\pi i(\alpha_1 + \alpha_3\lambda + (\alpha_6 + \alpha_4\mu)\theta)}, \quad \alpha_1, \dots, \alpha_6 \in \mathbb{R},$$

and

$$\chi_{(23; \theta, \lambda, \mu)} \left( \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \\ \alpha_5 & \alpha_6 \end{pmatrix} \right) = e^{2\pi i(\alpha_3 + \alpha_1\lambda + (\alpha_6 + \alpha_2\mu)\theta)}, \quad \alpha_1, \dots, \alpha_6 \in \mathbb{R}.$$

It is easy to see that an element of  $G_{12}(\theta; \lambda, \mu)^*$  is of the form

$$\psi_{(12; \theta, \lambda, \mu), \rho} \left( \left( I_2, \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \\ \alpha_5 & \alpha_6 \end{pmatrix} \right) \right) = \rho \cdot e^{2\pi i(\alpha_1 + \alpha_5\lambda + (\alpha_4 + \alpha_6\mu)\theta)},$$

where  $\alpha_1, \dots, \alpha_6 \in \mathbb{R}$  and  $\rho \in \mathbb{C}_1^\times$ . We observe that  $\psi_{(12;\theta,\lambda,\mu),\rho}$  is unitarily equivalent to  $\psi_{(12;\theta,\lambda,\mu)} := \psi_{(12;\theta,\lambda,\mu),1}$  for any  $\rho \in \mathbb{C}_1^\times$ . We also see that an element of  $G_{13}(\theta; \lambda, \mu)^*$  is of the form

$$\psi_{(13;\theta,\lambda,\mu),\rho} \left( \left( I_2, \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \\ \alpha_5 & \alpha_6 \end{pmatrix} \right) \right) = \rho \cdot e^{2\pi i(\alpha_1 + \alpha_3\lambda + (\alpha_6 + \alpha_4\mu)\theta)},$$

where  $\alpha_1, \dots, \alpha_6 \in \mathbb{R}$  and  $\rho \in \mathbb{C}_1^\times$ . We observe that  $\psi_{(13;\theta,\lambda,\mu),\rho}$  is unitarily equivalent to  $\psi_{(13;\theta,\lambda,\mu)} := \psi_{(13;\theta,\lambda,\mu),1}$  for any  $\rho \in \mathbb{C}_1^\times$ . And we also see that an element of  $G_{23}(\theta; \lambda, \mu)^*$  is of the form

$$\psi_{(23;\theta,\lambda,\mu),\rho} \left( \left( I_2, \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \\ \alpha_5 & \alpha_6 \end{pmatrix} \right) \right) = \rho \cdot e^{2\pi i(\alpha_3 + \alpha_1\lambda + (\alpha_6 + \alpha_2\mu)\theta)},$$

where  $\alpha_1, \dots, \alpha_6 \in \mathbb{R}$  and  $\rho \in \mathbb{C}_1^\times$ . We observe that  $\psi_{(23;\theta,\lambda,\mu),\rho}$  is unitarily equivalent to  $\psi_{(23;\theta,\lambda,\mu)} := \psi_{(23;\theta,\lambda,\mu),1}$  for any  $\rho \in \mathbb{C}_1^\times$ . By Theorem 7.1, we complete the proof.  $\square$

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