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NONCOMMUTATIVE CONTINUOUS FUNCTIONS

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ABSTRACT. By forming completions of families of noncommutative polynomials, we define a notion of noncommutative continuous function and locally bounded Borel function that give a noncommutative analogue of the functional calculus for elements of commutative C*-algebras and von Neumann algebras. These notions give a precise meaning to C*-algebras defined by generator and relations and we show how they relate to many parts of operator and operator algebra theory.

1. Introduction

It is convenient and common to think of the elements of a C*-algebra or a W*-algebra generated by elements $\{a_1, a_2, \ldots\}$ to be "functions" of $a_1, a_2, \ldots$. In the commutative case this can be made explicit. We want to put this notion on a useful and solid mathematical foundation in the noncommutative case.

Consider the algebra $\mathcal{C}$ of continuous complex-valued functions on the set $\mathbb{C}$ of complex numbers. The algebra $\mathcal{C}$ has an identity and an involution (conjugation). Furthermore, with the topology of uniform convergence on compact subsets, $\mathcal{C}$ is topologized by a countable family $\{||\ ||_n\}$ of $*$-seminorms defined by

$$||\phi||_n = \sup\{\,|\phi(z)|\ : z \in \mathbb{C}, |z| \le n\,\}.$$

The set of $*$-polynomials in z (i.e. polynomials in z and $\bar{z}$) is dense in $\mathcal{C}$. Hence, relative to the topology induced by the seminorms above, $\mathcal{C}$ is the closed unital $*$-algebra generated by the function $\mathbf{z}$ defined by $\mathbf{z}(\lambda) = \lambda$. However, this algebra is also closed under the operation of composition,

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which adds to its structure. There is also a natural functional calculus for normal operators, in the sense that if T is any normal operator on a Hilbert Space $\mathcal{H}$, then there is a unique unital $*$-homomorphism, $\phi \mapsto \phi(T)$, from $\mathcal{C}$ into $B(\mathcal{H})$ that sends $\mathbf{z}$ to T and is continuous with respect to the topology of uniform convergence on compact sets on $\mathcal{C}$ and the norm topology on $B(\mathcal{H})$. In fact, this functional calculus makes sense when T is any normal element of a unital C*-algebra. Moreover, this functional calculus satisfies the following:

1. $(\phi \circ \psi)(T) = \phi(\psi(T))$ for every $\phi, \psi \in \mathcal{C}$ and every normal operator T.
2. $C^*(T) = \{\phi(T) \ : \phi \in \mathcal{C}\}$.
3. $\pi(\phi(T)) = \phi(\pi(T))$ for every normal operator T, every $\phi \in \mathcal{C}$, and every unital $*$-homomorphism π of $C^*(T)$.

For non-unital algebras, analogous results hold when we replace $\mathcal{C}$ with $\{\phi \in \mathcal{C} \ : \phi(0) = 0\}$. Similar results hold for m variables (m a cardinal), if we replace $\mathbb{C}$ with a Cartesian product $\mathbb{C}^m$ of m copies of $\mathbb{C}$ with the product topology, and replace $\mathcal{C}$ with the set $\mathcal{C}_m$ of all continuous complex-valued functions on $\mathbb{C}^m$.

We also have the $*$-subalgebra $\mathcal{C}^b$ of bounded functions in $\mathcal{C}$. We can obtain the functions in $\mathcal{C}^b$ as truncations of functions in $\mathcal{C}$. More precisely, if $r > 0$, we can define the function $\tau_r \in \mathcal{C}$ by $\tau_r(z) = z$ if $|z| \leq r$, and $\tau_r(z) = \frac{rz}{|z|}$ if $|z| > r$. Then

$$\mathcal{C}^b = \{\tau_r \circ \phi \ : r > 0, \phi \in \mathcal{C}\}.$$

The truncation relates to the functional calculus so that if T is a normal operator, $\phi \in \mathcal{C}$, and $r \geq ||\phi(T)||$, then $(\tau_r \circ \phi)(T) = \phi(T)$.

One of the nice things about the functional calculus is that it reduces problems about normal elements (families) in C*-algebras to problems about continuous functions. Another nice thing about this functional calculus is that it gives a way of talking about "comparable" or "analogous" elements in two C*-algebras generated by normal elements, a, b, i.e., if $\phi \in \mathcal{C}$, then $\phi(a)$ in $C^*(a)$ somehow seems to "correspond" to $\phi(b)$ in $C^*(b)$.

In this note we provide a non-commutative analogue of the space $\mathcal{C}_m$. We show that a special case of our construction may be identified with the continuous decomposable functions defined in [3] and studied in [6]. We also provide an analogous construction where, in the single variable case, singly generated von Neumann algebras play the central role and the objects obtained are identifiable with the decomposable functions defined in [3].

Suppose $\mathcal{X}$ is a nonempty set and let $\mathbb{P}(\mathcal{X})$ denote the free complex unital involutive algebra generated by $\mathcal{X}$. A typical element of $\mathbb{P}(\mathcal{X})$ is a polynomial in free variables $x_1, x_1^*, x_2, x_2^*, \ldots, x_n, x_n^*$, where $x_i \in \mathcal{X}$ for each $i = 1, 2, \ldots, n$. We refer to the elements of $\mathbb{P}(\mathcal{X})$ as non-commutative $*$-polynomials. We define $\mathcal{F}(\mathcal{X})$ to be the class of functions $f : \mathcal{X} \to B(\mathcal{H}_f)$, where $B(\mathcal{H}_f)$ denotes the algebra of all operators on a Hilbert space $\mathcal{H}_f$. Thus the elements of $\mathcal{F}(\mathcal{X})$ are in one-to-one correspondence with the algebraic representations (i.e. unital $*$-homomorphisms) of $\mathbb{P}(\mathcal{X})$ as operators on some Hilbert space. Given $p \in \mathbb{P}(\mathcal{X})$ and $f \in \mathcal{F}(\mathcal{X})$, we write $p(f)$ to denote the image of p under the representation determined by f, i.e. $p(f)$ is the polynomial p with each indeterminate x replaced by the operator $f(x)$ and each x^* replaced by $f(x)^*$. For example, if $\mathcal{X} = \{x\}$ is a singleton, then $\mathcal{F}(\mathcal{X})$ can be identified with the class of all Hilbert Space operators. If $p(x)$ is the polynomial $x^2 + 5x(x^*)^3$, and $f \in \mathcal{F}(\mathcal{X})$ with $f(x) = T$, then $p(f) = T^2 + 5T(T^*)^3$.

Let $\mathcal{N}(\mathcal{X})$ denote the family of functions mapping $\mathcal{X}$ into $[0, \infty)$; the elements of $\mathcal{N}(\mathcal{X})$ will be called *$\mathcal{X}$-norms.* If $n_1, n_2 \in \mathcal{N}(\mathcal{X})$ we will write $n_1 \leq n_2$ in case $n_1(x) \leq n_2(x)$ for all $x \in \mathcal{X}$. Given $f \in \mathcal{F}(\mathcal{X})$ we define $n_f \in \mathcal{N}(\mathcal{X})$ by $n_f(x) = ||f(x)||$, and call n_f the $\mathcal{X}$-norm of f.

There are several ways of generating new elements of $\mathcal{F}(\mathcal{X})$ from old ones. The two most important for our considerations are via direct sums and unitary equivalence. Specifically, if f and g are elements of $\mathcal{F}(\mathcal{X})$, define $f \oplus g \in \mathcal{F}(\mathcal{X})$ by

$$(f \oplus g)(x) \equiv f(x) \oplus g(x),$$

so one has $\mathcal{H}_{f \oplus g} = \mathcal{H}_f \oplus \mathcal{H}_g$ as well. If $f \in \mathcal{F}(\mathcal{X})$ and $U : \mathcal{H} \to \mathcal{H}_f$ is a unitary, then $U^* f U \in \mathcal{F}(\mathcal{X})$ denotes the function defined by

$$(U^* f U)(x) = U^* f(x) U.$$

More generally, if $n \in \mathcal{N}(\mathcal{X})$ and $\{ f_\iota \ : \iota \in I \}$ is a subset of $\mathcal{F}(\mathcal{X})$ such that, for each $\iota \in I$, $n_{f_\iota} \leq n$, we can define the direct sum $\sum_{\iota \in I}^{\oplus} f_\iota$ by

$$\left(\sum_{\iota \in I}^{\oplus} f_\iota \right)(x) = \sum_{\iota \in I}^{\oplus} f_\iota(x)$$

for each $x \in \mathcal{X}$.

It will often be sufficient to consider only representations of $\mathbb{P}(\mathcal{X})$ as operators on a single Hilbert space $\mathcal{H}$. For this purpose let $\mathcal{F}(\mathcal{X}, B(\mathcal{H}))$ denote the subset of $\mathcal{F}(\mathcal{X})$ consisting of those functions mapping $\mathcal{X}$ into $B(\mathcal{H})$. We will let $\mathcal{H}_\mathcal{X}$ denote a fixed Hilbert space whose dimension is the smallest infinite cardinal greater than or equal to the cardinality of $\mathcal{X}$. Thus if $\mathcal{X} = \{x\}$ is a singleton, then $\mathcal{F}(\mathcal{X}, B(\mathcal{H}_{\mathcal{X}}))$ is in one-to-one correspondence with the operators on a fixed infinite dimensional separable Hilbert space.

2. Noncommutative Borel functions

We will now define some locally convex topologies on $\mathbb{P}(\mathcal{X})$ and identify their completions. We will see that in each of the completions the elements may be identified with a certain algebra of functions for which there is a naturally defined functional calculus, just as there is for $\mathcal{C}$. However, this functional calculus has the advantage of being defined *for all operators.* There are two key properties of this functional calculus that are satisfied by polynomials and which extend naturally to the functions in the completions:

1. $p(f \oplus g) = p(f) \oplus p(g)$ for every $p \in \mathbb{P}(\mathcal{X})$ and $f, g \in \mathcal{F}(\mathcal{X})$.
2. $p(U^* f U) = U^* p(f) U$ for every $p \in \mathbb{P}(\mathcal{X})$, $f \in \mathcal{F}(\mathcal{X})$, and unitary U.

Given $f \in \mathcal{F}(\mathcal{X})$ and $\alpha \in \mathcal{H}_f$, we define a family of seminorms $||\cdot||_{f,\alpha}$ on $\mathbb{P}(\mathcal{X})$ by

$$||p||_{f,\alpha} \equiv ||p(f)\alpha||.$$

We call the resulting locally convex topology the *point-strong topology* on $\mathbb{P}(\mathcal{X})$, and we denote this space by $\mathbb{P}_s(\mathcal{X})$. We define another family of seminorms $||\cdot||_{f,\alpha,*}$ on $\mathbb{P}(\mathcal{X})$ by

$$||p||_{f,\alpha,*} \equiv ||p(f)\alpha|| + ||p(f)^*\alpha||,$$

refer to the resulting topology as the *point-$*$-strong topology* on $\mathbb{P}(\mathcal{X})$, and denote the resulting space by $\mathbb{P}_{*-s}(\mathcal{X})$.

We might intuitively think of $\mathbb{P}(\mathcal{X})$ as a subset of

$$\prod_{f \in \mathcal{F}(\mathcal{X})} B(\mathcal{H}_f).$$

Then $\mathbb{P}_s(\mathcal{X})$ (respectively, $\mathbb{P}_{*-s}(\mathcal{X})$) can be viewed as $\mathbb{P}(\mathcal{X})$ endowed with the relative topology inherited from the product strong (respectively, $*$-strong) operator topology on $\prod_{f \in \mathcal{F}(\mathcal{X})} B(\mathcal{H}_f)$. We could then obtain the completions as the closures in $\prod_{f \in \mathcal{F}(\mathcal{X})} B(\mathcal{H}_f)$ with respect to these topologies. However, although this view is intuitively helpful, $\mathcal{F}(\mathcal{X})$ is not a set, and the cartesian product $\prod_{f \in \mathcal{F}(\mathcal{X})} B(\mathcal{H}_f)$ is not defined. Moreover, $B(\mathcal{H})$ is not complete in the strong operator topology (i.e., the completion can be identified with the set of all linear transformation on $\mathcal{H}$.) We can overcome the former problem with the following simple observation, which shows we can replace $\prod_{f\in\mathcal{F}(\mathcal{X})} B(\mathcal{H}_f)$ by the set $\prod_{f\in\mathcal{F}(\mathcal{X},B(\mathcal{H}_\mathcal{X}))} B(\mathcal{H}_f)$.

PROPOSITION 1. *Suppose $f \in \mathcal{F}(\mathcal{X})$ and $\alpha \in \mathcal{H}_f$. Then there exists $g \in \mathcal{F}(\mathcal{X}, B(\mathcal{H}_\mathcal{X}))$ and $\beta \in \mathcal{H}_\mathcal{X}$ such that*

$$||\cdot||_{f,\alpha} = ||\cdot||_{g,\beta} \text{ and } ||\cdot||_{f,\alpha,*} = ||\cdot||_{g,\beta,*}.$$

Proof. Let $\mathcal{H}$ be the norm closure of $\{p(f)\alpha \ : p \in \mathbb{P}(\mathcal{X})\}$. Then $\dim(\mathcal{H}) \le \dim(\mathcal{H}_\mathcal{X}) = d$. It is clear that we can replace f with the restriction of f to the reducing subspace $\mathcal{H}$. Once this is done, we can replace f with a direct sum of d copies of f and α with a vector with α in one coordinate and 0 in the other coordinates. Once this is done we have $\dim(\mathcal{H}_f) = \dim(\mathcal{H}_\mathcal{X})$, which clearly implies that we can find the required g and β. □

We can now identify $\mathbb{P}(\mathcal{X})$ as a subset of the set $\mathfrak{F}_\mathcal{X}$ of all functions from $\mathcal{F}(\mathcal{X}, B(\mathcal{H}_\mathcal{X}))$ into $B(\mathcal{H}_\mathcal{X})$. We can topologize $\mathfrak{F}_\mathcal{X}$ with the topologies of pointwise convergence in the strong and $*$-strong operator topologies and the relative topologies on $\mathbb{P}(\mathcal{X})$ give us $\mathbb{P}_s(\mathcal{X})$ and $\mathbb{P}_{*-s}(\mathcal{X})$, respectively. The algebra $\mathfrak{F}_\mathcal{X}$ is complete in the topology of pointwise $*$-strong operator convergence (i.e., if S, T are linear transformations on $\mathcal{H}_\mathcal{X}$ such that $(S\alpha, \beta) = (\alpha, T\beta)$ for all $\alpha, \beta \in \mathcal{H}_\mathcal{X}$, then the closed graph theorem implies that S and T are bounded). However, $\mathfrak{F}_\mathcal{X}$ is not complete in the topology of pointwise strong operator convergence. Nevertheless, the next proposition shows that the completion of $\mathbb{P}_s(\mathcal{X})$ can be identified with its point-strong closure in $\mathfrak{F}_\mathcal{X}$.

PROPOSITION 2. *Suppose that $\{p_\lambda\}$ is a Cauchy net in $\mathbb{P}_s(\mathcal{X})$. Then for every $f \in \mathcal{F}(\mathcal{X})$ there exists $T \in B(\mathcal{H}_f)$ such that $p_\lambda(f) \to T$ in the strong operator topology.*

Proof. Let $f \in \mathcal{F}(\mathcal{X})$ be given. If $\alpha \in \mathcal{H}_f$, then $\{p_\lambda(f)\alpha\}$ is a Cauchy net in $\mathcal{H}_f$ and consequently it converges to some vector $T\alpha \in \mathcal{H}_f$. It is clear that T is linear; we must show T is bounded. If T were not bounded, then there would exist vectors $\alpha_i \in \mathcal{H}_f$ such that $\sum_i ||\alpha_i||^2 < \infty$ but for which $\sum_i ||T\alpha_i||^2 = \infty$. Let $\mathcal{H}$ be the infinite Hilbert space direct sum of $\mathcal{H}_f$, let $g(x) = f(x) \oplus f(x) \oplus \ldots$, and let $\vec{\alpha} = (\alpha_1, \alpha_2, \ldots)$. Since $p_\lambda(g)\vec{\alpha} = (p_\lambda(f)\alpha_1, p_\lambda(f)\alpha_2, \ldots)$, we see that it is now impossible for the net $\{p_\lambda(g)\vec{\alpha}\}$ to converge to a vector in $\mathcal{H}$, contradicting the fact that $\{p_\lambda\}$ is a Cauchy net in $\mathbb{P}_s(\mathcal{X})$. Thus T is bounded. □

The preceding proposition implies the analogue for $\mathbb{P}_{*-s}(\mathcal{X})$. Hence, we can indeed identify the completion of $\mathbb{P}_s(\mathcal{X})$ with its point-strong closure in $\mathfrak{F}_{\mathcal{X}}$, and we can identify the completion of $\mathbb{P}_{*-s}(\mathcal{X})$ with its point-$*$-strong closure in $\mathfrak{F}_{\mathcal{X}}$. Although the topologies are different, we will see that the closures of $\mathbb{P}_s(\mathcal{X})$ and $\mathbb{P}_{*-s}(\mathcal{X})$ in $\mathfrak{F}_{\mathcal{X}}$ give the same easily identifiable subset of $\mathfrak{F}_{\mathcal{X}}$.

We define $\mathcal{B}\langle\mathcal{X}\rangle$ as the set of all $\phi \in \mathfrak{F}_{\mathcal{X}}$ such that, if $f, g \in \mathcal{F}(\mathcal{X}, B(\mathcal{H}_{\mathcal{X}}))$, M and N are closed linear subspaces of $\mathcal{H}_{\mathcal{X}}$ that reduce f and g (respectively), then

1. M reduces $\phi(f)$ and N reduces $\phi(g)$, and
2. if $U : M \to N$ is unitary, and $U^*(g|_N)U = f|_M$, then $U^*(\phi(g)|_N)U = \phi(f)|_M$.

Note that if $\phi \in \mathcal{B}\langle\mathcal{X}\rangle$, we can unambiguously extend ϕ to a function ψ whose domain is

$$sub\mathcal{F}(\mathcal{X}, B(\mathcal{H}_{\mathcal{X}})) = \bigcup \{\mathcal{F}(\mathcal{X}, B(M)) : M \text{ is a closed subspace of } \mathcal{H}_{\mathcal{X}}\}$$

defined, for $g \in \mathcal{F}(\mathcal{X}, B(M))$, by

$$\psi(g) \oplus [\phi(0)|_{M^\perp}] = \phi(g \oplus 0).$$

Moreover, the function ψ satisfies the properties

3. $\psi(f \oplus g) = \psi(f) \oplus \psi(g)$,
4. $\psi(U^*AU) = U^*\psi(A)U$, whenever M and N are closed linear subspaces of $\mathcal{H}_{\mathcal{X}}$, $A \in B(N)$ and $U : M \to N$ is unitary.

Conversely, if ψ is a function on $sub\mathcal{F}(\mathcal{X}, B(\mathcal{H}_{\mathcal{X}}))$ satisfying (3) and (4) above, then $\phi = \psi|_{\mathcal{F}(\mathcal{X}, B(\mathcal{H}_{\mathcal{X}}))}$ is in $\mathcal{B}\langle\mathcal{X}\rangle$. The functions ψ are the *decomposable functions* introduced in [3] and studied in [6], [7] and [8].

The following result shows that the completions of $\mathbb{P}_s(\mathcal{X})$ and $\mathbb{P}_{*-s}(\mathcal{X})$ are naturally identified with $\mathcal{B}\langle\mathcal{X}\rangle$.

PROPOSITION 3. *The point-strong closure of* $\mathbb{P}(\mathcal{X})$ *in* $\mathfrak{F}_{\mathcal{X}}$ *equals the point-$*$-strong closure of* $\mathbb{P}(\mathcal{X})$ *in* $\mathfrak{F}_{\mathcal{X}}$ *equals* $\mathcal{B}\langle\mathcal{X}\rangle$.

Proof. The preceding proposition implies that the point-$*$-strong closure of $\mathbb{P}(\mathcal{X})$ is contained in the point-strong closure of $\mathbb{P}(\mathcal{X})$. It is clear that the point-strong closure of $\mathbb{P}(\mathcal{X})$ is contained in $\mathcal{B}\langle\mathcal{X}\rangle$. Thus it suffices to show that $\mathcal{B}\langle\mathcal{X}\rangle$ is contained in the point-$*$-strong closure of $\mathbb{P}(\mathcal{X})$. Suppose $\phi \in \mathcal{B}\langle\mathcal{X}\rangle$, and let ψ denote its natural extension to $sub\mathcal{F}(\mathcal{X}, B(\mathcal{H}_{\mathcal{X}}))$ described above. Suppose $\mathcal{U}$ is a point-$*$-strong neighborhood of ϕ. Then there is an $\epsilon > 0$, $f_1, f_2, \ldots, f_n$ in $\mathcal{F}(\mathcal{X}, B(\mathcal{H}_{\mathcal{X}}))$ and $\alpha_1, \alpha_2, \ldots, \alpha_n$ in $\mathcal{H}_{\mathcal{X}}$ such that $\mathcal{U}$ contains all $\gamma \in \mathfrak{F}_{\mathcal{X}}$ such that

$$||\gamma(f_k)\alpha_k - \phi(f_k)\alpha_k|| + ||\gamma(f_k)^*\alpha_k - \phi(f_k)^*\alpha_k|| < \epsilon$$

for $1 \leq k \leq n$. Since $\mathcal{H}_{\mathcal{X}}$ is infinite dimensional, there is an f in $\mathcal{F}(\mathcal{X}, B(\mathcal{H}_{\mathcal{X}}))$ that is unitarily equivalent to $f_1 \oplus f_2 \oplus \cdots \oplus f_n$ with a unitary that sends $\alpha_1 \oplus \alpha_2 \oplus \cdots \oplus \alpha_n$ to a vector α. It follows from part (1) of the definition of $\mathcal{B}\langle\mathcal{X}\rangle$ that any subspace that reduces f must also reduce $\phi(f)$. It follows that $\phi(f)$ must belong to the von Neumann algebra generated by $f(\mathcal{X})$, whence there is a $p \in \mathbb{P}(\mathcal{X})$ such that

$$||p(f)\alpha - \phi(f)\alpha|| + ||p(f)^*\alpha - \phi(f)^*\alpha|| < \epsilon$$

and it follows that $p \in \mathcal{U}$. □

REMARK. It follows from the preceding Proposition that $\mathcal{B}\langle\mathcal{X}\rangle$ is the point-weak operator closure of $\mathbb{P}(\mathcal{X})$ in $\mathfrak{F}_{\mathcal{X}}$, but it seems unlikely that this coincides with the completion of $\mathbb{P}(\mathcal{X})$ with respect to the family $\{||\cdot||_{f,\alpha,\beta}\}$ defined by

$$||p||_{f,\alpha,\beta} = |(p(f)\alpha, \beta)|,$$

where $f \in \mathcal{F}(\mathcal{X})$ and $\alpha, \beta \in \mathcal{H}_f$.

We call the elements of $\mathcal{B}\langle\mathcal{X}\rangle$ *noncommutative Borel functions with variables in* $\mathcal{X}$. Note that we have a *functional calculus* for elements ϕ in $\mathcal{B}\langle\mathcal{X}\rangle$ and every element in $\mathcal{F}(\mathcal{X})$. Namely, if $\phi \in \mathcal{B}\langle\mathcal{X}\rangle$ then there is a Cauchy net $\{p_\lambda\}$ in $\mathbb{P}_{*-s}(\mathcal{X})$ converging in the point-$*$-strong topology on $\mathfrak{F}_{\mathcal{X}}$ to ϕ. However, if $f \in \mathcal{F}(\mathcal{X})$, then $\{p_\lambda(f)\}$ is $*$-strongly Cauchy in $B(\mathcal{H}_f)$, so we can define

$$\phi(f) = \lim_{\lambda} p_\lambda(f).$$

Note that, since every f is a direct sum of functions g such that $\dim\mathcal{H}_g \leq \dim\mathcal{H}_{\mathcal{X}}$, it follows that the definition of $\phi(f)$ is independent of the choice of $\{p_\lambda\}$. This functional calculus has many pleasant properties; the ones in the next proposition follow immediately from the definition.

PROPOSITION 4. *Suppose* $\phi \in \mathcal{B}\langle\mathcal{X}\rangle$. *Then*

1. *if* $f \in \mathcal{F}(\mathcal{X})$ *and* $U : \mathcal{H} \to \mathcal{H}_f$ *is unitary, then*
$$\phi(U^* f U) = U^* \phi(f) U,$$
2. *if* $n \in \mathcal{N}(\mathcal{X})$ *and* $\{f_\iota : \iota \in I\} \subset \mathcal{F}(\mathcal{X})$ *with each* $n_{f_\iota} \leq n$, *then*
$$\phi(\textstyle\sum_{\iota\in I}^{\oplus} f_\iota) = \sum_{\iota\in I}^{\oplus} \phi(f_\iota).$$

Note that condition (2) in the preceding proposition holds if and only if it holds for the direct sum of pairs of elements of $\mathcal{F}(\mathcal{X})$. The following proposition comes from the relationship between decomposable functions and the elements of $\mathcal{B}\langle\mathcal{X}\rangle$ described above. Since $\mathcal{F}(\mathcal{X})$ is not a set, it is not correct to talk about an operator-valued *function* on $\mathcal{F}(\mathcal{X})$. Instead, we can view $\mathcal{F}(\mathcal{X})$ as a *category* of mappings, and talk about a *functor* η from this category to the category of Hilbert space operators, i.e., for every $f \in \mathcal{F}(\mathcal{X})$, $\eta(f)$ is an operator on some Hilbert space.

PROPOSITION 5. *If η is an operator-valued functor on $\mathcal{F}(\mathcal{X})$ such that, for every $f, g \in \mathcal{F}(\mathcal{X})$,*

$$\eta(f \oplus g) = \eta(f) \oplus \eta(g),$$

then there is a unique $\phi \in \mathcal{B}\langle\mathcal{X}\rangle$ such that, for every $f \in \mathcal{F}(\mathcal{X})$,

$$\eta(f) = \phi(f).$$

In other words, there is a net $\{p_\lambda\}$ in $\mathbb{P}(\mathcal{X})$ such that, for every $f \in \mathcal{F}(\mathcal{X})$,

$$p_\lambda(f) \to \eta(f) \ *\text{-strongly}.$$

Proof. In order to show the restriction of η to $sub\mathcal{F}(\mathcal{X}, B(\mathcal{H}_\mathcal{X}))$ is a decomposable function, it suffices to show that, for each $f \in \mathcal{F}(\mathcal{X})$ and each unitary U,

$$\eta(U^* f U) = U^* \eta(f) U.$$

It follows from the hypothesis that, for each $f \in \mathcal{F}(\mathcal{X})$ and each reducing subspace M of $f(\mathcal{X})$, M must also reduce $\eta(f)$. It follows from the double commutant theorem that $\eta(f) \in W^*(f(\mathcal{X}))$ for each $f \in \mathcal{F}(\mathcal{X})$. Thus

$$\eta(f) \oplus \eta(U^* f U) = \eta(f \oplus U^* f U) \in W^*([f \oplus U^* f U](\mathcal{X})),$$

which implies that $\eta(U^* f U) = U^* \eta(f) U$. □

Note that the preceding proposition immediately shows that, via pointwise operations, $\mathcal{B}\langle\mathcal{X}\rangle$ is a $*$-algebra with identity.

The following examples show the power of the preceding proposition. Later we will see even more ways of constructing noncommutative Borel functions. The nice thing about the functional calculus viewed this way is that it gives a *formula* for certain constructions.

EXAMPLE. Suppose $\mathcal{X} = \{x\}$, and for each $f \in \mathcal{F}(\mathcal{X})$, $\eta(f)$ is the partial isometry in the polar decomposition of $f(x)$. It is clear that η satisfies the condition of the preceding proposition. Hence there is a noncommutative Borel function ϕ such that, for every $f \in \mathcal{F}(\mathcal{X})$, $\eta(f) = \phi(f)$. Thus there is a net $\{p_\lambda\}$ of $*$-polynomials such that, for every Hilbert space operator T, $\{p_\lambda(T)\}$ converges $*$-strongly to the partial isometry part of the polar decomposition of T.

EXAMPLE. Suppose $\mathcal{X}$ is arbitrary, and we define $\eta(f)$ to be the orthogonal projection onto the closed linear span of $\cup_{x \in \mathcal{X}} \mathrm{Ran} f(x)$. Again it is clear that η satisfies the condition of the previous proposition, so there is a noncommutative Borel function ϕ so that, for every $f \in \mathcal{F}(\mathcal{X})$, $\eta(f) = \phi(f)$.

EXAMPLE. The condition $\psi(U^*TU) = U^*\psi(T)U$ in the definition of decomposable function cannot be omitted in spite of its omission in the preceding proposition. Suppose $\mathcal{X} = \{x\}$, and suppose ψ is a function on $subB(\mathcal{H}_{\mathcal{X}})$ defined so that $\psi(T) = 0$ if T is an irreducible operator whose domain is $\mathcal{H}_{\mathcal{X}}$, and $\psi(T) = T$ otherwise. Then, whenever $T = A \oplus B$, $\psi(T) = \psi(A) \oplus \psi(B)$. However, ψ is not a decomposable function.

On the other hand, if η is a functor on $\mathcal{F}(\mathcal{X})$ defined by $\eta(f) = 0$ if f is irreducible and $\eta(f) = f$ if f is reducible, then we have $\eta(U^*fU) = U^*\eta(f)U$ for every $f \in \mathcal{F}(\mathcal{X})$ and U is unitary, but η does not satisfy the condition in the preceding theorem.

We can also talk about *multivalued noncommutative* Borel functions and *compositions*. Suppose $\mathcal{X}$ and $\mathcal{Y}$ are sets. We define $\mathcal{B}\langle\mathcal{Y}, X\rangle$ to be the set of all functions from $\mathcal{Y}$ into $\mathcal{B}\langle\mathcal{X}\rangle$. Suppose $\omega \in \mathcal{B}\langle\mathcal{Y}, X\rangle$ and $f \in \mathcal{F}(\mathcal{X})$. We define $\omega f \in \mathcal{F}(\mathcal{Y})$ by

$$(\omega f)(y) = [\omega(y)](f).$$

In view of the preceding proposition we can define, for $\phi \in \mathcal{B}\langle\mathcal{Y}\rangle$, the *composition* $\phi \circ \omega \in \mathcal{B}\langle\mathcal{X}\rangle$ as the noncommutative Borel function such that, for every $f \in \mathcal{F}(\mathcal{X})$

$$(\phi \circ \omega)(f) = \phi(\omega f).$$

For example, suppose $\mathcal{Y} = \{y_1, y_2\}$, $\omega(y_1) = \psi_1$, $\omega(y_2) = \psi_2$, and ϕ is the polynomial $p(y_1, y_2)$. Then, for any $f \in \mathcal{F}(\mathcal{X})$,

$$(\phi \circ \omega)(f) = p(\psi_1(f), \psi_2(f)).$$

Another type of composition is in the case in which a noncommutative Borel function ϕ has the property that $\phi(f)$ is always normal with spectrum contained in some subset K of $\mathbb{C}$, and $\zeta : K \to \mathbb{C}$ is Borel measurable and bounded on bounded sets. Then the composition $\zeta \circ \phi$ is defined by

$$(\zeta \circ \phi)(f) = \zeta(\phi(f)),$$

where the application of ζ comes from the Borel functional Calculus for normal operators.

A simple application of composition is truncation. Suppose $0 < r < \infty$, and define the function $\zeta_r : [0, \infty) \to [0, 1]$ by $\zeta_r(t) = 1$ when $0 \leq t \leq r$ and $\zeta_r(t) = r^2/t^2$ when $r < t$. If A is an operator, we define

$\tau_r(A) = [\zeta_r(AA^*)]A$. If $\phi \in \mathcal{B}\langle\mathcal{X}\rangle$ we define the r-truncation of ϕ to be $\tau_r \circ \phi$. It is clear that

$$\sup\{\, \|(\tau_r \circ \phi)(f)\| \;:\, f \in \mathcal{F}(\mathcal{X})\,\} \le r$$

and that $(\tau_r \circ \phi)(f) = \phi(f)$ whenever $\|\phi(f)\| \le r$.

We define $\mathcal{B}^b\langle\mathcal{X}\rangle$ to be the *bounded* elements of $\mathcal{B}\langle\mathcal{X}\rangle$, i.e., $\phi \in \mathcal{B}^b\langle\mathcal{X}\rangle$ if and only if $\phi \in \mathcal{B}\langle\mathcal{X}\rangle$ and

$$\sup\{\, \|\phi(f)\| \;:\, f \in \mathcal{F}(\mathcal{X})\,\} \equiv \|\phi\| < \infty.$$

It is clear that $\mathcal{B}^b\langle\mathcal{X}\rangle$ is a C*-algebra.

The next result generalizes the fact that if N is a normal operator on a separable Hilbert space, then $W^*(N)$ is the set of all operators of the form $f(N)$, where $f : \mathbb{C} \to \mathbb{C}$ is Borel measurable and bounded on bounded sets.

PROPOSITION 6. *Suppose* $f \in \mathcal{F}(\mathcal{X})$. *Then*

$$W^*(f(\mathcal{X})) = \{\phi(f) \;:\, \phi \in \mathcal{B}\langle\mathcal{X}\rangle\,\} = \{\,\phi(f) \;:\, \phi \in \mathcal{B}^b\langle\mathcal{X}\rangle\,\}.$$

Proof. It follows from Proposition 4 and the von Neumann double commutant theorem that $\{\phi(f) \;:\, \phi \in \mathcal{B}\langle\mathcal{X}\rangle\,\} \subset W^*(f(\mathcal{X}))$. Clearly

$$\{\,\phi(f) \;:\, \phi \in \mathcal{B}^b\langle\mathcal{X}\rangle\,\} \subset \{\phi(f) \;:\, \phi \in \mathcal{B}\langle\mathcal{X}\rangle\,\}.$$

Next suppose $A \in W^*(f(\mathcal{X}))$ and let $r = \|A\|$. By Kaplansky's density theorem there is an ultranet $\{p_\lambda\}$ of polynomials in $\mathbb{P}(\mathcal{X})$ such that $p_\lambda(f) \to A$ in the weak operator topology and such that $\|p_\lambda(f)\| \le r$ for every λ. Then $\phi_\lambda = \tau_r \circ p_\lambda$ defines an ultranet in $\mathcal{B}^b\langle\mathcal{X}\rangle$. Since the closed ball with radius r of all operators on a Hilbert space is compact in the weak operator topology, it follows that, for every $g \in \mathcal{F}(\mathcal{X})$, the ultranet $\{\phi_\lambda(g)\}$ is weak operator convergent to an operator $\phi(g)$. It follows from Proposition 5 that $\phi \in \mathcal{B}\langle\mathcal{X}\rangle$. However, it is clear that $\|\phi\| \le r$; whence, $\phi \in \mathcal{B}^b\langle\mathcal{X}\rangle$. Since $\phi(f) = A$, we have shown that

$$W^*(f(\mathcal{X})) \subset \{\phi(f) \;:\, \phi \in \mathcal{B}\langle\mathcal{X}\rangle\,\},$$

and the proof is complete. □

Note that every von Neumann algebra is generated by a subset that can be expressed in the form $f(\mathcal{X})$ for some $\mathcal{X}$. Thus our results apply to arbitrary von Neumann algebras.

3. Parts, functionals and ideals

In this section we want to discuss the notion of part (e.g., the *normal part* of an operator), closed ideals in $\mathcal{B}\langle\mathcal{X}\rangle$, continuous linear functionals on $\mathcal{B}\langle\mathcal{X}\rangle$, and central projections in $\mathcal{B}\langle\mathcal{X}\rangle$, as well as their interrelationships.

We begin with the notion of *disjointness*. A (direct) *summand* of an element f of $\mathcal{F}(\mathcal{X})$ is the restriction of f to a *nonzero* reducing subspace. We say that two elements $f, g \in \mathcal{F}(\mathcal{X})$ are *disjoint*, denoted by $f \diamond g$, if f and g have no unitarily equivalent summands, i.e., there do not exist nonzero subspaces M and N such that M reduces f and N reduces g with $f|_M$ unitarily equivalent to $g|_N$. Disjointness has been studied in terms of representations of C*-algebras and von Neumann algebras (see [1]), and it has been studied for single operators by J. Ernest [5]. At the opposite extreme, we say that f is *weakly contained* in g, denoted by $f \ll g$, if no summand of f is disjoint from g. We say that f and g are *weakly equivalent*, denoted by $f \approx g$, if $f \ll g$ and $g \ll f$.

More generally, if $\mathcal{E}$ is a nonempty subclass of $\mathcal{F}(\mathcal{X})$, we say f is *disjoint from* $\mathcal{E}$, denoted $f \diamond \mathcal{E}$ if, for every $g \in \mathcal{E}$, $f \diamond g$, and we say $f \ll \mathcal{E}$ if no summand of f is disjoint from $\mathcal{E}$. We say that two summands f_1, f_2 of f are orthogonal, denoted by $f_1 \perp f_2$, if the Hilbert spaces they live on are orthogonal. The following proposition, which is taken from [5] and [1], contains most of the important properties of weak containment and disjointness.

Proposition 7. *Suppose $f, g \in \mathcal{F}(\mathcal{X})$, f_1, f_2 are summands of f, and $\mathcal{E}$ is a nonempty subclass of $\mathcal{F}(\mathcal{X})$. Then*

1. *if $f_1 \diamond f_2$, then $f_1 \perp f_2$,*
2. *$f \ll \mathcal{E}$ if and only if, f is a direct sum of summands of elements of $\mathcal{E}$,*
3. *if $\mathcal{E}$ is bounded by some $n \in \mathcal{N}(\mathcal{X})$, then $f \diamond \mathcal{E} \iff f \diamond \sum^{\oplus}_{h\in\mathcal{E}} h$ and $f \ll \mathcal{E} \iff f \ll \sum^{\oplus}_{h\in\mathcal{E}} h$,*
4. *$f \diamond g \iff W^*((f \oplus g)(\mathcal{X})) = W^*(f(\mathcal{X})) \oplus W^*(g(\mathcal{X}))$,*
5. *(lebesgue decomposition Theorem) There are $f_a, f_s \in \mathcal{F}(\mathcal{X})$ such that $f = f_a \oplus f_s$, $f_a \ll \mathcal{E}$, and $f_s \diamond \mathcal{E}$,*
6. *$W^*(f(\mathcal{X}))$ is a factor von Neumann algebra if and only if no two summands of f are disjoint,*
7. *$f \ll g$ if and only if, for some cardinal m, f is unitarily equivalent to a summand of $g^{(m)}$,*
8. *$f \approx g$ if and only if, for some cardinal m, $f^{(m)}$ and $g^{(m)}$ are unitarily equivalent.*

It is well known that every contraction operator can be uniquely decomposed as the direct sum of a unitary operator (the unitary part) and an operator having no unitary direct summands (the completely non-unitary part). This is an instance of a *part property* of operators, the class of unitary operators occupying what we call a *part class* of operators. Part properties for operators were studied and completely characterized in [3]. We show that a similar characterization holds in our more general setting. We say that a nonempty subclass $\mathcal{P}$ of $\mathcal{F}(\mathcal{X})$ is a *part class* if $\mathcal{P}$ is closed under unitary equivalence, and every $f \in \mathcal{F}(\mathcal{X})$ can be uniquely decomposed into the direct sum of an element in $\mathcal{P}$ (the $\mathcal{P}$-part of f) and an element having no direct summands in $\mathcal{P}$ (the completely non-$\mathcal{P}$-part of f).

A nonempty subclass $\mathcal{E}$ of $\mathcal{F}(\mathcal{X})$ is *equationally defined* if there is a nonempty subset $\mathcal{W}$ of $\mathcal{B}\langle\mathcal{X}\rangle$ such that

$$\mathcal{E} = \{ f \in \mathcal{F}(\mathcal{X}) \ : \forall \phi \in \mathcal{W} \quad \phi(f) = 0 \}.$$

PROPOSITION 8. *Suppose $\mathcal{P}$ is a subclass of $\mathcal{F}(\mathcal{X})$ that is closed under unitary equivalence. The following are equivalent:*

1. *$\mathcal{P}$ is a part class,*
2. *the direct sum of a point-norm bounded subset of $\mathcal{F}(\mathcal{X})$ is in $\mathcal{P}$ if and only if each summand is in $\mathcal{P}$,*
3. *$\mathcal{P}$ is equationally defined,*
4. *there is a (selfadjoint) projection ρ in the center of $\mathcal{B}\langle\mathcal{X}\rangle$ such that, $\mathcal{P} = \{ f \in \mathcal{F}(\mathcal{X}) \ : \rho(f) = 0 \}$.*

Proof. (3) $\implies$ (2) This clearly follows from part (2) of Proposition 4.

(2) $\implies$ (1) Let $\mathcal{M}$ be the direct sum of a maximal orthogonal collection of subspaces $\mathcal{L}$ that reduce f for which the restriction of f to $\mathcal{L}$ is in $\mathcal{P}$. Write $f = f_{\mathcal{M}} \oplus g$ relative to $\mathcal{M} \oplus \mathcal{M}^{\perp}$. It follows from (2) that $f \in \mathcal{P}$ and $g \lozenge \mathcal{P}$. It follows from Proposition 7 that any summand of f that is in $\mathcal{P}$ must be orthogonal to g. Hence $\mathcal{M}$ is the *unique* maximal reducing subspace $\mathcal{L}$ for f such that the restriction of f to $\mathcal{L}$ is in $\mathcal{P}$. Hence the decomposition of f into a summand in $\mathcal{P}$ and a sumand disjoint from $\mathcal{P}$ is unique. Thus $\mathcal{P}$ is a part class.

(1) $\implies$ (4) For each $f \in \mathcal{F}(\mathcal{X})$ we define $\rho(f) \in B(\mathcal{H}_f)$ as follows: by the hypothesis in (1) we may write $f = f_{\mathcal{M}} \oplus g$ relative to a unique decomposition $\mathcal{M} \oplus \mathcal{M}^{\perp}$ of $\mathcal{H}_f$ such that $f_{\mathcal{M}} \in \mathcal{P}$ and g has no summand in $\mathcal{P}$. We define $\rho(f)$ to be the orthogonal projection onto $\mathcal{M}^{\perp}$. We can easily verify that $\rho(U^*fU) = U^*\rho(f)U$ and $\rho(f_1 \oplus f_2) = \rho(f_1) \oplus \rho(f_2)$, from which we deduce by Proposition 5 that $\rho \in \mathcal{B}\langle\mathcal{X}\rangle$. This proves (4).

Since the proof of (4) $\Longrightarrow$ (3) is trivial we are done. □

COROLLARY 9. *Assume that* $\mathcal{G} \subseteq \mathcal{F}(\mathcal{X})$. *The smallest part class containing* $\mathcal{G}$ *is* $\{ f \in \mathcal{F}(\mathcal{X}) \ : f \ll \mathcal{G} \}$.

COROLLARY 10. *A subset* $\mathcal{G} \subseteq \mathcal{F}(\mathcal{X})$ *is a part class if and only if* $\mathcal{G}$ *is the class of all* f*'s in* $\mathcal{F}(\mathcal{X})$ *that are unitarily equivalent to direct sums of elements of* $\mathcal{G} \cap \mathcal{F}(\mathcal{X}, B(\mathcal{H}_{\mathcal{X}}))$.

COROLLARY 11. *There is a one-to-one correspondence between the central projections in* $\mathcal{B}\langle\mathcal{X}\rangle$ *and all part classes.*

Using Proposition 6 we know that each $f \in \mathcal{F}(\mathcal{X})$ induces a surjective unital $*$-homomorphism $\pi_f : \mathcal{B}\langle\mathcal{X}\rangle \to W^*(f(\mathcal{X}))$ defined by $\pi_f(\phi) = \phi(f)$. We can use the preceding theorem to show that central projections in $W^*(f(\mathcal{X}))$ can be lifted to central projections in $\mathcal{B}\langle\mathcal{X}\rangle$. Moreover, elements in part classes can always be lifted to elements in the same class, e.g., projections can be lifted to projections and normal elements can be lifted to normal elements.

PROPOSITION 12. *Suppose* $f \in \mathcal{F}(\mathcal{X})$. *Then*

1. *if* P *is a central projection in* $W^*(f(\mathcal{X}))$, *then there is a central projection* $\rho \in \mathcal{B}\langle\mathcal{X}\rangle$ *such that* $\rho(f) = P$,
2. *if* ψ *is a noncommutative Borel function of one variable such that* $\psi(\lambda) = 0$ *for some* $\lambda \in \mathbb{C}$, *if* $A \in W^*(f(\mathcal{X}))$, *and* $\psi(A) = 0$, *then there is a* $\phi \in \mathcal{B}\langle\mathcal{X}\rangle$ *such that* $\phi(f) = A$ *and* $\psi \circ \phi = 0$.

Proof. 1. Suppose $f : \mathcal{X} \to B(\mathcal{H})$ and P is a central projection in $W^*(f(\mathcal{X}))$. Then, relative to the orthogonal decomposition $\mathcal{H} = \ker P \oplus \operatorname{ran} P$, $f = g \oplus h$. Thus $W^*(f(\mathcal{X})) = W^*(g(\mathcal{X})) \oplus W^*(h(\mathcal{X}))$, which means $g \diamond h$. Let $\mathcal{P} = \{ e \in \mathcal{F}(\mathcal{X}) \ : e \ll g \}$. Then, by Corollary 9, $\mathcal{P}$ is a part class, and, by Proposition 8, there is a central projection $\rho \in \mathcal{B}\langle\mathcal{X}\rangle$ such that $\mathcal{P} = \{ e \in \mathcal{F}(\mathcal{X}) \ : \rho(e) = 0 \}$. Hence, $\rho(g) = 0$. On the other hand, since ρ is central, $\rho(h)$ is a projection commuting with h, and if h_0 is the summand $h|_{\ker \rho(h)}$, then $\rho(h_0) = 0$, which implies $h_0 \ll g$. Since $g \diamond h$, it follows that $\ker \rho(h) = \{0\}$, implying $\rho(h) = 1$. Thus $\rho(f) = \rho(g) \oplus \rho(h) = P$.

2. Suppose A and ψ are as given. Choose $\eta \in \mathcal{B}\langle\mathcal{X}\rangle$ so that $\eta(f) = A$, and let

$$\mathcal{Q} = \{ e \in \mathcal{B}\langle\mathcal{X}\rangle \ : \psi(\eta(e)) = 0 \}.$$

Then $\mathcal{Q}$ is a part-class and there is a central projection $\rho \in \mathcal{B}\langle\mathcal{X}\rangle$ such that

$$\mathcal{Q} = \{ e \in \mathcal{B}\langle\mathcal{X}\rangle \ : \rho(e) = 0 \}.$$

The desired ϕ is defined by

$$\phi = (1-\rho)\eta + \lambda\rho.$$

□

Proposition 6 makes $\mathcal{B}\langle\mathcal{X}\rangle$ (or $\mathcal{B}^b\langle\mathcal{X}\rangle$) seem like a von Neumann algebra. Using the notion of composition, it is clear that every normal element ($\phi\phi^* = \phi^*\phi$) in $\mathcal{B}\langle\mathcal{X}\rangle$ has spectral projections. Moreover, we can use Proposition 6 to decompose $\mathcal{B}\langle\mathcal{X}\rangle$ into the different types (i.e., $I, I_n, II_1, II_\infty, III$).

PROPOSITION 13. *For each* $\natural$ *in* $\{II_1, II_\infty, III, I, I_\infty, I_1, I_2, \ldots\}$ *there is a central projection* $P_\natural$ *in* $\mathcal{B}\langle\mathcal{X}\rangle$ *such that for every* $f \in \mathcal{F}(\mathcal{X})$, $P_\natural(f)$ *is the central projection in* $W^*(f(\mathcal{X}))$ *onto the type* $\natural$ *part of* $W^*(f(\mathcal{X}))$.

Proof. Note that "$P_\natural(f)$ is the central projection in $W^*(f(\mathcal{X}))$ onto the type $\natural$ part of $W^*(f(\mathcal{X}))$" defines an operator-valued functor on $\mathcal{F}(\mathcal{X})$. Clearly, $P_\natural(f \oplus g) = P_\natural(f) \oplus P_\natural(g)$ always holds. Thus, by Proposition 6, $P_\natural \in \mathcal{B}\langle\mathcal{X}\rangle$. Since the operations in $\mathcal{B}\langle\mathcal{X}\rangle$ are pointwise, $P_\natural$ is a central projection. □

We next look at continuous linear functionals on $\mathcal{B}\langle\mathcal{X}\rangle$. There are two natural topologies on $\mathcal{B}\langle\mathcal{X}\rangle$, namely, the point-strong and point-$*$-strong topologies. There is also the point-weak topology determined by the seminorms $p_{f,u,v}$ ($f \in \mathcal{F}(\mathcal{X})$, and $u, v \in \mathcal{H}_f$) defined by

$$p_{f,u,v}(\phi) = |(\phi(f), u, v)|.$$

Although these topologies are distinct, they have the same continuous linear functionals, and thus the same closed convex sets.

PROPOSITION 14. *Suppose* $\tau : \mathcal{B}\langle\mathcal{X}\rangle \to \mathbb{C}$ *is a linear functional that is continuous with respect to the point-$*$-strong topology. Then there is an* $f \in \mathcal{F}(\mathcal{X}, B(\mathcal{H}_\mathcal{X}))$, *and* $u, v \in \mathcal{H}_\mathcal{X}$ *such that, for every* $\phi \in \mathcal{B}\langle\mathcal{X}\rangle$,

$$\tau(\phi) = (\phi(f)u, v).$$

Proof. Assume τ is a point-$*$-strong continuous linear functional on $\mathcal{B}\langle\mathcal{X}\rangle$. It follows that there exists a positive number $M > 0$ and there exist finitely many seminorms $||\cdot||_{f_i,\alpha_i,*}$ ($i = 1, \ldots, k$) such that $f_i \in \mathcal{F}(\mathcal{X}, B(\mathcal{H}_\mathcal{X}))$ and

$$|\tau(\phi)| \leq M\sqrt{\sum_{i=1}^{k} ||\phi||^2_{f_i,\alpha_i,*}}$$

for all $\phi \in \mathcal{B}\langle\mathcal{X}\rangle$. We can let $M = 1$ by replacing each α_i with $M\alpha_i$. Let $f = f_1 \oplus \ldots \oplus f_k$ and $\alpha = \alpha_1 \oplus \ldots \oplus \alpha_k$, then we have

$$|\mathrm{Re}\,\tau(\phi)| \leq |\tau(\phi)| \leq M\sqrt{||\phi(f)\alpha||^2 + ||\phi(f)^*\alpha||^2}$$

for all $\phi \in \mathcal{B}\langle\mathcal{X}\rangle$, and if we use a unitary operator to identify $\mathcal{H}_{\mathcal{X}}^{(k)}$ with $\mathcal{H}_{\mathcal{X}}$, we may replace f and α with elements of $\mathcal{F}(\mathcal{X}, B(\mathcal{H}_{\mathcal{X}}))$ and $\mathcal{H}_{\mathcal{X}}$, respectively. Assuming this has been done we let $\mathcal{K}$ be the closure of

$$\mathcal{K}_0 = \{\, (\phi(f)\alpha, \phi(f)^*\alpha) \ : \phi \in \mathcal{B}\langle\mathcal{X}\rangle \,\}$$

in $\mathcal{H}_{\mathcal{X}} \oplus \mathcal{H}_{\mathcal{X}}$, so $\mathcal{K}$ is a *real* Hilbert subspace of $\mathcal{H}_{\mathcal{X}} \oplus \mathcal{H}_{\mathcal{X}}$. The mapping

$$(\phi(f)\alpha, \phi(f)^*\alpha) \to \mathrm{Re}\tau(\phi)$$

is a real continuous functional on $\mathcal{K}_0$ that extends to a real continuous functional on $\mathcal{K}$. It follows that there exist vectors $\beta_1, \beta_2 \in \mathcal{H}_{\mathcal{X}}$ such that

$$\begin{aligned} \mathrm{Re}\,\tau(\phi) &= < \phi(f)\alpha, \beta_1 > + < \phi(f)^*\alpha, \beta_2 > \\ &= \mathrm{Re} < \phi(f)\alpha, \beta_1 > + \mathrm{Re} < \phi(f)\beta_2, \alpha > \\ &= \mathrm{Re} < \phi(f) \oplus \phi(f)(\alpha + \beta_2), (\beta_1 + \alpha) > . \end{aligned}$$

Since the real part of the mapping $\phi \mapsto < \phi(f) \oplus \phi(f)(\alpha \oplus \beta_2), (\beta_1 \oplus \alpha) >$ equals that of τ we conclude that these two functionals are equal. Using a unitary to identify $\mathcal{H}_{\mathcal{X}} \oplus \mathcal{H}_{\mathcal{X}}$ with $\mathcal{H}_{\mathcal{X}}$ we may replace $f \oplus f$ with an element of $\mathcal{F}(\mathcal{X}, B(\mathcal{H}_{\mathcal{X}}))$ and replace $\alpha \oplus \beta_2$ and $\beta_1 \oplus \alpha$ with elements of $\mathcal{H}_{\mathcal{X}}$, completing the proof. □

We can now characterize the closed ideals in $\mathcal{B}\langle\mathcal{X}\rangle$. Note that this result automatically implies its analog for von Neumann algebras.

PROPOSITION 15. *Suppose $\mathcal{I} \subset \mathcal{B}\langle\mathcal{X}\rangle$. The following are equivalent:*

1. *$\mathcal{I}$ is a point-strong closed two-sided ideal in $\mathcal{B}\langle\mathcal{X}\rangle$,*
2. *there is a part-class $\mathcal{P}$ in $\mathcal{F}(\mathcal{X})$ such that*

$$\mathcal{I} = \{\, \phi \in \mathcal{B}\langle\mathcal{X}\rangle \ : \forall f \in \mathcal{P} \quad \phi(f) = 0 \,\},$$

3. *there is a central projection $P \in \mathcal{B}\langle\mathcal{X}\rangle$ such that*

$$\mathcal{I} = P\mathcal{B}\langle\mathcal{X}\rangle.$$

Proof. (1) $\implies$ (2) Let $\mathcal{P}$ be the class of all $f \in \mathcal{F}(\mathcal{X})$ such that $\phi(f) = 0$ for every $\phi \in \mathcal{I}$. Let $\mathcal{J}$ be the set of all $\psi \in \mathcal{B}\langle\mathcal{X}\rangle$ such that, $\psi(g) = 0$ for every $g \in \mathcal{P}$. Clearly, $\mathcal{I} \subset \mathcal{J}$ and $\mathcal{J}$ is point-strong closed. Assume that $\mathcal{I} \neq \mathcal{J}$ and choose $\psi \in \mathcal{J}$ with $\psi \notin \mathcal{I}$. It follows from the Hahn-Banach theorem that there is a point-strong continuous linear functional τ on $\mathcal{B}\langle\mathcal{X}\rangle$ such that $\tau|_{\mathcal{I}} = 0$ and $\tau(\psi) = 1$. It follows from the preceding proposition that there is an $f \in \mathcal{F}(\mathcal{X}, B(\mathcal{H}_{\mathcal{X}}))$, and $u, v \in \mathcal{H}_{\mathcal{X}}$ such that, for every $\phi \in \mathcal{B}\langle\mathcal{X}\rangle$,

$$\tau(\phi) = (\phi(f)u, v).$$

Let $M = \bar{sp}\{\phi(f)u \ : \phi \in \mathcal{I}\}$. Since $\mathcal{I}$ is an ideal in $\mathcal{B}\langle\mathcal{X}\rangle$, M reduces $W^*(f(\mathcal{X}))$. Relative to $\mathcal{H}_{\mathcal{X}} = M \oplus M^\perp$, write $f = g \oplus h$ and $u = y \oplus w$. It follows from the definition of u, v and M that $v \in M^\perp$ and $\phi(f)w = 0$ for all $\phi \in \mathcal{I}$. Since $\psi(f) = \psi(g) \oplus \psi(h)$, it follows that

$$\psi(f)w = \psi(h)w \neq 0.$$

Let $N = \bar{sp}(W^*(f(\mathcal{X}))w)$ and let f_0 be f restricted to N. It follows from the fact that $\mathcal{I}$ is an ideal that for every $\phi \in \mathcal{I}$ and every $\rho \in \mathcal{B}\langle\mathcal{X}\rangle$ that

$$\phi(f_0)[\rho(f)w] = (\phi\rho)(f)w = 0.$$

Thus, $\phi(f_0) = 0$ for every $\phi \in \mathcal{I}$. However, $\psi(f_0) \neq 0$, since $\psi(f_0)w = \psi(f)w \neq 0$. This contradicts $\psi \in \mathcal{J}$. Thus (2) must be true.

(2) $\Longrightarrow$ (3) Assume (2) is true, and let P be the central projection in $\mathcal{B}\langle\mathcal{X}\rangle$ such that, for every $f \in \mathcal{F}(\mathcal{X})$, $P(f)$ is the orthogonal projection onto the $\mathcal{P}$-part of f (see Proposition 8). It follows from (2) that

$$\phi \in \mathcal{I} \iff \forall f \in \mathcal{P}\phi(f) = 0 \iff$$
$$\forall f \in \mathcal{F}(\mathcal{X})(1-P)\phi(f) = 0 \iff (1-P)\phi = 0.$$

It follows that $\mathcal{I} = P\mathcal{B}\langle\mathcal{X}\rangle$, which is (3).

(3) $\Longrightarrow$ (1) This is obvious. □

We can put together some of the preceding results into a Galois-type framework. If $\mathcal{W} \subset \mathcal{F}(\mathcal{X})$, we define

$$\mathcal{W}^\perp = \{\phi \in \mathcal{B}\langle\mathcal{X}\rangle \ : \forall f \in \mathcal{W} \ \phi(f) = 0\}.$$

Clearly, $\mathcal{W}^\perp$ is a point-strong closed two-sided ideal in $\mathcal{B}\langle\mathcal{X}\rangle$. If $\mathcal{S} \subset \mathcal{B}\langle\mathcal{X}\rangle$, we define

$$\mathcal{S}^\perp = \{f \in \mathcal{F}(\mathcal{X}) \ : \forall \phi \in \mathcal{S} \ \phi(f) = 0\}.$$

It is clear that $\mathcal{S}^\perp$ is a part class. The usual Galois relationships hold, e.g., $\mathcal{W} \subset \mathcal{W}^{\perp\perp}$, $\mathcal{S} \subset \mathcal{S}^{\perp\perp}$, $\mathcal{W}^\perp = \mathcal{W}^{\perp\perp\perp}$, $\mathcal{S}^\perp = \mathcal{S}^{\perp\perp\perp}$, and $\perp$ reverses inclusions. The gist of the preceding results can be summarized in terms of this Galois correspondence.

PROPOSITION 16. *There is a one-to-one correspondence between part classes in $\mathcal{F}(\mathcal{X})$ and point-strong closed two-sided ideals in $\mathcal{B}\langle\mathcal{X}\rangle$ that associates each part class $\mathcal{P}$ with the ideal $\mathcal{P}^\perp$. The inverse map associates each point-strong closed two-sided ideal $\mathcal{I}$ with $\mathcal{I}^\perp$. More precisely,*

1. $\mathcal{P}^{\perp\perp} = \mathcal{P}$ for every part class $\mathcal{P}$,
2. $\mathcal{I}^{\perp\perp} = \mathcal{I}$ for every point-strong closed two-sided ideal $\mathcal{I}$,
3. if P is the central projection associated in Proposition 8 with the part class $\mathcal{P}$, then

$$\mathcal{P}^{\perp} = (1 - P)\mathcal{B}\langle\mathcal{X}\rangle.$$

More generally, if $\mathcal{W} \subset \mathcal{F}(\mathcal{X})$ and $\mathcal{S} \subset \mathcal{B}\langle\mathcal{X}\rangle$, then

1. $\mathcal{W}^{\perp\perp} = \{ f \in \mathcal{F}(\mathcal{X}) \ : f \ll \mathcal{W} \}$,
2. $\mathcal{S}^{\perp\perp}$ is the point-strong closed two-sided ideal generated by $\mathcal{S}$.

The minimal projections and minimal central projections in $\mathcal{B}\langle\mathcal{X}\rangle$ can easily be characterized.

PROPOSITION 17. *The following are true.*

1. *A projection P in $\mathcal{B}\langle\mathcal{X}\rangle$ is minimal if and only if there is an irreducible $f \in \mathcal{F}(\mathcal{X})$ and a $\phi \in \mathcal{B}\langle\mathcal{X}\rangle$ such that $\phi(f)$ is a rank-one projection such that*

$$P = \phi Q,$$

where Q is the central projection corresponding to the part-class

$$\{ g \in \mathcal{F}(\mathcal{X}) \ : g \ll f \}.$$

2. *A projection P in $\mathcal{B}\langle\mathcal{X}\rangle$ is a minimal central projection if and only if there is an $f \in \mathcal{F}(\mathcal{X})$ such that $W^*(f(\mathcal{X}))$ is a factor such that P is the central projection corresponding to the part-class*

$$\{ g \in \mathcal{F}(\mathcal{X}) \ : g \ll f \}.$$

The following corollary is a consequence of the fact that minimal central projections correspond to maximal ideals.

COROLLARY 18. *The maximal point-strong closed ideals of $\mathcal{B}\langle\mathcal{X}\rangle$ are precisely the sets $\{f\}^{\perp}$ with $W^*(f(X))$ a factor.*

We conclude this section with a general technique for constructing elements of $\mathcal{B}\langle\mathcal{X}\rangle$.

LEMMA 19. *Suppose $n \in \mathcal{N}(\mathcal{X})$ and $\mathcal{G} = \{ f_\lambda \ : \lambda \in \Lambda \}$ is a subset of $\{ f \in \mathcal{F}(\mathcal{X}, B(\mathcal{H}_\mathcal{X})) \ : n_f \le n \}$ that is disjoint in the sense that $\alpha, \beta \in \Lambda$, $\alpha \ne \beta$ implies $f_\alpha \diamond f_\beta$. Suppose, for each $\lambda \in \Lambda$, $A_\lambda \in W^*(f_\lambda(\mathcal{X}))$ and*

$$M = \sup_{\lambda\in\Lambda} ||A_\lambda|| < \infty.$$

Then there is a $\phi \in \mathcal{B}^b\langle\mathcal{X}\rangle$ with $||\phi|| = M$ such that, $\phi(f_\lambda) = A_\lambda$ for each $\lambda \in \Lambda$. Moreover, if $\mathcal{G}$ is a maximal disjoint subset of $\{f \in$

$\mathcal{F}(\mathcal{X}, B(\mathcal{H}_{\mathcal{X}})) : n_f \leq n\}$, then the restriction of ϕ to $\{f \in \mathcal{F}(\mathcal{X}, B(\mathcal{H}_{\mathcal{X}})) : n_f \leq n\}$ is unique.

Proof. Let $f = \sum_{\lambda\in\Lambda}^{\oplus} f_\lambda$ and $A = \sum_{\lambda\in\Lambda}^{\oplus} A_\lambda$. Then, by Proposition 7,

$$W^*(f(\mathcal{X})) = \sum_{\lambda\in\Lambda}^{\oplus} W^*(f_\lambda(\mathcal{X})).$$

Thus $A \in W^*(f(\mathcal{X}))$ and $||A|| = M$. Hence, by Proposition 6, there is a $\phi \in \mathcal{B}^b\langle\mathcal{X}\rangle$ with $||\phi|| = M$ such that $\phi(f) = A$. Thus, for each $\lambda \in \Lambda$, $\phi(f_\lambda) = A_\lambda$. If $\mathcal{G}$ is maximal, then

$$f \ll \{ f_\lambda \ : \lambda \in \Lambda \}$$

for each $f \in \mathcal{F}(\mathcal{X})$ with $n_f \leq n$, which implies the asserted uniqueness. □

The finiteness condition on $\mathcal{X}$ in the following proposition can be dropped at the expense of replacing $\{n_k\}$ with a well-ordered (any ordering) cofinal subset of $\mathcal{N}(\mathcal{X})$.

PROPOSITION 20. *Suppose $\mathcal{X}$ is finite and $\{n_k\}$ is a cofinal sequence in $\mathcal{N}(\mathcal{X})$. Assume, for each positive integer k, that $\mathcal{G}_k$ is a disjoint collection such that whenever $f \in \mathcal{G}_k$ and $1 \leq j < k$, f has no summand g with $n_g \leq n_j$. Suppose, for each $f \in \cup_k\mathcal{G}_k$ that $A_f \in W^*(f(\mathcal{X}))$ and*

$$M_k = \sup_{f\in\mathcal{G}_k} ||A_f|| < \infty.$$

Then there is a $\phi \in \mathcal{B}\langle\mathcal{X}\rangle$ such that, $\phi(f) = A_f$ for each $f \in \cup_k\mathcal{G}_k$.

Proof. It follows from the preceding lemma that, for each positive integer k, there is a $\phi_k \in \mathcal{B}\langle\mathcal{X}\rangle$ such that $\phi_k(f) = A_f$ for each $f \in \mathcal{G}_k$. For each k, define

$$\mathcal{P}_k = \{ f \in \mathcal{F}(\mathcal{X}) \ : f \ll \mathcal{G}_k \}.$$

Then, by Proposition 8, there is a central projection $P_k \in \mathcal{B}\langle\mathcal{X}\rangle$ such that $P_k(f) = 1$ when $f \in \mathcal{P}_k$ and $P_k(f) = 0$ when $f \diamond \mathcal{G}_k$. The assumption on the $\mathcal{G}_k$'s implies that $\{P_k\}$ is an orthogonal sequence in $\mathcal{B}\langle\mathcal{X}\rangle$. Clearly, the sum

$$\sum_{k=1}^{\infty} P_k\phi_k$$

is point-strong convergent to the required ϕ. □

We can apply the preceding result to subnormal tuples. We call an element $f \in \mathcal{F}(\mathcal{X})$ *normal* if $W^*(f(\mathcal{X}))$ is commutative. An element $g \in \mathcal{F}(\mathcal{X}, B(H))$ is *subnormal* if there is a Hilbert space K containing H and a normal $f \in \mathcal{F}(\mathcal{X}, B(K))$ such that each $f(x)$ leaves H invariant, and the restriction of f to H is g. In this case we say that f is a *normal extension* of g; if K is the smallest invariant subspace for $f(\mathcal{X})$ that contains H, then f is a *minimal normal extension* of g. A subnormal g is *pure* if g has no normal direct summands. It was shown in [11] that if $\mathcal{X}$ is finite and g is a subnormal element in $\mathcal{F}(\mathcal{X})$, then there are subsets $\{D_x : x \in \mathcal{X}\}$ and $\{E_x : x \in \mathcal{X}\}$ of $W^*(g(\mathcal{X}))$ such that the function f defined by

$$f(x) = \begin{pmatrix} g(x) & D_x \\ 0 & E_x^* \end{pmatrix}$$

is a normal extension of g, and $n_f \leq n$. Moreover, it was shown in [11] that if g is pure, then f is the minimal normal extension of g We can use noncommutative Borel functions to show that there is a *formula* for the D_x's and the E_x's.

PROPOSITION 21. *Suppose $\mathcal{X}$ is finite. Then there are subsets $\{\phi_x : x \in \mathcal{X}\}$ and $\{\psi_x : x \in \mathcal{X}\}$ of $\mathcal{B}\langle\mathcal{X}\rangle$ such that*

1. *for every pure subnormal g in $\mathcal{F}(\mathcal{X})$*
$$f(x) = \begin{pmatrix} g(x) & \phi_x(g) \\ 0 & \psi_x^*(g) \end{pmatrix}$$
is a minimal normal extension of g, and
2. *for every normal g in $\mathcal{F}(\mathcal{X})$ and each $x \in \mathcal{X}$ we have $\phi_x(g) = \psi_x^*(g) = 0$.*

Proof. Define the sequence $\{n_k\}$ in $\mathcal{N}(\mathcal{X})$ by $n_k(x) = k$. Since $\mathcal{X}$ is finite, $\{n_k\}$ is cofinal in $\mathcal{N}(\mathcal{X})$. Let $\mathcal{G}_1$ be a maximal disjoint collection of pure subnormal elements g in $\mathcal{F}(\mathcal{X})$ with $n_g \leq n_1$, and for each $k > 1$ let $\mathcal{G}_k$ be a maximal disjoint collection of pure subnormal elements g in $\mathcal{F}(\mathcal{X})$ with $n_g \leq n_k$ and $g \diamond \{h \in \mathcal{F}(\mathcal{X}) : n_h \leq n_{k-1}\}$. It follows from Proposition 6 that there are subsets $\{\phi_x : x \in \mathcal{X}\}$ and $\{\psi_x : x \in \mathcal{X}\}$ of $\mathcal{B}\langle\mathcal{X}\rangle$ such that, for each positive integer k and each $g \in \cup_{k=1}^{\infty}\mathcal{G}_k$ we have

$$f(x) = \begin{pmatrix} g(x) & \phi_x(g) \\ 0 & \psi_x^*(g) \end{pmatrix}$$

is a minimal normal extension of g. However, if g is an arbitrary pure subnormal element of $\mathcal{F}(\mathcal{X})$, we can write

$$g = g_1 \oplus g_2 \oplus \cdots ,$$

with $n_{g_k} \leq n_k$ for each positive integer k, and such that, for $k > 1$,

$$g_k \diamond \{ h \in \mathcal{F}(\mathcal{X}) \ : n_h \leq n_{k-1} \}.$$

Since each g_k is a pure subnormal element, it follows from the definition of the $\mathcal{G}_k$'s that, for each positive integer k,

$$g_k \ll \mathcal{G}_k.$$

It follows that (1) above is true.

Since the class $\mathcal{P}$ of pure subnormal elements in $\mathcal{F}(\mathcal{X})$ is a part class, there is a central projection $P \in \mathcal{B}\langle\mathcal{X}\rangle$ such that

$$\mathcal{P}^\perp = (1 - P)\mathcal{B}\langle\mathcal{X}\rangle.$$

If we replace each ϕ_x and ψ_x with $P\phi_x$ and $P\psi_x$, respectively, then (1) remains true and (2) is true, since $P(g) = 0$ for every normal g. □

4. Noncommutative continuous functions

We will now define two new families of seminorms on $\mathbb{P}(\mathcal{X})$. The first family of seminorms is indexed by the elements of $\mathcal{F}(\mathcal{X})$; given $f \in \mathcal{F}(\mathcal{X})$ and $p \in \mathbb{P}(\mathcal{X})$, define $||p||_f \equiv ||p(f)||$. The second family of seminorms is indexed by the elements of $\mathcal{N}(\mathcal{X})$; given $n \in \mathcal{N}(\mathcal{X})$ and $p \in \mathbb{P}(\mathcal{X})$, define

$$||p||_n \equiv \sup \{ ||p||_f \ : n_f \leq n \}.$$

Given $f \in \mathcal{F}(\mathcal{X})$ we have $||p||_f \leq ||p||_{n_f}$, so the two locally convex topologies are related. We will soon see that the two topologies are actually equivalent. We begin with an analogue of Proposition 1.

PROPOSITION 22. *If $g \in \mathcal{F}(\mathcal{X})$, then there exists $f \in \mathcal{F}(\mathcal{X}, B(\mathcal{H}_\mathcal{X}))$ such that $||p||_f = ||p||_g$ for all $p \in \mathbb{P}(\mathcal{X})$.*

Proof. Let $\mathcal{A} \equiv \{p(g) \ : p \in \mathbb{P}(\mathcal{X}) \}$ and use Zorn's lemma to obtain a maximal collection of vectors $\alpha_i \in \mathcal{H}_g$ ($i \in I$ for some index set I) for which the family of subspaces $\{ \mathcal{H}_i \equiv \overline{\mathcal{A}\alpha_i} \ : i \in I \}$ form an orthogonal family. By maximality we have

$$\mathcal{H}_g = \bigoplus_{i \in I} \mathcal{H}_i.$$

Furthermore, for each $i \in I$ it is true that $\mathcal{H}_i$ is a reducing subspace for every operator $p(g)$ and the dimension of each space $\mathcal{H}_i$ does not exceed the dimension of $\mathcal{H}_\mathcal{X}$.

Let $\mathbb{Q}(\mathcal{X}) \subset \mathbb{P}(\mathcal{X})$ denote the set of polynomials with complex rational coefficients (i.e., coefficients in $\mathbb{Q} + i\mathbb{Q}$), so the cardinality of $\mathbb{Q}(\mathcal{X})$

does not exceed the smallest infinite cardinal greater than or equal to that of $\mathcal{X}$. For each $p \in \mathbb{Q}(\mathcal{X})$ there exists a countable subset $I_p \subseteq I$ such that $||p(g)||$ is equal to the norm of $p(g)$ restricted to the space $\bigoplus_{i \in I_p} \mathcal{H}_i$. If $J = \bigcup_{p \in \mathbb{Q}(\mathcal{X})} I_p$, then the dimension of $\bigoplus_{i \in J} \mathcal{H}_i$ still does not exceed the dimension of $\mathcal{H}_{\mathcal{X}}$ and we have that $||p(g)||$ is equal to the norm of $p(g)$ restricted to the space $\bigoplus_{i \in J} \mathcal{H}_i$ for all $p \in \mathbb{Q}(\mathcal{X})$, and hence for all $p \in \mathbb{P}(\mathcal{X})$. Write $g = g_1 \oplus g_2$ relative to the decomposition $\mathcal{H}_g = (\bigoplus_{i \in J} \mathcal{H}_i) \bigoplus \left(\bigoplus_{i \notin J} \mathcal{H}_i\right)$, let $U : \bigoplus_{i \in J} \mathcal{H}_i \to \mathcal{H}_{\mathcal{X}}$ be an isometry, and let $f = Ug_1U^* \oplus 0$ relative to the decomposition $\mathcal{H}_{\mathcal{X}} = \operatorname{Ran}(U) \oplus \operatorname{Ran}(U)^{\perp}$. We now have for every $p \in \mathbb{P}(\mathcal{X})$ that

$$||p(g)|| = ||p(g_1)|| = ||Up(g_1)U^*|| = ||p(Ug_1U^*)|| = ||p(f)||.$$

□

PROPOSITION 23. *For every $n \in \mathcal{N}(\mathcal{X})$ there exists $f \in \mathcal{F}(\mathcal{X}, B(\mathcal{H}_{\mathcal{X}}))$ such that $n(f) = n$ and $||p||_f = ||p||_n$ for all $p \in \mathbb{P}(\mathcal{X})$.*

Proof. Assume $n \in \mathcal{N}(\mathcal{X})$. As in the preceeding proof, let $\mathbb{Q}(\mathcal{X})$ denote the set of polynomials $p \in \mathbb{P}(\mathcal{X})$ with coefficients in $\mathbb{Q}+i\mathbb{Q}$. Clearly, the cardinality of $\mathbb{Q}(\mathcal{X})$ equals $\dim(\mathcal{H}_{\mathcal{X}})$. It follows from Proposition 1 and the definition of $||p||_n$ (taking countable direct sums) that, for each $p \in \mathbb{Q}(\mathcal{X})$ there is an $f_p \in \mathcal{F}(\mathcal{X}, B(\mathcal{H}_{\mathcal{X}}))$ with $n_{f_p} \leq n$ and $||p||_f = ||p||_n$. We then choose $f \in \mathcal{F}(\mathcal{X}, B(\mathcal{H}_{\mathcal{X}}))$ that is unitarily equivalent to the direct sum of all the f_p's ($p \in \mathbb{Q}(\mathcal{X})$). Then $||p||_f = ||p||_n$ for all $p \in \mathbb{Q}(\mathcal{X})$, and thus for all $p \in \mathbb{P}(\mathcal{X})$. □

We use the symbol $\mathbb{P}_n(\mathcal{X})$ to denote the locally convex space that arises from either of the two equivalent family of seminorms. As with $\mathbb{P}_s(\mathcal{X})$ and $\mathbb{P}_{*-s}(\mathcal{X})$, the completion of $\mathbb{P}_n(\mathcal{X})$ may be identified with the closure of $\mathbb{P}(\mathcal{X})$ in

$$\prod_{f \in \mathcal{F}(\mathcal{X}, B(\mathcal{H}_{\mathcal{X}}))} B(\mathcal{H}_f),$$

this time when each $B(\mathcal{H}_f)$ carries to operator norm topology. We will use the symbol $\mathcal{C}\langle\mathcal{X}\rangle$ to denote this closure, which reflects our view of this space as a noncommutative version of $\mathcal{C}$. Since the norm topology is coarser than the strong operator topology, we can (and do) view $\mathcal{C}\langle\mathcal{X}\rangle$ as a subset of $\mathcal{B}\langle\mathcal{X}\rangle$. It is a consequence of Proposition 22 that the closure of $\mathbb{P}(\mathcal{X})$ in $\prod_{f \in \mathcal{F}(\mathcal{X}, B(\mathcal{H}_{\mathcal{X}}))} B(\mathcal{H}_f)$ is isomorphic to $\mathcal{C}\langle\mathcal{X}\rangle$. We now characterize which elements of $\mathcal{B}\langle\mathcal{X}\rangle$ are actually in $\mathcal{C}\langle\mathcal{X}\rangle$. If $\mathcal{S}$ is a subset of a C^*-algebra, we let $C^*(\mathcal{S})$ denote the smallest unital

C^*-algebra containing $\mathcal{S}$ and let $C_0^*(\mathcal{S})$ denote the smallest C^*-algebra containing $\mathcal{S}$.

PROPOSITION 24. *If $\psi \in \mathcal{B}\langle\mathcal{X}\rangle$, then the following are equivalent.*

1. $\psi \in \mathcal{C}\langle\mathcal{X}\rangle$,
2. *there is a net $\{p_\lambda\}$ in $\mathbb{P}(\mathcal{X})$ such that*
$$||\psi(f) - p_\lambda(f)|| \to 0$$
uniformly on norm-bounded subclasses of $\mathcal{F}(\mathcal{X})$,
3. *for every $f \in \mathcal{F}(\mathcal{X}, B(\mathcal{H}_\mathcal{X}))$ we have $\psi(f) \in C^*(f(\mathcal{X}))$,*
4. *for every $f \in \mathcal{F}(\mathcal{X}, B(\mathcal{H}_\mathcal{X}))$ and every representation*
$$\pi : C^*(f(\mathcal{X}) \cup \{\psi(f)\}) \to B(\mathcal{H}_\mathcal{X})$$
we have $\pi(\psi(f)) = \psi(\pi \circ f)$,
5. *for every $n \in \mathcal{N}(\mathcal{X})$, the restriction of ψ to*
$$\{ f \in \mathcal{F}(\mathcal{X}, B(\mathcal{H}_\mathcal{X})) \ : n_f \le n \}$$
is continuous with respect to the topology of pointwise norm convergence on $\mathcal{F}(\mathcal{X}, B(\mathcal{H}_\mathcal{X}))$ and the norm topology on $B(\mathcal{H}_\mathcal{X})$,
6. *for every $n \in \mathcal{N}(\mathcal{X})$, the restriction of ψ to*
$$\{ f \in \mathcal{F}(\mathcal{X}, B(\mathcal{H}_\mathcal{X})) \ : n_f \le n \}$$
is continuous with respect to the topology of pointwise $*$-strong convergence on $\mathcal{F}(\mathcal{X}, B(\mathcal{H}_\mathcal{X}))$ and the weak operator topology on $B(\mathcal{H}_\mathcal{X})$.

Proof. (3) $\Rightarrow$ (1) Let $\mathcal{U}$ be a neighborhood of ψ in $\mathcal{B}\langle\mathcal{X}\rangle$ with respect to the topology of pointwise norm convergence. By Proposition 1 we may assume there exist $f_1, f_2, \ldots, f_k \in \mathcal{F}(\mathcal{X}, B(\mathcal{H}_\mathcal{X}))$ and $\epsilon > 0$ such that
$$\mathcal{U} = \{ \rho \in \mathcal{B}\langle\mathcal{X}\rangle \ : ||\rho(f_i) - \psi(f_i)|| < \epsilon \text{ for } 1 \le i \le k \}.$$
Let $f = f_1 \oplus f_2 \oplus \cdots \oplus f_k$ and choose g in $\mathcal{F}(\mathcal{X}, B(\mathcal{H}_\mathcal{X}))$ so that g is unitarily equivalent to f. Then $\psi(g) \in C^*(g(\mathcal{X}))$ by our hypothesis, and consequently
$$\psi(f) = \psi(f_1) \oplus \psi(f_2) \oplus \cdots \oplus \psi(f_k) \in C^*(f(\mathcal{X})).$$
It follows that there exists a polynomial p such that
$$\max_{1 \le i \le k} ||\psi(f_i) - p(f_i)|| = ||\psi(f) - p(f)|| < \epsilon,$$
so $\psi \in \mathcal{C}\langle\mathcal{X}\rangle$.

(5) $\Rightarrow$ (3) Suppose $f \in \mathcal{F}(\mathcal{X}, B(\mathcal{H}_\mathcal{X}))$ and $\{U_\lambda\}$ is a net of unitary operators such that

$$||U_\lambda f(x) - f(x)U_\lambda|| \to 0$$

for every $x \in \mathcal{X}$. It follows that $U_\lambda^* f U_\lambda \to f$ in the topology of pointwise norm convergence. Since $n_f = n_{(U_\lambda^* f U_\lambda)}$ for all λ, we have by (5) that

$$||\psi(f) - \psi(U_\lambda^* f U_\lambda)|| = ||\psi(f) - U_\lambda^* \psi(f) U_\lambda|| = ||U_\lambda \psi(f) - \psi(f) U_\lambda|| \to 0.$$

It follows from the asymptotic double commutant theorem [7] that

$$\psi(f) \in C^*(f(\mathcal{X})).$$

(4) $\Rightarrow$ (3) Suppose $f \in \mathcal{F}(\mathcal{X}, B(\mathcal{H}_\mathcal{X}))$ and $\psi(f)$ is not in $C^*(f(\mathcal{X}))$. It follows that there is a continuous linear functional η on $C^*(f(\mathcal{X}) \cup \{\psi(f)\})$ such that $\eta(\psi(f)) \neq 0$ while η vanishes on $C^*(f(\mathcal{X}))$. Hence there is a representation

$$\pi : C^*(f(\mathcal{X}) \cup \{\psi(f)\}) \to B(\mathcal{H}_\pi)$$

and vectors $\alpha, \beta \in \mathcal{H}_\pi$ such that

$$\eta(T) = \langle \pi(T)\alpha, \beta \rangle$$

for every $T \in C^*(f(\mathcal{X}) \cup \{\psi(f)\})$. (One sees this by writing an arbitrary functional as a combination of positive functionals, then mimicking the GNS construction for the positive functionals; see [1] for details.) Since the cardinality of the $*$-algebra generated by $f(\mathcal{X}) \cup \{\psi(f)\}$ over the complex rationals is at most the dimension of $\mathcal{H}_\mathcal{X}$, we can assume that the dimension of $\mathcal{H}_\pi$ is at most the dimension of $\mathcal{H}_\mathcal{X}$. It follows that we may assume $\mathcal{H}_\pi = \mathcal{H}_\mathcal{X}$. By (4) we know that $\pi(\psi(f)) = \psi(\pi \circ f)$, so

$$\eta(\psi(f)) = \langle \pi(\psi(f))\alpha, \beta \rangle = \langle \psi(\pi \circ f)\alpha, \beta \rangle.$$

Since η vanishes on $C^*(f(\mathcal{X}))$ we know that $C^*((\pi \circ f)(\mathcal{X}))\alpha \perp \beta$. But $\psi \in \mathcal{B}\langle \mathcal{X} \rangle$ so $\psi(\pi \circ f)$ is a strong limit of operators in $C^*((\pi \circ f)(\mathcal{X}))$, and hence $\psi(\pi \circ f)\alpha \perp \beta$, contradicting $\eta(\psi(f)) \neq 0$. Thus $\psi(f) \in C^*(f(\mathcal{X}))$, which proves (3).

(6) $\Rightarrow$ (4) Assume that $f \in \mathcal{F}(\mathcal{X}, B(\mathcal{H}_\mathcal{X}))$ and

$$\pi : C^*(f(\mathcal{X}) \cup \{\psi(f)\}) \to B(\mathcal{H}_\pi)$$

is a representation. Let $m = \dim(\mathcal{H}_\mathcal{X})$. By replacing f with an element of $\mathcal{F}(\mathcal{X}, B(\mathcal{H}_\mathcal{X}))$ that is unitarily equivalent to a direct sum of m copies of f, we can assume that f is unitarily equivalent to a direct sum of m copies of f. Since $\psi \in \mathcal{B}\langle \mathcal{X} \rangle$ we know that $\psi(f)$ is unitarily equivalent (same unitary) to a direct sum of m copies of $\psi(f)$. Let ι denote the identity representation on $C^*(f(\mathcal{X}) \cup \{\psi(f)\})$. It follows from [8] that

ι is approximately equivalent to $\iota \oplus \pi$, so there exists a net $\{U_\lambda\}$ of unitaries such that

$$U_\lambda^* T U_\lambda \to \pi(T)$$

for every $T \in C^*(f(\mathcal{X}) \cup \{\psi(f)\})$. Hence $U_\lambda^* f U_\lambda \to \pi \circ f$ in the pointwise $*$-strong topology and, for every λ, we have $n_{(U_\lambda^* f U_\lambda)} = n_f$. It follows from (6) that $\psi(U_\lambda^* f U_\lambda) \to \psi(\pi \circ f)$. However, $\psi(U_\lambda^* f U_\lambda) = U_\lambda^* \psi(f) U_\lambda \to \pi(\psi(f))$ in the $*$-strong operator topology. Hence $\pi(\psi(f)) = \psi(\pi \circ f)$ and (4) is established.

A consequence of Proposition 23 is that convergence in $\mathcal{C}\langle\mathcal{X}\rangle$ happens uniformly on bounded subsets of $\mathcal{F}(\mathcal{X}, B(\mathcal{H}_\mathcal{X}))$, i.e., on sets of the form

$$\{ f \in \mathcal{F}(\mathcal{X}, B(\mathcal{H}_\mathcal{X})) \ : n_f \le n \}.$$

In view of this the implications (1) $\Leftrightarrow$ (2), (1) $\Rightarrow$ (5) and (1) $\Rightarrow$ (6) are obvious. This completes the proof. □

Suppose T is a normal operator on a separable Hilbert space. It is well-known that

1. $C^*(T) = \{ \phi(T) \ : \phi : \mathbb{C} \to \mathbb{C}$ a continuous function $\}$,
2. $C_0^*(T) = \{ \phi(T) \ : \phi : \mathbb{C} \to \mathbb{C}$ a continuous function and $\phi(0) = 0\}$.

The following result is a generalization of this fact and its extension obtained in [6].

Proposition 25. *Suppose $f \in \mathcal{F}(\mathcal{X})$. Then*

1. $C^*(f(\mathcal{X})) = \{ \phi(f) \ : \phi \in \mathcal{C}\langle\mathcal{X}\rangle \}$,
2. $C_0^*(f(\mathcal{X})) = \{ \phi(f) \ : \phi \in \mathcal{C}\langle\mathcal{X}\rangle$ *and* $\phi(0) = 0 \}$.

Proof. (1) Assume that $T \in C^*(f(\mathcal{X}))$, and with no loss of generality assume $||T|| < 1$. We can choose a polynomial p_0 such that $||p_0(f)|| < 1$ and $||T - p_0(f)|| < \frac{1}{2}$. Let ϕ_0 be the truncation of p_0 by 1. Then $\phi_0 \in \mathcal{C}^b\langle\mathcal{X}\rangle$, $||\phi_0|| \le 1$ and $\phi_0(f) = p_0(f)$. Next choose a polynomial p_1 so $||p_1(f)|| < \frac{1}{2}$ and so that $||T - \phi_0(f) - p_1(f)|| < \frac{1}{4}$, and let ϕ_1 be the truncation of p_1 by $\frac{1}{2}$. Then $\phi_1 \in \mathcal{C}^b\langle\mathcal{X}\rangle$, $||\phi_1|| \le \frac{1}{2}$ and $\phi_1(f) = p_1(f)$. Proceeding inductively, we can construct a sequence $\{\phi_n\}$ in $\mathcal{C}^b\langle\mathcal{X}\rangle$ with $||\phi_n|| \le \frac{1}{2^n}$ such that $\sum_{n=0}^\infty \phi_n(f)$ converges in norm to T. However, $\sum_{n=0}^\infty \phi_n$ converges in point-norm to an element $\phi \in \mathcal{C}^b\langle\mathcal{X}\rangle$, and it follows that $\phi(f) = T$. Thus we have that

$$C^*(f(\mathcal{X})) \subseteq \{ \phi(f) \ : \phi \in \mathcal{C}\langle\mathcal{X}\rangle \},$$

and the reverse inclusion is obvious from Proposition 24.

(2) This follows easily from (1). □

One nice thing that the preceding proposition gives us is a way to consider corresponding elements in C*-algebras. For example, if we are looking at $C^*(S)$ and $C^*(T)$, and we think of S as corresponding to T, then $1 + 2S - 3iS^*S^3$ naturally corresponds to $1 + 2T - 3iT^*T^3$. However, a general element in $C^*(S)$ is $\phi(S)$ for some $\phi \in \mathcal{C}\langle\mathcal{X}\rangle$ (here $\mathcal{X}$ is a singleton) and the naturally corresponding element of $C^*(T)$ is $\phi(T)$. Furthermore, if π is a unital $*$-homomorphism from $C^*(S)$ to $C^*(T)$ that sends S to T, then $\pi(\phi(S)) = \phi(T)$ for every $\phi \in \mathcal{C}\langle\mathcal{X}\rangle$. This leads to the following generalization of corollary 3.2 in [6].

PROPOSITION 26. *Suppose $f, g \in \mathcal{F}(\mathcal{X})$. The following are equivalent:*

1. *there is a unital $*$-homomorphism π from $C^*(f(\mathcal{X}))$ to $C^*(g(\mathcal{X}))$ such that $\pi \circ f = g$,*
2. *for every $\phi \in \mathcal{C}\langle\mathcal{X}\rangle$ one has*

$$\phi(f) = 0 \;\Rightarrow\; \phi(g) = 0.$$

Proof. The implication (1) $\Rightarrow$(2) is obvious from the fact that $\pi(\phi(f)) = \phi(\pi \circ f)$ for every $\phi \in \mathcal{C}\langle\mathcal{X}\rangle$. To prove the reverse implication, simply define π on the generators by $\pi(\phi(f)) = \phi(g)$. Statement (2) implies that π is a well-defined $*$-homomorphism. □

We next describe point-norm continuous linear functionals on $\mathcal{C}\langle\mathcal{X}\rangle$.

PROPOSITION 27. *Assume τ is a point-norm continuous linear functional on $\mathcal{C}\langle\mathcal{X}\rangle$. There exists $f \in \mathcal{F}(\mathcal{X})$ and $\alpha, \beta \in \mathcal{H}_f$ such that*

$$\tau(\phi) = \langle\phi(f)\alpha, \beta\rangle$$

for all $\phi \in \mathcal{C}\langle\mathcal{X}\rangle$.

Proof. Assume τ is a point-norm continuous linear functional on $\mathcal{C}\langle\mathcal{X}\rangle$. There exists $M > 0$ and $g \in \mathcal{F}(\mathcal{X}, B(\mathcal{H}_{\mathcal{X}}))$ such that

$$|\tau(\phi)| \leq M||\phi(g)||$$

for all $\phi \in \mathcal{C}\langle\mathcal{X}\rangle$. It follows that the mapping $\phi(g) \mapsto \tau(\phi)$ is a continuous linear functional on the C*-algebra generated by $g(\mathcal{X})$, so there exists a representation $\pi : C^*(g(\mathcal{X})) \to B(\mathcal{H})$ and vectors $\alpha, \beta \in \mathcal{H}$ such that

$$\tau(\phi) = \langle\pi(\phi(f))\alpha, \beta\rangle.$$

Letting $f = \pi \circ g$ and using Proposition 24 completes the proof. □

Although $\mathcal{B}\langle\mathcal{X}\rangle$ contains many projections, $\mathcal{C}\langle\mathcal{X}\rangle$ does not.

PROPOSITION 28. *The only projections in $\mathcal{C}\langle\mathcal{X}\rangle$ are the constant 0 and constant 1 functions.*

Proof. Assume that $\phi \in \mathcal{C}\langle \mathcal{X} \rangle$ is a non-zero projection and define

$$f_0 : \mathcal{X} \to B(\mathcal{H}_{\mathcal{X}})$$

to be the function that takes every $x \in \mathcal{X}$ to the zero operator in $B(\mathcal{H}_{\mathcal{X}})$. Now f_0 is a direct sum of constant zero functions $\hat{f}_0 : \mathcal{X} \to \mathbb{C}$. Since $\phi(\hat{f}_0)$ is a projection on a one dimensional space it must be either 0 or 1. By Proposition 4 we know $\phi(f_0)$ is a direct sum of copies of $\phi(\hat{f}_0)$, and hence must either be the zero operator or identity operator in $B(\mathcal{H}_{\mathcal{X}})$. Given any $g \in \mathcal{F}(\mathcal{X}, B(\mathcal{H}_{\mathcal{X}}))$, the set

$$\{rg \ : 0 \le r \le 1\}$$

is connected in the pointwise norm topology. By Proposition 24 we know that

$$\{\phi(rg) \ : 0 \le r \le 1\}$$

is a norm connected subset of $B(\mathcal{H}_{\mathcal{X}})$ that consists of projections. Hence this set is a singleton, since it contains $\phi(f_0)$ which is either the zero operator or the identity operator. It follows that $\phi(g) = \phi(f_0)$ for every $g \in \mathcal{F}(\mathcal{X}, B(\mathcal{H}_{\mathcal{X}}))$, so ϕ is either the constant 0 function or the constant 1 function. □

We now wish to consider elements in $\mathcal{B}\langle \mathcal{X} \rangle$ that are continuous in other topologies. We first need a few definitions. Suppose $x_1, \ldots, x_n \in \mathcal{X}$ and let $\mathcal{M}$ be the set of all non* monomials in the variables $x_1, \ldots, x_n$. We define a noncommutative *entire function* ϕ to be a formal expression of the form

$$\phi = \alpha_0 + \sum_{m \in \mathcal{M}} \alpha_m m$$

such that, for every $f \in \mathcal{F}(\{x_1, \ldots, x_n\})$ (equivalently, for every $f \in \mathcal{F}(\mathcal{X})$) the sum $\phi(f)$ is norm convergent. This is equivalent to saying that the sum $\sum_{m \in \mathcal{M}} |\alpha_m| m$ converges when $x_1, \ldots, x_n$ are allowed to take on arbitrary complex values. Recall that if $\mathcal{H}$ is a Hilbert space and $\mathcal{S} \subset B(\mathcal{H})$, then AlgLat$(\mathcal{S})$ is the set of all $T \in B(\mathcal{H})$ leaving invariant every closed linear subspace left invariant by every element in $\mathcal{S}$.

Proposition 29. *Suppose $\phi \in \mathcal{B}\langle \mathcal{X} \rangle$. The following are equivalent:*

1. *ϕ is point-strong operator to strong operator continuous on bounded subsets of $\mathcal{F}(\mathcal{X}, B(\mathcal{H}_{\mathcal{X}}))$,*
2. *ϕ is a noncommutative entire function in finitely many variables from $\mathcal{X}$,*
3. *$\phi(f) \in$ AlgLat$(f(\mathcal{X}))$ for every $f \in \mathcal{F}(\mathcal{X}, B(\mathcal{H}_{\mathcal{X}}))$,*

4. *for every* $f \in \mathcal{F}(\mathcal{X}, B(\mathcal{H}_{\mathcal{X}}))$ *and every invertible operator* $S \in B(\mathcal{H}_{\mathcal{X}})$,
$$\phi(S^{-1}fS) = S^{-1}\phi(f)S.$$

Proof. (1) $\Longrightarrow$ (3) Let $m = \dim(\mathcal{H}_{\mathcal{X}})$. Suppose (1), and suppose $f \in \mathcal{F}(\mathcal{X}, B(\mathcal{H}_{\mathcal{X}}))$ and $M \in \mathrm{Lat}(f(\mathcal{X}))$. We want to show that $M \in \mathrm{Lat}(\phi(f))$. By replacing f with a direct sum of m copies of f and M with a direct sum of m copies of M, we can assume that $\dim(M) = \dim(\mathcal{H}_{\mathcal{X}})$. By [12] we can choose a net $\{U_\lambda\}$ of unitaries from M onto $\mathcal{H}_{\mathcal{X}}$ that converge in the strong operator topology to the inclusion map from M to $\mathcal{H}_{\mathcal{X}}$. It is easily shown that an operator $T \in B(\mathcal{H}_{\mathcal{X}})$ leaves M invariant if and only if $U_\lambda^* T U_\lambda \to T|_M$ in the strong operator topology. Thus, $U_\lambda^* f U_\lambda \to f|_M$ in the point-strong topology, which by (1), implies $U_\lambda^* \phi(f) U_\lambda \to \phi(f)|_M$, and it therefore follows that $M \in \mathrm{Lat}(\phi(f))$.

(3) $\Longrightarrow$ (2) It follows from (3) that for any $f \in \mathcal{F}(\mathcal{X}, B(\mathcal{H}_{\mathcal{X}}))$
$$\phi(f)^{(\infty)} = \phi(f^{(\infty)}) \in \mathrm{AlgLat}(f^{(\infty)}(\mathcal{X})) = \mathcal{A}_s(f^{(\infty)}(\mathcal{X})),$$
where $f^{(\infty)} = f \oplus f \oplus \cdots$ and $\mathcal{A}_s(f^{(\infty)}(\mathcal{X}))$ denotes the strong operator closed unital algebra generated by $f^{(\infty)}(\mathcal{X})$. It follows, for every $f \in \mathcal{F}(\mathcal{X}, B(\mathcal{H}_{\mathcal{X}}))$, that $\phi(f)$ is a strong-operator limit of non* polynomials in f. Let $\mathcal{S}$ denote the free unital semigroup generated by $\mathcal{X}$, let $\mathcal{H} = \ell^2(\mathcal{S})$, and let f be the representation of $\mathcal{X}$ in the left regular representation of $\mathcal{S}$, i.e., for every $x, y \in \mathcal{S}$ and $h \in \mathcal{H}$, we have
$$(f(x)h)(y) = h(xy).$$

Since the right regular representation of $\mathcal{S}$ is in the commutant of the left regular representation, and since the unit e of $\mathcal{S}$ is a cyclic vector for the right regular representation, e must be a separating vector for the strong-operator closed unital algebra $\mathcal{A}$ generated by $f(\mathcal{X})$. We know $\phi(f) \in \mathcal{A}$. Write
$$\phi(f)(e) = a_e e + \sum_{n=1}^{\infty} \sum_{\sigma \in \mathcal{S}_n} a_\sigma \sigma,$$
where $\mathcal{S}_n = \mathcal{X}^n$ is the set of monomials of degree n. For each positive integer N we know $\phi(f \oplus (Nf)) = \phi(f) \oplus \phi(Nf) \in \mathcal{A}_s([f \oplus (Nf)](\mathcal{X}))$, and since $e \oplus e$ is a separating vector, we must have
$$\phi(Nf) = a_e e + \sum_{n=1}^{\infty} \sum_{\sigma \in \mathcal{S}_n} N^n a_\sigma \sigma.$$
Since the coefficients must be square-summable for each N, and since $\mathcal{X}$ is finite, it easily follows (from the Cauchy-Schwartz inequality) that

$a_e e+\sum_{n=1}^{\infty}\sum_{\sigma\in\mathcal{Y}_n} a_\sigma\sigma$ defines a noncommutative entire function. Hence $\psi(h) = a_e e+\sum_{n=1}^{\infty}\sum_{\sigma\in\mathcal{Y}_n} a_\sigma\sigma(h)$ defines an element of $\mathcal{B}\langle\mathcal{X}\rangle$ satisfying (1).

Now suppose g is arbitrary and choose $N > 2m\max\{||g(x)|| : x \in \mathcal{X}\}$, where m is the cardinality of $\mathcal{X}$. If

$$p = c_e e + \sum_{n=1}^{\infty}\sum_{\sigma\in\mathcal{X}_n} c_\sigma\sigma$$

is any non* polynomial, we have

$$\begin{aligned} ||p(h)|| &\leq |c_e| + \textstyle\sum_{n=1}^{\infty}\sum_{\sigma\in\mathcal{X}_n}|c_\sigma|\left(\frac{N}{2m}\right)^n \\ &\leq ||p(Nf)||\left[1+\textstyle\sum_{n=1}^{\infty}\sum_{\sigma\in\mathcal{X}_n}\left(\frac{1}{2m}\right)^n\right]^{\frac{1}{2}} \\ &= ||p(Nf)(e)||\left[1+\textstyle\sum_{n=1}^{\infty}m^n\left(\frac{1}{2m}\right)^n\right]^{\frac{1}{2}} \\ &\leq ||p(Nf)||\sqrt{2}. \end{aligned}$$

It follows that the map $\pi(p(Nf)) = p(h)$ extends to a linear mapping from $\mathcal{A}_s((Nf)(\mathcal{X}))$ to $\mathcal{A}_s(h(\mathcal{X}))$ that is strong-operator to norm continuous. Hence, it follows that $\mathcal{A}_s((Nf\oplus h)(\mathcal{X}))$ is the graph of π. Since $\phi(Nf)\oplus\phi(h) = \phi(Nf\oplus h)$ is in $\mathcal{A}_s((Nf\oplus h)(\mathcal{X}))$, it follows that $\phi(h) = \pi(\phi(Nf)) = \pi(\psi(Nf)) = \psi(h)$.

(2) $\Longrightarrow$ (4), (2) $\Longrightarrow$ (1) These are obvious.

(4) $\Longrightarrow$ (3) Suppose $M \in \mathrm{Lat}(f(\mathcal{X}))$ and let P be the orthogonal projection onto M. Define a sequence $\{S_n\}$ of invertible operators by

$$S_n = P + \frac{1}{n}(1-P).$$

Clearly, $S_n^{-1}fS_n \to fP + (1-P)f$ in point-norm, so $\{S_n^{-1}fS_n\}$ is a bounded sequence. Hence $g = \sum_n^{\oplus} S_n^{-1}fS_n \in \mathcal{F}(\mathcal{X})$. Thus $\phi(g)$ is a bounded operator. However, by (4),

$$\phi(g) = \sum_n^{\oplus} S_n^{-1}\phi(f)S_n,$$

which implies $\{S_n^{-1}\phi(f)S_n\}$ is bounded. But

$$(1-P)S_n^{-1}\phi(f)S_nP = n(1-P)\phi(f)P$$

implies $(1-P)\phi(f) = 0$, or $M \in \mathrm{Lat}(\phi(f))$. This proves (3). □

Suppose $f_0, f_1, \ldots$ are arbitrary functions from $\mathbb{C}$ to $\mathbb{C}$. Using a variant on the Lagrange interpolation polynomial we can, for any $\lambda = (F, n)$ with F a finite subset of $\mathbb{C}$ and $n \in \mathbb{N}$, find a polynomial p_λ so that

$$p_\lambda^{(k)}(z) = f_k(z)$$

for $0 \leq k \leq n$ and $z \in F$. We therefore have a net $\{p_\lambda\}$ of polynomials such that,

$$p_\lambda^{(k)}(z) \to f_k(z)$$

for any $z \in \mathbb{C}$ and any integer $k \geq 0$. Using the Jordan canonical form, it follows that, for each finite complex matrix T, there is a matrix $\phi(T)$ such that

$$p_\lambda(T) \to \phi(T).$$

However, the preceding theorem shows that if, for every operator T on ℓ^2 there is an operator $\phi(T)$ such that $p_\lambda(T) \to \phi(T)$ in the strong operator topology, then f_0 must be entire and $f_n = f_0^{(n)}$ for each positive integer n. The preceding theorem answers two very natural questions. The first is what happens when you complete the non* polynomials and the second is what happens if you insist on preserving similarity rather than unitary equivalence.

COROLLARY 30. *Suppose $\mathcal{X}$ is finite and $\mathcal{P}(\mathcal{X})$ is the set of non* polynomials on $\mathcal{X}$. Then the point-strong closure (completion) of $\mathcal{P}(\mathcal{X})$ equals the point-$*$-strong closure (completion) of $\mathcal{P}(\mathcal{X})$ equals the point-weak closure of $\mathcal{P}(\mathcal{X})$ equals the noncommutative entire functions.*

Here is the analog of the preceding theorem for point-weak continuity.

PROPOSITION 31. *Suppose $\phi \in B\langle\mathcal{X}\rangle$ and $\phi(0) = 0$. The following are equivalent.*

1. ϕ is point-weak to weak operator topology continuous on bounded subsets of $\mathcal{F}(\mathcal{X}, \mathcal{B}\langle\mathcal{H}_\mathcal{X}\rangle)$.

2. ϕ has the form

$$\phi = \sum_{k=1}^{n} [\alpha_k x_k + \beta_k x_k^*]$$

where $\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n \in \mathbb{C}$, and $x_1, \ldots, x_n \in \mathcal{X}$.

3. ϕ is $\mathbb{R}$-linear and $\phi|\mathcal{F}(\mathcal{X}, \mathbb{C})$ is point-norm continuous on bounded sets.

Proof. (2) $\Rightarrow$ (1) This is obvious.

(1) $\Rightarrow$ (3) Use the first paragraph of the original proof.

(3) $\Rightarrow$ (2) Suppose $x \in \mathcal{X}$. We define $\delta_x \in \mathcal{F}(\mathcal{X}, B(B(\mathcal{H}_\mathcal{X})))$ by

$$\delta_x(y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

and, for $T \in B\left(B(\mathcal{H}_{\mathcal{X}})\right)$, we define $T\delta_x$ by

$$(T\delta_x)(y) = \begin{cases} 0 & \text{if } x = y \\ T & \text{if } x \neq y. \end{cases}$$

It is clear that, for each $f \in \mathcal{F}\left(\mathcal{X}, B\left(B(\mathcal{H}_{\mathcal{X}})\right)\right)$, that

$$\sum_{x\in\mathcal{X}} f(x)\,\delta_x$$

converges point-norm to f. It follows from the linearity of ϕ that

$$\phi(f) = \sum_{x\in\mathcal{X}} \phi\left(f(x)\,\delta_x\right).$$

However, we can identify the elements of the form $f(x)\,\delta_x$ with the elements of $\mathcal{F}\left(\{x\}, B(\mathcal{H}_{\mathcal{X}})\right) = B\left(B(\mathcal{H}_{\mathcal{X}})\right)$, and ϕ with an $\mathbb{R}$-linear map ψ on $B\left(B(\mathcal{H}_{\mathcal{X}})\right)$. If we restrict ψ to $\mathbb{C}1$, the $\mathbb{R}$-linearity implies that $\psi(z) = \alpha_x z + \beta_x \bar{z}$, and it follows that $\psi(D) = \alpha_x D + \beta_x D^*$. However, the linear span of the diagonal operators (in fact operators of the form P or iP with P a projection) is all of $B\left(B(\mathcal{H}_{\mathcal{X}})\right)$ (see [4]). Hence we have

$$\phi\left(f(x)\,\delta_x\right) = \alpha_x f(x) + \beta_x f(x)^*.$$

Suppose there is a sequence $x_1, x_2, \ldots$ of distinct elements of $\mathcal{X}$ for which $|\dot{\alpha}_{x_n}| + |\beta_{x_n}| \neq 0$, and choose $\lambda_n \in \mathbb{C}$ such that, for each n,

$$\left|\sum_{k=1}^{n} \alpha_{x_k}\lambda_k + \beta_k\bar{\lambda}_k\right| \geq n.$$

Define $f \in \mathcal{F}\left(\mathcal{X}, B\left(B(\mathcal{H}_{\mathcal{X}})\right)\right)$ by $f(x_k) = \lambda_k$ for $k = 1, 2, 3, \ldots$ and $f(x) = 0$ otherwise. It follows that

$$\phi(f) = \sum_{k=1}^{\infty} \alpha_{x_k}\lambda_k + \beta_k\bar{\lambda}_k,$$

but the sum does not converge. Hence the set of $x \in \mathcal{X}$ such that $|\dot{\alpha}_x| + |\beta_x| \neq 0$ must be finite, say $\{x_1, \ldots, x_n\}$. This implies (2). □

COROLLARY 32. *If $\phi \in B\langle\mathcal{X}\rangle$ is point-weak to weak operator continuous then ϕ has the form*

$$\phi = \alpha_0 + \sum_{k=1}^{n} \alpha_{x_k} x_k + \beta_k x_k^*.$$

REMARK. If $\phi, \psi \in \mathcal{C}\langle\mathcal{X}\rangle$, it is easily seen that $\phi = \psi$ if and only if $\phi(f) = \psi(f)$ for every *irreducible* $f \in \mathcal{F}(\mathcal{X})$, since, given an arbitrary $f \in \mathcal{F}(\mathcal{X})$, there is a $g \in \mathcal{F}(\mathcal{X})$ and a net $\{U_\lambda\}$ of unitaries such that g is a direct sum of irreducible elements and $U_\lambda^* g U_\lambda \to f$ in the point-$*$-strong topology. However, there is an $h \in \mathcal{F}(\mathcal{X})$ such that h is a direct sum of irreducible finite dimensional elements of $\mathcal{F}(\mathcal{X})$ and a $*$-homomorphism π on $C^*(h(\mathcal{X}))$ such that $f = \pi \circ h$. Thus it follows that $\phi = \psi$ if and only if $\phi(f) = \psi(f)$ for every irreducible $f \in \mathcal{F}(\mathcal{X})$ with $\dim \mathcal{H}_f < \infty$.

5. Continuous part properties and ideals

We call a part class $\mathcal{P}$ in $\mathcal{F}(\mathcal{X})$ a *continuous part class* if there is a family $C \subseteq \mathcal{C}\langle\mathcal{X}\rangle$ such that $C^\perp = \mathcal{P}$, i.e., $\mathcal{P}$ is equationally defined by a family of noncommutative *continuous* functions. We say that a part class $\mathcal{P}$ is *bounded* if there exists $n \in \mathcal{N}(\mathcal{X})$ such that $n_f \leq n$ for every $f \in \mathcal{P}$. Note that a part class $\mathcal{P}$ is bounded if and only if there exists $f \in \mathcal{P}$ such that $\mathcal{P} = \{f\}^{\perp\perp}$. Indeed, if $\mathcal{P} = \{f\}^{\perp\perp}$, then $n_g \leq n_f$ for all $g \in \mathcal{P}$, and if $\mathcal{P}$ is bounded we may take

$$f = \bigoplus_{g \in \mathcal{F}(\mathcal{X}, B(\mathcal{H}_\mathcal{X})) \cap \mathcal{P}} g.$$

The following result is the key to characterizing the bounded continuous part classes.

PROPOSITION 33. *Let $\mathcal{L}$ be any convex subset of $\mathcal{C}\langle\mathcal{X}\rangle$. Then the point-norm closure of $\mathcal{L}$ in $\mathcal{C}\langle\mathcal{X}\rangle$ is equal to $\overline{\mathcal{L}} \cap \mathcal{C}\langle\mathcal{X}\rangle$, where $\overline{\mathcal{L}}$ denotes the point strong closure of $\mathcal{L}$ in $\mathcal{B}\langle\mathcal{X}\rangle$ (which is the same as the point $*$-strong closure).*

Proof. We must prove that $\overline{\mathcal{L}} \cap \mathcal{C}\langle\mathcal{X}\rangle$ is contained in the point-norm closure of $\mathcal{L}$ (the opposite inclusion is trivial). Assume that τ is any point-norm continuous functional on $\mathcal{C}\langle\mathcal{X}\rangle$. Thus there exists $f \in \mathcal{F}(\mathcal{X})$ and $\alpha, \beta \in \mathcal{H}_f$ such that

$$\tau(\phi) = \langle \phi(f)\alpha, \beta \rangle$$

for all $\phi \in \mathcal{C}\langle\mathcal{X}\rangle$ (by Proposition 27). If $\phi \in \overline{\mathcal{L}} \cap \mathcal{C}\langle\mathcal{X}\rangle$ and $\{\phi_\lambda\}$ is a net in $\mathcal{L}$ converging point strongly to ϕ, then $\tau(\phi) = \lim_\lambda \langle \phi_\lambda(f)\alpha, \beta \rangle$ is in the closure of $\tau(\mathcal{L})$. Thus ϕ is in the point-norm closure of $\mathcal{L}$ by the Hahn Banach theorem.

The fact that the closure of $\mathcal{L}$ in $\mathcal{B}\langle\mathcal{X}\rangle$ is the same relative to either the point strong or point $*$-strong topologies is also a consequence of the Hahn-Banach theorem and Proposition 14. □

PROPOSITION 34. *Assume that $\mathcal{I}$ is a point-norm closed ideal in $\mathcal{C}\langle\mathcal{X}\rangle$. It follows that $\mathcal{I} = \mathcal{I}^{\perp\perp} \cap \mathcal{C}\langle\mathcal{X}\rangle$. In particular, there is a one to one correspondence between continuous part classes, point-norm closed ideals in $\mathcal{C}\langle\mathcal{X}\rangle$, and strongly closed ideals $\mathcal{J}$ in $\mathcal{B}\langle\mathcal{X}\rangle$ that have $\mathcal{J} \cap \mathcal{C}\langle\mathcal{X}\rangle$ point strongly dense in $\mathcal{J}$.*

Proof. The fact that $\mathcal{I} = \mathcal{I}^{\perp\perp} \cap \mathcal{C}\langle\mathcal{X}\rangle$, which was proved in Proposition 33, ensures that the mapping $\mathcal{I} \mapsto \mathcal{I}^{\perp\perp}$ is injective. □

COROLLARY 35. *Point-norm closed ideals in $\mathcal{C}\langle\mathcal{X}\rangle$ and point strongly closed ideals in $\mathcal{B}\langle\mathcal{X}\rangle$ are $*$-ideals.*

COROLLARY 36. *If $\mathcal{C} \subseteq \mathcal{C}\langle\mathcal{X}\rangle$, then the point-norm closed ideal in $\mathcal{C}\langle\mathcal{X}\rangle$ generated by $\mathcal{C}$ is $\mathcal{C}^{\perp\perp} \cap \mathcal{C}\langle\mathcal{X}\rangle$.*

PROPOSITION 37. *Suppose $\mathcal{P}$ is a bounded part class in $\mathcal{F}(\mathcal{X})$. The following are equivalent:*

1. *$\mathcal{P}$ is a continuous part class,*
2. *$\mathcal{P}$ is closed under pointwise norm limits,*
3. *$\mathcal{P}$ is closed under approximate equivalence,*
4. *$\mathcal{P}$ is closed under pointwise $*$-strong limits,*
5. *$\mathcal{P}$ is closed under representations; i.e., if $f \in \mathcal{P}$ and*
$$\pi : C^*(f(\mathcal{X})) \to B(\mathcal{H})$$
is a unital representation of $C^*(f(\mathcal{X}))$, then $\pi \circ f \in \mathcal{P}$,
6. *there exists $f \in \mathcal{F}(\mathcal{X})$ such that*
$$\mathcal{P} = \{\pi(f) \ : \pi \text{ a unital representation of } C^*(f(\mathcal{X}))\ \}.$$

Proof. (1) ⇒ (2) follows from Proposition 24, and (2) ⇒ (3) is clear.

(3) ⇒ (4) We will use the non-separable extension of Voiculescu's theorem given in [8], for which we need to write rank T to indicate the dimension of the closure of the range of an operator T. Assume that $\{f_\lambda\}_{\lambda \in I}$ is a net in $\mathcal{P}$ that converges point $*$-strongly to f. If m denotes the dimension of $\mathcal{H}_f$ and if $g = \bigoplus_{\lambda \in I} f_\lambda^{(m)}$, then for every $p \in \mathbb{P}(\mathcal{X})$ we have
$$||p(f)|| \leq \sup_\lambda ||p(f_\lambda)|| = ||p(g)||.$$
It follows that there is a unital representation of $C^*(g(\mathcal{X}))$ that maps every $g(x)$ to $f(x)$. If this representation is summed with the identity

representation we obtain, for each $\phi \in \mathcal{C}\langle\mathcal{X}\rangle$, the unital representation

$$\phi(g) \overset{\pi}{\mapsto} \phi(f) \oplus \phi(g).$$

It is clear that rank $\phi(g) \leq$ rank $(\phi(f) \oplus \phi(g))$, and Proposition 26 (2) together with the choice of m ensures that rank $\phi(g) \geq$ rank $(\phi(f) \oplus \phi(g))$ for every $\phi \in \mathcal{C}\langle\mathcal{X}\rangle$. We now have that rank $\pi(T) =$ rank T for every $T \in C^*(g(\mathcal{X}))$, so π is approximately equivalent to the identity representation by theorem 3.14 of [8]. Since $g \in \mathcal{P}$ and $\pi \circ g = f \oplus g$ is approximately equivalent to g, our hypothesis in (3) gives us $f \oplus g \in \mathcal{P}$, and hence $f \in \mathcal{P}$.

(4) $\Rightarrow$ (5) First notice that (4) $\Rightarrow$ (2) is trivial, so we have already established the equivalence of (2) through (4). Hence it is enough to prove that (3) $\Rightarrow$ (5) . Assume that $f \in \mathcal{P}$ and $\pi : C^*(f(\mathcal{X})) \to B(\mathcal{H})$ is a unital representation. Let m denote the cardinality of $\mathcal{H}$. If id denotes the identity representation of $C^*(f(\mathcal{X}))$, then as in the previous paragraph we see that $id^{(m)}$ is approximately equivalent to $id^{(m)} \oplus \pi$. It follows that $f^{(m)}$ is approximately equivalent to $f^{(m)} \oplus (\pi \circ f)$, so $f^{(m)} \oplus (\pi \circ f) \in \mathcal{P}$ by the hypothesis in (3), and $\pi \circ f \in \mathcal{P}$ as desired.

(5) $\Rightarrow$ (6) Choose any $f \in \mathcal{F}(\mathcal{X})$ such that $\mathcal{P} = \{f\}^{\perp\perp}$. By the hypothesis in (5) we see

$$\{\pi(f) \ : \pi \text{ a unital representation of } C^*(f(\mathcal{X}))\ \} \subseteq \mathcal{P},$$

and a direct verification using Proposition 8 (2) reveals that

$$\{\pi(f) \ : \pi \text{ a unital representation of } C^*(f(\mathcal{X}))\ \}$$

is itself a part class containing f, hence

$$\mathcal{P} = \{f\}^{\perp\perp} \subseteq \{\pi(f) \ : \pi \text{ a unital representation of } C^*(f(\mathcal{X}))\ \}.$$

(6) $\Rightarrow$ (1) Assume that

$$\mathcal{P} = \{\pi(f) \ : \pi \text{ a unital representation of } C^*(f(\mathcal{X}))\ \}.$$

We intend to prove that $\mathcal{P} = (\{f\}^{\perp} \cap \mathcal{C}\langle\mathcal{X}\rangle)^{\perp}$, so assume $g \in (\{f\}^{\perp} \cap \mathcal{C}\langle\mathcal{X}\rangle)^{\perp}$. It follows that for every $\phi \in \mathcal{C}\langle\mathcal{X}\rangle$ one has

$$\phi(f) = 0 \ \Rightarrow \ \phi(g) = 0,$$

hence

$$g \in \{\pi(f) \ : \pi \text{ a unital representation of } C^*(f(\mathcal{X}))\ \}$$

by Proposition 26 (2). Thus the inclusion

$$(\{f\}^{\perp} \cap \mathcal{C}\langle\mathcal{X}\rangle)^{\perp} \subseteq \mathcal{P}$$

is established, and since the opposite inclusion follows from Proposition 16 and

$$\{f\}^{\perp} \cap \mathcal{C}\langle\mathcal{X}\rangle \subseteq \{f\}^{\perp},$$

our proof is complete. □

LEMMA 38. *If $\mathcal{X}$ is finite, then every point-norm closed ideal in $\mathcal{C}\langle\mathcal{X}\rangle$ is singly generated.*

Proof. Since $\mathcal{X}$ is finite, the point-norm topology on $\mathcal{C}\langle\mathcal{X}\rangle$ is given by countably many seminorms, and is hence metrizable. Also the set of polynomials with coefficients in $\mathbb{Q} + i\mathbb{Q}$ is a countable dense set. Hence every closed ideal $\mathcal{I}$ of $\mathcal{C}\langle\mathcal{X}\rangle$ is separable. Choose a dense sequence $\{\phi_k\}$ in $\mathcal{I}$, and define

$$\phi = \sum_{k=1}^{\infty} \frac{\phi_k \phi_k^*}{2^k[1+d_k]^2},$$

where $d_k = \sup\{\,||\phi(f)|| \ : \forall x \in \mathcal{X}\ n_f(x) \leq k\,\}$. Clearly, $\phi \in \mathcal{I}$ and $\{\phi\}^{\perp} = \{\phi_1, \phi_2, \ldots\}^{\perp} = \mathcal{I}^{\perp}$. Thus the ideal generated by ϕ is $\{\phi\}^{\perp\perp} = \mathcal{I}^{\perp\perp} = \mathcal{I}$. □

The following proposition, which is a consequence of the preceding two results, illustrates the need for a characterization of unbounded continuous part classes analogous to the one in Proposition 37 for bounded part classes.

PROPOSITION 39. *Assume $\mathcal{X}$ is finite. Then every continuous part class has the form $\{\phi\}^{\perp}$ for some $\phi \in \mathcal{C}\langle\mathcal{X}\rangle$.*

Since part classes correspond to central projections in $\mathcal{B}\langle\mathcal{X}\rangle$, it follows that part classes form a complete Boolean algebra. If $\mathcal{P}$ and $\mathcal{Q}$ are part classes, then $\mathcal{P}' = \{\, f \in \mathcal{F}(\mathcal{X}) \ : f \perp \mathcal{P}\,\}$, $\mathcal{P} \wedge \mathcal{Q} = \mathcal{P} \cap \mathcal{Q}$, and $\mathcal{P} \vee \mathcal{Q} = \{\, f \in \mathcal{F}(\mathcal{X}) \ : f \ll \mathcal{P} \cup \mathcal{Q}\,\}$. In other words, $f \in \mathcal{P} \vee \mathcal{Q}$ if and only if $f \in \mathcal{P} \cup \mathcal{Q}$ or f is a direct sum $g \oplus h$ with $g \in \mathcal{P}$ and $h \in \mathcal{Q}$. It is clear that if $\mathcal{P}$ and $\mathcal{Q}$ are continuous part classes, then $\mathcal{P} \cap \mathcal{Q}$ is a continuous part class. However, it is not so clear that the same is true for $\mathcal{P} \vee \mathcal{Q}$. However, we can at least say the following. (The proof is essentially the same as the one in [10] and is omitted.)

PROPOSITION 40. *Suppose $\mathcal{X}$ is countable and $\mathcal{P}$ and $\mathcal{Q}$ are bounded continuous part classes. Then $\mathcal{P} \vee \mathcal{Q}$ is a bounded continuous part class.*

6. Generators and relations

There are many cases in which C*-algebras are defined in terms of generators and relations. This can be tricky business since the relations cannot be arbitrary. Suppose we let $\mathcal{X}$ denote the generators and $\mathcal{R}$ denote the relations, and we let $C^*(\mathcal{X} : \mathcal{R})$ denote the C*-algebra generated by $\mathcal{X}$ subject to the relations $\mathcal{R}$. There are two versions of this problem, one in the category of nonunital C*-algebras, and one in the category of unital C*-algebras. We use the notation $C_0^*(\mathcal{X} : \mathcal{R})$ for the nonunital version and $C^*(\mathcal{X} : \mathcal{R})$ for the unital version. What we want is that if $f \in \mathcal{F}(\mathcal{X})$ and f satisfies the relations in $\mathcal{R}$, then f extends uniquely to a $*$-homomorphism π_f from $C_0^*(\mathcal{X} : \mathcal{R})$ into $B(\mathcal{H}_f)$ (we insist that π_f be unital when we are dealing with $C^*(\mathcal{X} : \mathcal{R})$). What are the conditions on $\mathcal{R}$ to make this construction work?

First of all the relations must imply that each of the generators has a finite norm. Thus, for example $C^*(\{x\} : x = x^*)$ is not well defined. This trickiness even foils the experts, e.g., in [18, p.25] it is mentioned that the relation $x \geq 0$ can be obtained by insisting $x = x^*$, adding another generator y and adding the relations $y = y^*$ and $y^2 = x$. However, the latter relations are represented by $x = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $y = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, but y is not in the C*-algebra generated by x. To make this trick work, we must add a sequence $\{y_n\}$ of new generators, assumed to be selfadjoint so that $y_1^2 = x$ and $y_{n+1}^2 = y_n$ for $n \geq 1$.

The real question is: what is an acceptable relation? Are the relations "$||x|| \leq 1$ and x is nilpotent" or "$||x|| \leq 1$ and the spectrum of x is $\{0\}$" acceptable? It turns out that these, by themselves, are not acceptable relations. If $C_0^*(\mathcal{X} : \mathcal{R})$ actually makes sense, then there must be a faithful representation π of $C_0^*(\mathcal{X} : \mathcal{R})$ on some Hilbert space $\mathcal{H}$. Then $f = \pi|_{\mathcal{X}} \in \mathcal{F}(\mathcal{X})$. Moreover, if $g \in \mathcal{F}(\mathcal{X})$, then g satisfies the relations in $\mathcal{R}$ if and only if there is a $*$-homomorphism $\rho : C_0^*(\mathcal{X} : \mathcal{R}) \to B(\mathcal{H}_g)$ such that $\rho \circ f = g$. It follows that the class $\mathcal{P}$ of all $g \in \mathcal{F}(\mathcal{X})$ satisfying the relations in $\mathcal{R}$ is precisely

$$\{\, \rho \circ f \ : \rho \text{ is a } *\text{-homomorphism on } C_0^*(f(\mathcal{X}))\,\}.$$

It follows from the preceding section that $\mathcal{P}$ is a bounded continuous part class. Hence there is a point-norm closed ideal $\mathcal{I}$ of $\mathcal{C}\langle\mathcal{X}\rangle$ such that $\mathcal{P} = \mathcal{I}^\perp \cap \mathcal{C}_0\langle\mathcal{X}\rangle$. Hence the relations in $\mathcal{R}$ can be expressed in terms of equations involving noncommutative continuous functions. We now can define what a relation is, namely, an equation

$$\phi(f) = 0,$$

where ϕ is a noncommutative continuous function. If we are in the nonunital case, we only need to use relations involving ϕ's for which $\phi(0) = 0$. Note that an inequality $\phi(f) \geq 0$ can be expressed as $[(\phi^*\phi)^{\frac{1}{2}} - \phi](f) = 0$ and a relation $||\phi(f)|| \leq 1$ can be written $[1 - \phi^*\phi](f) \geq 0$, or

$$[([1 - \phi^*\phi]^2)^{\frac{1}{2}} - [1 - \phi^*\phi]](f) = 0.$$

To check whether relations are really acceptable, given that they imply the norm boundedness of the generators, it is sufficient to check any of the conditions in Proposition 37 on the representations of the relations. For example, if J_n is the $n \times n$ nilpotent Jordan block, the direct sum J of all the J_n's is not nilpotent; in fact the spectrum of J is the closed unit disk. Thus "x is nilpotent" or "the spectrum of x is $\{0\}$" are not acceptable relations since the class of representatives is not closed under direct sums. Another way we could eliminate these bogus relations is using the fact that the set of nilpotent operators on ℓ^2 is not norm closed [14]. Thus the determination of which "relations" are acceptable and which are not depends on a knowledge of single operator theory.

We summarize the remarks in the preceding paragraph in the following proposition. We call a collection $\mathcal{K}$ of noncommutative continuous functions *null bounded* if there is an element $n \in \mathcal{N}(\mathcal{X})$ such that $n_f \leq n$ for every $f \in \mathcal{K}^{\perp}$.

PROPOSITION 41. *A family $\mathcal{R}$ of relations on the variables $\mathcal{X}$ has the property that there is a universal unital C*-algebra generated by $\mathcal{X}$ subject to the family $\mathcal{R}$ of relations, if and only if there is a null-bounded subset $\mathcal{K} \subset \mathcal{C}\langle\mathcal{X}\rangle$ such that, for every $f \in \mathcal{F}(\mathcal{X})$, the following are equivalent*:

1. *$f(\mathcal{X})$ satisfies the relations in $\mathcal{R}$,*
2. *$\phi(f) = 0$ for every $\phi \in \mathcal{K}$.*

Hence the relations in $\mathcal{R}$ can be reformulated as

$$\phi \equiv 0 \text{ for all } \phi \in \mathcal{K}.$$

In the case when $\mathcal{X}$ is finite it is always possible to take the set $\mathcal{R}$ to be a single relation. In other words, every finitely generated C*-algebra is finitely presented.

Corollary 42. *If $\mathcal{X}$ is finite, then every acceptable family $\mathcal{R}$ of relations on $\mathcal{X}$ can be expressed as a single equation*

$$\phi(f) = 0$$

for some null-bounded noncommutative continuous function ϕ.

It turns out that every C*-algebra can be defined by generators and relations.

Proposition 43. *Suppose $\mathcal{A}$ is a unital C*-algebra generated by a subset $\mathcal{X}$. Let*

$$\mathcal{P} = \{ f \in \mathcal{F}(\mathcal{X}) \ : f \text{ is the restriction of a representation of } \mathcal{A} \text{ to } \mathcal{X} \}.$$

Then the identity mapping on $\mathcal{X}$ extends to a $*$-homomorphism from $\mathcal{A}$ to $C^*(\mathcal{X} : \mathcal{P}^\perp)$.

7. Stable relations

In the previous section we saw that relations can be expressed in the form $\phi(f) = 0$, where ϕ is a noncommutative continuous function. Thus we can identify each set of relations with a set of noncommutative continuous functions. If $\mathcal{R} = \{\phi_1, \phi_2, \ldots, \phi_k\}$ is a set of noncommutative continuous functions (considered as relations), we say that an $f \in \mathcal{F}(\mathcal{X})$ is a *representation of* $\mathcal{R}$ if $\phi_j(f) = 0$ for $1 \le j \le k$. If $\delta > 0$, we say f is a *δ-representation* of $\mathcal{R}$ if $||\phi_j(f)|| \le \delta$ for $1 \le j \le k$.

Stable and weakly stable relations are defined and studied by T. Loring [18] (based partly on his previous work [15],[16],[17] where the terminology is somewhat different). Since relations are given by noncommutative continuous functions, we will translate Loring's terminology into our framework. Suppose $\mathcal{X}$ is finite. We call a finite family $\mathcal{R} = \{\phi_1, \phi_2, \ldots, \phi_k\}$ of noncommutative continuous functions *weakly stable* in the category of nonunital C*-algebras if, for every pair of positive elements $n, \epsilon \in \mathcal{N}(\mathcal{X})$ there exists a positive number $\delta > 0$ such that if $f \in \mathcal{F}(\mathcal{X})$, $n_f \le n$, and f is a δ-representation of $\mathcal{R}$, then there is a representation g of $\mathcal{R}$ such that

1. $g(\mathcal{X}) \subset C_0^*(f(\mathcal{X}))$,
2. $n_{f-g} < \epsilon$ (recall that this means $||f(x) - g(x)|| < \epsilon(x)$ for all $x \in \mathcal{X}$).

What this condition says is that every f that is bounded by n for which $(\phi_1(f), \phi_2(f), \ldots, \phi_k(f))$ is small may be approximated by a g

whose values are in $C_0^*(f(\mathcal{X}))$ such that $(\phi_1(g), \phi_2(g), \ldots, \phi_k(g)) = (0, 0, \ldots, 0)$.

As in [9], we can show, in terms of noncommutative continuous functions, that the choice of g above can be made in a canonical and continuous way. Note that the condition on g requiring $g(\mathcal{X}) \subset C_0^*(f(\mathcal{X}))$, implies that, for each $x \in \mathcal{X}$, there is a noncommutative continuous function ψ (depending on x, ϵ, n, and f) with $\psi(0) = 0$ such that $g(x) = \psi(f)$. We show that the weak stability condition actually gives us the function ψ independent of f.

PROPOSITION 44. *Suppose $\mathcal{X}$ is a finite set and $\mathcal{R} = \{\phi_1, \phi_2, \ldots, \phi_k\} \subset \mathcal{C}_0\langle\mathcal{X}\rangle$. We have that $\mathcal{R}$ is weakly stable if and only if for every pair n, ϵ of positive elements of $\mathcal{N}(\mathcal{X})$ there exists a $\delta > 0$ and a subfamily*

$$\{\psi_{n,\epsilon,x} \ : x \in \mathcal{X}\} \subset \mathcal{C}_0\langle\mathcal{X}\rangle$$

such that whenever f is a δ-representation of $\mathcal{R}$ and $n_f \leq n$, then

$$g(x) = \psi_{n,\epsilon,x}(f)$$

for $x \in \mathcal{X}$, defines a representation g of $\mathcal{R}$ with $n_{f-g} \leq \epsilon$.

Proof. To prove the non-trivial implication, assume that $\mathcal{R}$ is weakly stable. Let $\hat{f}$ be the direct sum of all the $f \in \mathcal{F}(\mathcal{X}, B(\mathcal{H}_\mathcal{X}))$ such that $n_f \leq n$ and f is a δ-representation of $\mathcal{R}$. Then $n_{\hat{f}} \leq n$ and $\hat{f}$ is a δ-representation of $\mathcal{R}$. Thus, since $\mathcal{R}$ is weakly stable, there is a representation $\hat{g}$ of $\mathcal{R}$ such that $\hat{g}(\mathcal{X}) \subset C_0^*(\hat{f}(\mathcal{X}))$, $n_{\hat{f}-\hat{g}} \leq \epsilon$. It follows from Proposition 25 that, for each $x \in \mathcal{X}$, there is a noncommutative continuous function $\psi_{n,\epsilon,x} \in \mathcal{C}_0\langle\mathcal{X}\rangle$ such that $\hat{g}(x) = \psi_{n,\epsilon,x}(\hat{f})$. Since every δ-representation f of $\mathcal{R}$ with $n_f \leq n$ has the form $f = \pi \circ \hat{f}$ for some representation π of $C_0^*(f(\mathcal{X}))$, it follows that the definition $g(x) = \psi_{n,\epsilon,x}(f)$ for each $x \in \mathcal{X}$ is the same as $g = \pi \circ \hat{g}$, which gives the desired conclusion. □

Weak stability can also be stated in terms of sequences. From this viewpoint one sees that the set $\mathcal{R} = \{\phi_1, \phi_2, \ldots, \phi_k\}$ is weakly stable if, for every bounded sequence $\{f_m\}$ in $\mathcal{F}(\mathcal{X})$ such that $||\phi(f_m)|| \to 0$ for each $\phi \in \mathcal{R}$, there is a sequence $\{g_m\}$ that is eventually in $\mathcal{R}^\perp$ such that $g_m(\mathcal{X}) \subset C_0^*(f_m(\mathcal{X}))$ for each $m = 1, 2, \ldots$, and $n_{f_m - g_m} \to 0$ pointwise. This sequential statement leads to a formulation of stability in terms of certain liftings, and proves that stability is independent of the representative relations. Given a sequence $\{\mathcal{A}_m\}$ of C*-algebras, we let $C(\sum_{m=1}^\infty \mathcal{A}_m)$ denote the C*-algebra $\prod_{m=1}^\infty \mathcal{A}_m / \sum_{m=1}^\infty \mathcal{A}_m$. Let $\eta : \prod_{m=1}^\infty \mathcal{A}_m \to C(\sum_{m=1}^\infty \mathcal{A}_m)$ denote the quotient map.

PROPOSITION 45. *Suppose $\mathcal{X}$ is finite and $\mathcal{R}$ is a finite subset of $\mathcal{C}_0\langle\mathcal{X}\rangle$. The following are equivalent:*

1. *$\mathcal{R}$ is weakly stable,*
2. *for every sequence $\{\mathcal{A}_m\}$ of C^*-algebras and every element $f : \mathcal{X} \to C(\sum_{m=1}^{\infty} \mathcal{A}_m)$, there exists $g = \{g_m\} : \mathcal{X} \to C(\sum_{m=1}^{\infty} \mathcal{A}_m)$ such that $f(x) = \eta(g(x))$ for all $x \in \mathcal{X}$ and such that, eventually $\phi(g_m) = 0$ for every $\phi \in \mathcal{R}$,*
3. *for every positive $n \in \mathcal{N}(\mathcal{X})$, every sequence $\{\mathcal{A}_m\}$ of C^*-algebras, and every $*$-homomorphism*
$$\pi : C_0^*(\mathcal{X} : \mathcal{R}, n_{\mathcal{X}} \leq n) \to C(\sum_{m=1}^{\infty} \mathcal{A}_m),$$
there exists an N and a $*$-homomorphism
$$\rho : C_0^*(\mathcal{X} : \mathcal{R}, n_{\mathcal{X}} \leq n) \to \prod_{m=N}^{\infty} \mathcal{A}_m$$
such that $\pi = \eta \circ \rho$.

The definition of stability of relations is more complicated. The subset $\mathcal{R} = \{\phi_1, \phi_2, \ldots, \phi_k\}$ of $\mathcal{C}_0\langle\mathcal{X}\rangle$ is *stable* if, for every positive $\epsilon, n \in \mathcal{N}(\mathcal{X})$ there is a $\delta > 0$ such that: given C*-algebras A and B, a surjective homomorphism $\pi : A \to B$ and a δ-representation $f : \mathcal{X} \to A$ of $\mathcal{R}$ with $n_f \leq n$, such that $\pi \circ f$ is a representation of $\mathcal{R}$, there is a representation $g : \mathcal{X} \to A$ of $\mathcal{R}$ with $n_{g-f} \leq \epsilon$, such that $\pi \circ f = \pi \circ g$.

Weak stability says, given a positive $n, \epsilon \in \mathcal{N}(\mathcal{X})$ there is a continuous formula that transforms any δ-representation f of $\mathcal{R}$ with $n_f \leq n$ into an actual representation of $\mathcal{R}$ that is within ϵ of f. Stability says that this continuous formula has the additional property that it fixes representations h of $\mathcal{R}$ with $n_h \leq n$. This, in turn is equivalent to being able to find a formula that only depends on n.

PROPOSITION 46. *Suppose $\mathcal{X}$ is finite and $\mathcal{R} = \{\phi_1, \phi_2, \ldots, \phi_k\} \subset \mathcal{C}_0\langle\mathcal{X}\rangle$. The following are equivalent:*

1. *$\mathcal{R}$ is stable,*
2. *for every positive elements $n, \epsilon \in \mathcal{N}(\mathcal{X})$ there exists a family*
$$\{\psi_{n,\epsilon,x} \ : x \in \mathcal{X}\} \subset \mathcal{C}_0\langle\mathcal{X}\rangle$$
and a $\delta > 0$ such that
   (a) *if f is a δ-representation of $\mathcal{R}$ and $n_f \leq n$, then*
$$g(x) = \psi_{n,\epsilon,x}(f)$$
for $x \in \mathcal{X}$, defines a representation g of $\mathcal{R}$ with $n_{f-g} \leq \epsilon$,

(b) *if h is a representation of $\mathcal{R}$ with $n_h \leq n$, then $h(x) = \psi_{n,\epsilon,x}(f)$ for every $x \in \mathcal{X}$,*

3. *for every positive element $n \in \mathcal{N}(\mathcal{X})$ there exists a family*

$$\{\psi_{n,x} \ : x \in \mathcal{X}\} \subset \mathcal{C}_0\langle\mathcal{X}\rangle$$

such that, for every positive $\epsilon \in \mathcal{N}(\mathcal{X})$ there is a $\delta > 0$ such that if f is a δ-representation of $\mathcal{R}$ and $n_f \leq n$, then

$$g(x) = \psi_{n,x}(f)$$

for $x \in \mathcal{X}$, defines a representation g of $\mathcal{R}$ with $n_{f-g} \leq \epsilon$.

The last theorem gives us an easy way to construct examples of relations that are weakly stable but not stable (see [19]).

COROLLARY 47. *Suppose K is a compact nonempty subset of $\mathbb{R}$, and $\zeta : \mathbb{R} \to \mathbb{R}$ is defined so that $\zeta(t)$ is the distance from t to K. Let $\mathcal{X} = \{x\}$, and let $\mathcal{R}$ be the relations $x = x^*$ and $\zeta(\frac{x+x^*}{2}) = 0$. Then $\mathcal{R}$ is weakly stable, but $\mathcal{R}$ is stable if and only if K has only finitely many connected components.*

Proof. Since any approximate representation T of $\mathcal{R}$ must be close to ReT, we need only check the weak stability (respectively, stability) conditions for hermitian elements. However, a hermitian operator T is a δ-representation of $\mathcal{R}$ if and only if $\sigma(T)$ is contained in the δ-neighborhood $U_\delta = \zeta^{-1}((0,\delta))$ of K. Hence weak stability is equivalent to the statement that, for each $\epsilon > 0$ there is a $\delta > 0$ and a continuous map $\gamma : U_\delta \to K$ such that $|t - \gamma(t)| \leq \epsilon$ for all $t \in U_\delta$. This is true for every K. However, stability requires that, in addition, the function γ satisfies $\gamma(t) = t$ for all $t \in K$, and this is equivalent to K having finitely many connected components. □

We say a subset $\mathcal{R} = \{\phi_1, \phi_2, \ldots, \phi_k\}$ of $\mathcal{C}_0\langle\mathcal{X}\rangle$ is *strongly null-bounded* if there is a $\delta_0 > 0$ and an $n_0 \in \mathcal{N}(\mathcal{X})$ such that $n_f \leq n_0$ whenever f is a δ_0-representation of $\mathcal{R}$. Note that strong null-boundedness depends on the relations, for example the relations $x = x^*$ and $||x|| \leq 1$ are equivalent to the relations $x = x^*$ and $\zeta(\frac{x+x^*}{2}) = 0$, where $\zeta : \mathbb{R} \to \mathbb{R}$ is a function in $C_0(\mathbb{R})$ such that $\zeta^{-1}(\{0\}) = [-1,1]$, and also equivalent to $x = x^*$ and $\xi(\frac{x+x^*}{2}) = 0$, where $\xi : \mathbb{R} \to \mathbb{R}$ is the function $\xi(t) = \max\{0, |t| - 1\}$. The last relations are strongly null-bounded, but the second ones are not.

COROLLARY 48. *Suppose $\mathcal{R} = \{\phi_1, \phi_2, \ldots, \phi_k\}$ is a strongly null-bounded subset of $\mathcal{C}_0\langle\mathcal{X}\rangle$. Then $\mathcal{R}$ is stable if and only if there is a* *family*

$$\{\psi_x \ : x \in \mathcal{X}\ \} \subset \mathcal{C}_0\langle\mathcal{X}\rangle$$

such that, for every positive $\epsilon \in \mathcal{N}(\mathcal{X})$ *there is a* $\delta > 0$ *such that, for every* δ*-representation* f *of* $\mathcal{R}$*,*

$$g(x) = \psi_x(f)$$

for $x \in \mathcal{X}$*, defines a representation* g *of* $\mathcal{R}$ *with* $n_{f-g} \leq \epsilon$*.*

All of the preceding results in this section are for weak stability and stability for relations in the category of nonunital C*-algebras. If we wish to consider the unital case, all of the results in this section remain true in the unital case if $\mathcal{C}_0\langle\mathcal{X}\rangle$ is replaced with $\mathcal{C}\langle\mathcal{X}\rangle$.

In [18] T. Loring characterized stability and weak stability in terms of lifting properties, which implies that two finite null-bounded families of relations that generate the same universal C*-algebra are either both stable (respectively, weakly stable) or both not. Hence stability and weak stability are properties of the C*-algebras. Since every finitely generated C*-algebra can be defined by a single relation $\phi(f) = 0$, stability and weak stability makes sense for all finitely generated C*-algebras.

In [18] T. Loring showed that $C_0(0,1]$ is stable. If $\mathcal{X} = \{x\}$, $\phi_1(x) = x^*x^2 - x$, $\phi_2(x) = xx^*x - x$, $\mathcal{R} = \{\phi_1, \phi_2\}$, using arguments like the ones in [18] (i.e., looking at irreducible representations of $\mathcal{R}$), it is easy to see that $C_0^*(x, \mathcal{R})$ is isomorphic to $C_0^*(S \oplus 0)$, where S is the unilateral shift operator on ℓ^2. With a little more work, it can be shown that $\mathcal{R}$ is stable. On the other hand, again by considering irreducible representations, it can be shown that $C_0(0,1] \otimes C_0^*(S \oplus 0)$ is isomorphic to $C_0^*(P \oplus S)$, where P is a positive operator with spectrum $[0,1]$, S is the unilateral shift operator, and $P \oplus S$ is the spacial tensor product. However, it is well known [2] that $C_0^*(P \oplus S)$ is isomorphic to $C_0^*(\{x\} : x(x^*x) = (x^*x)x, ||x|| \leq 1)$ (i.e., x is a *quasinormal* contraction). However, quasinormality, like normality is highly nonstable. No Fredholm quasinormal operator can have positive index. For each positive integer n let T_n be the adjoint of the unilateral weighted shift whose weights are $\{\min\{1, \frac{k}{n}\} : k \geq 1\}$. Then each T_n is a compact perturbation of the adjoint of S, so the distance from T_n to the quasinormal operators is at least the distance from T_n to the operators with non-positive index, which is 1. However, $||T_n^*T_n - T_nT_n^*|| \to 0$, which means that asymptotically T_n is a δ-representation of $\mathcal{R}_1$. Thus $C_0(0,1] \otimes C_0^*(S \oplus 0)$ is not weakly stable. Hence no version of stability is preserved under tensor products.

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