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OPTIMAL CONTROL PROBLEMS FOR THE SEMILINEAR SECOND ORDER EVOLUTION EQUATIONS

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Abstract. In this paper, we study the optimal control for the damped semilinear hyperbolic systems with unknown parameters

$$(C(t)y')' + A_2(t,q)y' + A_1(t,q)y = f(t,q,y,u).$$

We will prove the existence of weak solution of this system and is to find the optimal control pair $(\bar{q},\bar{u}) \in Q_\tau \times \mathcal{U}_{ad}$ such that $\inf_{u\in\mathcal{U}_{ad}} \sup_{q\in Q_\tau} J(q,u) = J(\bar{q},\bar{u})$.

1. Introduction

The optimal control problems have been extensively studied by many authors [1, 6, 8, 12, 13, 19] and also identification problem for damping parameters in the second order hyperbolic systems have been dealt with by many authors [10, 11, 18]. In this paper, we consider the following systems

$$\text{(1.1)} \qquad \begin{cases} (C(t)y')' + A_2(t,q)y' + A_1(t,q)y = f(t,q,y,u) \quad \text{in } (0,T), \\ y(0,q,u) = y_0 \in V, y'(0,q,u) = y_1 \in H \end{cases}$$

and the cost criterion given by a general lower semicontinuous integral functional of the form

$$\text{(1.2)} \qquad J(q,u) = \int_0^T g(t,y,u)dt,$$

where V and H are real Hilbert spaces, $C(t)$ is a linear operator which is given by a bilinear form on H, $A_1(t,q)$ and $A_2(t,q)$ are differential operators containing unknown parameter $q \in Q_\tau$ which are given by some bilinear forms on Hilbert spaces, f is a forcing term with unknown

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parameters $q \in Q_T, u \in \mathcal{U}_{ad}$, g is an integrand defined on $[0,T] \times H \times Y$ determining the cost criterion and Y is a separable reflexive Banach space. The optimal control problem subject to (1.1) with (1.2) is the following "min-max" problem;

$$\inf_{u \in \mathcal{U}_{ad}} \sup_{q \in Q_T} J(q,u),$$

rather than a minimization problem. In this paper we will study the existence of weak solutions for (1.1) and the optimal control to the system (1.1) with (1.2). In other words, we will find $(\bar{q},\bar{u}) \in Q_T \times \mathcal{U}_{ad}$ satisfying $\inf_{u \in \mathcal{U}_{ad}} \sup_{q \in Q_T} J(q,u) = J(\bar{q},\bar{u})$. It is not easy to find the elements $(\bar{q},\bar{u})$ belonging to an admissible set $Q_T \times \mathcal{U}_{ad}$ of parameters subject to (1.1) with (1.2). Hence we will show the existence of such $(\bar{q},\bar{u})$ when Q_T is compact and $\mathcal{U}_{ad}$ is endowed with the w^*−topology in $L^\infty(0,T;Y)$. Recently, inspired by the optimal control theoretical studies of Euler-Bernoulli Beam Eqations with Kelvin-Voigt Damping and Love-Kirchoff Plate Equations with various damping terms, there appeared numerous papers studying optimal control theory and identification problem for the autonomous case of (1.1) on the Gelfand triple spaces. Banks et al. [7] and Banks and Kunisch [8] treated the existence of minimizing parameters by using the methods of approximations. When $A_1(t,q) \equiv \gamma A_2(t,q), \gamma \geq 0$ in (1.1), Ahmed [1, 2] studied the identification problem estimating q via output least-square identification problem based on the transposition method and some authors treated the problem with the cost criterion of the quadratic form (see [11, 17]) for the system (1.1) replacing $C(t)$ and $f(t,q,y,u)$ by identity operator and $f(t)$ or $f(t,y)$, respectively. In Papageorgiou [14, 16], the cost criterion is a general form concerned with one parameter. Specially, in this paper we study the optimal control problems to (1.1) with (1.2) on the Gelfand five fold. Using the Gelfand five fold structure we may have some advantages that the operators $A_1(t,q)$ and $A_2(t,q)$ can be defined with free differential orders in spatial sense. This paper is composed of the pairs of preliminaries as section 2, existence and uniqueness of the solutions for (1.1) as section 3 and sufficient conditions for (1.1) with (1.2) as section 4.

2. Preliminaries

First we explain the notations used in this paper. Let H be a real Hilbert space. The norm on H will be denoted by $|\cdot|_H$ and the corresponding inner product by $(\cdot,\cdot)_H$. Let us introduce underlying Hilbert spaces to describe the damped second order equations. For $i = 1, 2$, let V_i be a real separable Hilbert space. V_i^* denotes the dual space of V_i, $|| \cdot ||_{V_i}$ denotes the norm on V_i and $< \cdot, \cdot >_{V_i^*, V_i}$ denotes the dual pairing between V_i^* and V_i. Assume that each pair (V_i, H) is a Gelfand triple space with a notation, $V_i \hookrightarrow H \equiv H^* \hookrightarrow V_i^*$, which means that an embedding $V_i \hookrightarrow H$ is continuous and V_i is dense in H, so that the embedding $H \hookrightarrow V_i^*$ is also continuous and the identified $H \equiv H^*$ is dense in V_i^*. From now on, we write $V_1 = V$ for notational convenience. We suppose that V is continuously embedded in V_2. Then we see that $V \hookrightarrow V_2 \hookrightarrow H \equiv H^* \hookrightarrow V_2^* \hookrightarrow V^*$ and the equalities $< \phi, \varphi >_{V^*, V} = < \phi, \varphi >_{V_2^*, V_2}$ for $\phi \in V_2^*, \varphi \in V$ and $< \phi, \varphi >_{V^*, V} = (\phi, \varphi)_H$ for $\phi \in H, \varphi \in V$. We shall give an exact description of damped second order equations. We suppose that Q is algebraically contained in a linear topological vector space with topology τ and $Q_\tau = (Q, \tau)$ is closed. Let $T > 0$ be fixed.

We consider the following Cauchy problem for semilinear damped second order evolution equations

$$(2.1) \quad \begin{cases} (C(t)y')' + A_2(t,q)y' + A_1(t,q)y = f(t,q,y,u) \text{ in } (0,T), \\ y(0;q,u) = y_0 \in V, y'(0;q,u) = y_1 \in H. \end{cases}$$

We will need the following assumptions concerning the data of (2.1):

(HA) $a_i(t, q; \phi, \varphi), t \in [0, T], q \in Q_\tau, i = 1, 2 : V_i \times V_i \to R$: a bilinear form such that

(1) $a_i(t, q; \phi, \varphi) = a_i(t, q; \varphi, \phi)$ for all $\phi, \varphi \in V_i$,

(2) there exists $c_{i1} > 0$ such that $|a_i(t, q; \phi, \varphi)| \leq c_{i1} ||\phi||_{V_i} ||\varphi||_{V_i}$ for all $\phi, \varphi \in V_i$,

(3) there exists $\alpha_i > 0$ and $\lambda_i \in R$ such that $a_i(t, q; \phi, \phi) + \lambda_i |\phi|_H^2 \geq \alpha_i ||\phi||_{V_i}^2$ for all $\phi \in V_i$, the function $t \mapsto a_i(t, q; \phi, \varphi)$ is continuously differentiable in $[0, T]$,

(4) there exists c_{i2} such that $|a_i'(t, q; \phi, \varphi)| \leq c_{i2} ||\phi||_{V_i} ||\varphi||_{V_i}$ for all $\phi, \varphi \in V_i$, where $' = \frac{d}{dt}$. Then we can define the operator $A_i(t, q) \in \mathcal{L}(V_i, V_i^*)$ for $t \in [0, T]$ deduced by the relation S,

(5) $a_i(t, q; \phi, \varphi) = < A_i(t, q)\phi, \varphi >_{V_i^*, V_i}$ for all $\phi, \varphi \in V_i$,

(6) $a_i'(t, q; \phi, \varphi) = < A_i'(t, q)\phi, \varphi >_{V_i^*, V_i}$ for all $\phi, \varphi \in V_i$.

(HC) $c(t; \phi, \varphi), t \in [0, T] : H \times H \to R$: a bilinear form such that

(1) $c(t; \phi, \varphi) = c(t; \varphi, \phi)$ for all $\phi, \varphi \in H$,

(2) there exists $c_{31} > 0$ such that $|c(t; \phi, \varphi)| \leq c_{31} |\phi|_H |\varphi|_H$ for all $\phi, \varphi \in H$,

(3) there exists $\alpha_3 > 0$ such that $|c(t; \phi, \phi)| \geq \alpha_3 |\phi|_H^2$ for all $\phi \in H$, the function $t \mapsto c(t; \phi, \varphi)$ is continuously differentiable in $[0, T]$,

(4) there exists $c_{32} > 0$ such that $|c'(t;\phi,\varphi)| \le c_{32}|\phi|_H|\varphi|_H$ for all $\phi,\varphi \in H$. Also, we can define the operators $C(t), C'(t) \in \mathcal{L}(H,H)$ for $t \in [0,T]$ deduced by the relations,
(5) $c(t;\phi,\varphi) = (C(t)\phi,\varphi)_H$ for all $\phi,\varphi \in H$,
(6) $c'(t;\phi,\varphi) = (C'(t)\phi,\varphi)_H$ for all $\phi,\varphi \in H$.

(HU) (1) Y is a separable reflexive Banach space.
(2) $U : [0,T] \to CC(Y) = \{$class of nonempty, closed, convex subsets of $Y\}$ is a measurable multifunction satisfying $U(t) \subset \mathcal{U}$ for almost all $t \in [0,T]$, where $\mathcal{U}$ is a fixed weakly compact convex subset of Y. For admissible controls, we choose the set $\mathcal{U}_{ad} = \{u \in L^\infty(0,T;Y) : u(t) \in U(t) \text{ a.e. }\}$ and endow the relative w^*-topology on $\mathcal{U}_{ad}$ as a subset of $L^\infty(0,T;Y)$. Since $L^\infty(0,T;Y) = L^1(0,T;Y^*)^*$ and $L^1(0,T;Y^*)$ is separable, it is well known that $\mathcal{U}_{ad}$ topologized as above is compact and metrizable.

(Hf) $f : [0,T] \times Q_\tau \times V_2 \times Y \to V_2^*$ such that
(1) $t \mapsto f(t,q,y,u)$ is measurable,
(2) there exists a $\beta \in L^2(0,T;R^+)$ such that $||f(t,q,y_1,u) - f(t,q,y_2,u)||_{V_2^*} \le \beta(t)||y_1 - y_2||_{V_2}$ a.e.t, uniformly with respect to $q \in Q_\tau, u \in Y$,
(3) there exists a $\gamma \in L^2(0,T;R^+)$ such that $||f(t,q,0,u)||_{V_2^*} \le \gamma(t)$ a.e.t.

We write $g' = \frac{dg}{dt}$ and define a Hilbert space, which will be a space of solutions as

$$W(0,T) = \{g | g \in L^2(0,T;V), g' \in L^2(0,T;V_2), (C(\cdot)g')' \in L^2(0,T;V^*)\}.$$

The norm of $W(0,T)$ is given by

$$||g||_{W(0,T)} = (||g||^2_{L^2(0,T;V)} + ||g'||^2_{L^2(0,T;V_2)} + ||(C(\cdot)g')'||^2_{L^2(0,T;V^*)})^{\frac{1}{2}}.$$

We denote by $\mathcal{D}'(0,T)$ the space of distributions on $(0,T)$.

DEFINITION 2.1. A function y is said to be a weak solution of (2.1) if $y \in W(0,T)$ and y satisfies

$$\begin{aligned} (2.2) \quad & \langle (C(\cdot)y'(\cdot))', \phi\rangle_{V^*,V} + a_2(\cdot,q;y'(\cdot),\phi) + a_1(\cdot,q;y(\cdot),\phi) \\ = \quad & \langle f(\cdot,q,y(\cdot),u),\phi\rangle_{V_2^*,V_2} \text{ for all } \phi \in V \text{ in the sense of } \mathcal{D}(0,T), \\ (2.3) \quad & y(0,q,u) = y_0 \in V, y'(0,q,u) = y_1 \in H. \end{aligned}$$

3. Existence and uniqueness

We state the existence and uniqueness results of a weak solution of (2.1).

THEOREM 3.1. *Assume that* (HA), (HC), (HU) *and* (Hf) *hold. Then the problem* (2.1) *has a unique weak solution* y *in* $W(0,T)$.

COROLLARY 3.1. *Assume that* (HA), (HC) *hold and* $f(t,q,u) \in L^2(0,T;V_2^*)$. *Then*

$$(3.1) \qquad \begin{cases} (C(t)y')' + A_2(t,q)y' + A_1(t,q)y = f(t,q,u) \ \text{ in } (0,T), \\ y(0;q,u) = y_0 \in V, y'(0;q,u) = y_1 \in H \end{cases}$$

has a unique weak solution y *in* $W(0,T)$.

Existence Proof of Theorem 3.1. We divide the existence proof into four steps.

Step 1. Approximate solutions.
We use the Faedo-Galerkin approximation as in [6, 12]. Since V is real separable, there exists a basis $\{w_m\}_{m=1}^{\infty}$ in V such that

(i) $\{w_m\}_{m=1}^{\infty}$ is a complete orthonormal system in H,

(ii) the set of all finite linear combinations, $\{\sum_{j=1}^{m} \xi_j w_j | \xi_j \in \mathcal{R}, m \in \mathcal{N}\}$ is dense in V, where $\mathcal{N}$ is the set of natural numbers and $\mathcal{R}$ is the set of real numbers. For each $m \in \mathcal{N}$ we define an approximate solution of the problem (2.1) by $y_m(t,q,u) = \sum_{j=1}^{m} g_{jm}(t,q,u)w_j$. From now on, we write $y_m(t) = y_m(t,q,u), f(t,q,y,u) = f(t,y)$ for notational convenience. In above $y_m(t)$ satisfies

$$(3.2) \qquad \begin{aligned} & \frac{d}{dt}c(t;y_m'(t),w_j) + a_2(t,q;y_m'(t),w_j) + a_1(t,q;y_m(t),w_j) \\ = \ & \langle f(t,y_m(t)),w_j\rangle_{V_2^*,V_2}, \ \ t \in [0,T], 1 \leq j \leq m, \\ & y_m(0) = y_{0m} \in V, y_m'(0) = y_{1m} \in H. \end{aligned}$$

By (i) and (ii), we can find real numbers ξ_{im}^0 and ξ_{im}^1, $i = 1,2,\cdots,m, m \in \mathcal{N}$ such that

$$(3.3) \qquad \begin{aligned} y_{0m} &= \sum_{i=1}^{m} \xi_{im}^0 w_i \to y_0 \quad \text{in } V \ \text{ as } m \to \infty, \\ y_{1m} &= \sum_{i=1}^{m} \xi_{im}^1 w_i \to y_1 \quad \text{in } H \ \text{ as } m \to \infty. \end{aligned}$$

Then the equation (3.2) can be written as m vector differential equation

$$\frac{d}{dt}(\tilde{C}(t)\frac{d}{dt}\vec{g}_m) + \tilde{A}_2(t,q)\frac{d}{dt}\vec{g}_m + \tilde{A}_1(t,q)\vec{g}_m = \vec{f}(t,\vec{g}_m)$$

with initial values $\vec{g}_m(0) = [\xi_{1m}^0, \xi_{2m}^0, \cdots, \xi_{mm}^0]^t$ and $\frac{d}{dt}\vec{g}_m(0) = [\xi'_{1m}, \xi'_{2m}, \cdots, \xi'_{mm}]^t$. Here $\vec{g}_m = [g_{1m}, \cdots, g_{mm}]^t$, $\tilde{A}_1(t,q) = \big(a_1(t,q;w_i,w_j); i = 1,2,\cdots,m, j = 1,2,\cdots,m\big)$, $\tilde{A}_2(t,q) = \big(a_2(t,q;w_i,w_j); i = 1,2,\cdots,m, j = 1,2,\cdots,m\big), \tilde{C}(t) = \big(c(t;w_i,w_j); i = 1,2,\cdots,m, j = 1,2,\cdots,m\big)$ and $\vec{f}(t, \vec{g}_m) = [\langle f(t,\sum_{i=1}^m g_{im}w_i), w_1\rangle_{V_2^*,V_2}, \cdots, \langle f(t,\sum_{i=1}^m g_{im}w_i), w_m\rangle_{V_2^*,V_2}]^t$, where $[\cdots]^t$ denotes the transpose of $[\cdots]$. Then the elements of $\tilde{A}_1(t,q)$ and $\tilde{A}_2(t,q)$ are C^1-class and the nonlinear term of the vector function $\vec{f}$ is Lipshitz continuous. Indeed, for $\vec{g}_m = \sum_{i=1}^m g_{im}w_i, \vec{h}_m = \sum_{i=1}^m h_{im}w_i$, it follows by the assumption (Hf)(2) that

$$\begin{aligned}
& |\vec{f}(t,\vec{g}_m) - \vec{f}(t,\vec{h}_m)|^2 \\
= & \sum_{j=1}^m |\langle f(t,\sum_{i=1}^m g_{im}w_i) - f(t,\sum_{i=1}^m h_{im}w_i), w_j\rangle_{V_2^*,V_2}|^2 \\
\le & \beta(t)^2(\sum_{j=1}^m ||w_j||_{V_2}^2)^2 \sum_{i=1}^m |g_{im} - h_{im}|^2 \\
= & \beta(t)^2(\sum_{j=1}^m ||w_j||_{V_2}^2)^2 |\vec{g}_m - \vec{h}_m|^2.
\end{aligned}$$

Here we use the Hölder's inequality. Therefore this second order vector differential equation admits a unique solution $\vec{g}_m$ on $[0,T]$, by reducing this to a first order system and applying Carathéodory type existence theorem. Hence we can construct the approximate solutions $y_m(t)$ of (3.2).

Step 2. A priori estimates.
In this step we shall derive a priori estimates of $y_m(t)$. We multiply both sides of equation (3.2) by $g'_{jm}(t)$ and sum over j to have

$$\begin{aligned}
(3.4)\qquad & (\frac{d}{dt}[C(t)y'_m(t)], y'_m(t))_H + a_2(t,q;y'_m(t),y'_m(t)) \\
& + a_1(t,q;y_m(t),y'_m(t)) \\
= & \langle f(t,y_m(t)), y'_m(t)\rangle_{V_2^*,V_2}.
\end{aligned}$$

It is easily verified by the differentiation of a_1 and symmetry (HA)(1) that
(3.5)

$$a_1(t,q;y_m(t),y_m'(t)) = \frac{1}{2}\frac{d}{dt}a_1(t,q;y_m(t),y_m(t)) - \frac{1}{2}a_1'(t,q;y_m(t),y_m(t))$$

and
(3.6)

$$(\frac{d}{dt}[C(t)y_m'(t)],y_m'(t))_H = \frac{1}{2}\frac{d}{dt}c(t;y_m'(t),y_m'(t)) + \frac{1}{2}c'(t;y_m'(t),y_m'(t)).$$

To simplify notations let $\lambda_i = |\lambda_i|, i = 1,2$ in all estimations in what follows. Then by substituting (3.5) and (3.6) for (3.4), we have

$$\begin{aligned}(3.7)\quad & \frac{d}{dt}c(t;y_m'(t),y_m'(t)) + c'(t;y_m'(t),y_m'(t)) + \frac{d}{dt}a_1(t,q;y_m(t),y_m(t)) \\ & -a_1'(t,q;y_m(t),y_m(t)) + 2a_2(t,q;y_m'(t),y_m'(t)) \\ = \quad & 2\langle f(t,y_m(t)),y_m'(t)\rangle_{V_2^*,V_2}.\end{aligned}$$

Integrating (3.7) on $[0,t]$, we have

$$\begin{aligned}(3.8)\quad & c(t;y_m'(t),y_m'(t)) + a_1(t,q;y_m(t),y_m(t)) \\ & +2\int_0^t a_2(\sigma,q;y_m'(\sigma),y_m'(\sigma))d\sigma \\ = \quad & c(0,q;y_{1m},y_{1m}) + a_1(0,q;y_{0m},y_{0m}) + \int_0^t a_1'(\sigma,q;y_m(\sigma),y_m(\sigma))d\sigma \\ & -\int_0^t c'(\sigma;y_m'(\sigma),y_m'(\sigma))d\sigma + 2\int_0^t \langle f(\sigma,y_m(\sigma)),y_m'(\sigma)\rangle_{V_2^*,V_2}d\sigma.\end{aligned}$$

Let $\epsilon > 0$ be an arbitrary real number and k_2 be the constant such that $||\phi||_{V_2} \le k_2||\phi||_V$ for all $\phi \in V$. From (Hf)(2), (3) we obtain

$$\begin{aligned}(3.9)\quad & 2|\int_0^t \langle f(\sigma,y_m(\sigma)),y_m'(\sigma)\rangle d\sigma| \\ = \quad & 2|\int_0^t \langle f(\sigma,y_m(\sigma)) - f(\sigma,0) + f(\sigma,0),y_m'(\sigma)\rangle_{V_2^*,V_2}d\sigma| \\ \le \quad & 2\int_0^t \beta(\sigma)||y_m(\sigma)||_{V_2}||y_m'(\sigma)||_{V_2}d\sigma + 2\int_0^t \gamma(\sigma)||y_m'(\sigma)||_{V_2}d\sigma \\ \le \quad & \frac{1}{\epsilon}||\gamma||^2_{L^2(0,T;\mathcal{R}^+)} + \frac{k_2^2}{\epsilon}\int_0^t \beta(\sigma)^2||y_m(\sigma)||_V^2 d\sigma + 2\epsilon\int_0^t ||y_m'(\sigma)||^2_{V_2}d\sigma.\end{aligned}$$

Since the equality, $y_m(t) = y_{0m} + \int_0^t y_m'(s)ds$ implies

$$|y_m(t)|_H^2 \le 2|y_m(0)|_H^2 + 2T\int_0^t |y_m'(s)|_H^2 ds$$

and since $||y_{0m}||_V \le c_1||y_0||_V$ and $|y_{1m}|_H \le c_2|y_1|_H$ for some $c_1, c_2 > 0$ (see (3.3)), it follows from (3.8), using (HA)(1)-(4),(HC)(2), (4) and (3.9),

$$\begin{aligned}(3.10)\qquad & \alpha_1||y_m(t)||_V^2 + \alpha_3|y_m'(t)|_H^2 + 2(\alpha_2-\epsilon)\int_0^t ||y_m'(\sigma)||_{V_2}^2 d\sigma \\ \le\ & c_1^2(c_{11}+2\lambda_1 k_1^2)||y_0||_V^2 + c_{31}c_2^2|y_1|_H^2 + \frac{1}{\epsilon}||\gamma||^2_{L^2(0,T;\mathcal{R}^+)} \\ & + \int_0^t (c_{12} + \frac{k_2^2}{\epsilon}\beta^2(\sigma))||y_m(\sigma)||_V^2 d\sigma \\ & + (2\lambda_2 + c_{32} + 2T\lambda_1)\int_0^t |y_m'(\sigma)|_H^2 d\sigma,\end{aligned}$$

where k_1 is the embedding constant such that $|\phi|_H \le k_1||\phi||_V$ for all $\phi \in V$. Let us divide (3.10) by $\alpha = \min\{\alpha_1, \alpha_3\} > 0$. We choose ϵ sufficiently small such that $\eta = 2\alpha^{-1}(\alpha_2 - \epsilon) > 0$. Then (3.10) implies

$$\begin{aligned}(3.11)\qquad & ||y_m(t)||_V^2 + |y_m'(t)|_H^2 + \eta\int_0^t ||y_m'(\sigma)||_{V_2}^2 d\sigma \\ \le\ & C + \int_0^t \tilde\beta(\sigma)(||y_m(\sigma)||_V^2 + |y_m'(\sigma)|_H^2)d\sigma,\end{aligned}$$

where $C = \frac{1}{\alpha}[c_1^2(c_{11}+2\lambda_1 k_1^2)||y_0||_V^2 + c_{31}c_2^2|y_1|_H^2 + \frac{1}{\epsilon}||\gamma||^2_{L^2(0,T;\mathcal{R}^+)}]$ and $\tilde\beta(\sigma) = \frac{1}{\alpha}(c_{12} + \frac{k_2^2}{\epsilon}\beta^2(\sigma) + 2\lambda_2 + c_{32} + 2T\lambda_1)$. Thus it follows by Bellman-Gronwall's inequality that

$$(3.12)\qquad ||y_m(t)||_V^2 + |y_m'(t)|_H^2 \le C\exp\Big(\int_0^t \tilde\beta(\sigma)d\sigma\Big) \le C\exp(B),$$

where $B = ||\tilde\beta||_{L^1(0,T;\mathcal{R}^+)}$. By substuting (3.12) for (3.11), we have

$$\begin{aligned}(3.13)\qquad & ||y_m(t)||_V^2 + |y_m'(t)|_H + \eta\int_0^t ||y_m'(\sigma)||_{V_2}^2 d\sigma \\ \le\ & C + C\exp(B)B < \infty.\end{aligned}$$

Step 3. Passage to the limit.
The estimate (3.13) implies that

$$(3.14)\qquad \{y_m\} \text{ is bounded in } L^\infty(0,T;V) \subset L^2(0,T;V)$$

and

(3.15) $\{y'_m\}$ is bounded in $L^2(0,T;V_2)\cap L^\infty(0,T;H)$.

Since the forms a_1, a'_1, a_2, c and c' are continuous in t for all $q \in Q_\tau$, we deduce that $\{A_1(\cdot,q)y_m\}$ and $\{A'_1(\cdot,q)y_m\}$ lie in a bounded set of $L^\infty(0,T;V^*) \subset L^2(0,T;V^*), \{A_2(\cdot,q)y'_m\}$ lies in a bounded set of $L^2(0,T;V_2^*)$ and $\{C(\cdot)y'_m\}$ and $\{C'(\cdot)y_m\}$ lie in a bounded set of $L^2(0,T;H)$. Therefore, by the extraction theorem of Rellich's we can find a subsequence $\{y_{mk}\}$ of $\{y_m\}$ and find $z \in L^\infty(0,T;V) \subset L^2(0,T;V), \bar{z} \in L^2(0,T;V_2) \cap L^\infty(0,T;H), z_1 \in L^2(0,T;V^*), z_2 \in L^2(0,T;V^*), z_3 \in L^2(0,T;V_2^*), z_4 \in L^2(0,T;H)$ and $z_5 \in L^2(0,T;H)$ such that

(3.16) $y_{mk} \to z$ weak-star in $L^\infty(0,T:V)$ and weakly in $L^2(0,T;V)$,

(3.17) $y'_{mk} \to \bar{z}$ weakly in $L^2(0,T;V_2)$,

(3.18) $A_1(\cdot,q)y_{mk} \to z_1$ weakly in $L^2(0,T;V^*)$,

(3.19) $A'_1(\cdot,q)y_{mk} \to z_2$ weakly in $L^2(0,T;V^*)$,

(3.20) $A_2(\cdot,q)y'_{mk} \to z_3$ weakly in $L^2(0,T;V_2^*)$,

(3.21) $C(\cdot)y'_{mk} \to z_4$ weakly in $L^2(0,T;H)$,

(3.22) $C'(\cdot)y_{mk} \to z_5$ weakly in $L^2(0,T;H)$.

It can be seen from (Hf)(2), (3) and (3.13) that $\{f(t,y_{mk})\}$ is bounded in $L^2(0,T;V_2^*)$. Hence we can find a subsequence $\{m_l\}$ of $\{m_k\}$ and $Y \in L^2(0,T;V_2^*)$ satisfying

(3.23) $$f(\cdot,y_{ml}) \to Y \text{ weakly in } L^2(0,T;V_2^*).$$

On the other hand, we have that for $t \in [0,T)$

(3.24) $$y_{ml}(t) = y_{ml}(0) + \int_0^t y'_{ml}(\sigma)d\sigma$$

in the V (and hence V_2 and H) sense. Moreover, $y_{ml}(0) = y_{0m_l} \to y_0$ in the V and hence V_2 sense, whereas for each t,

$$\int_0^t y'_{m_l}(\sigma)d\sigma \to \int_0^t \bar{z}(\sigma)d\sigma \quad \text{weakly in } V_2 \text{ by (3.17).}$$

Hence taking the limit in the weak V_2 sense in (3.24) we obtain

(3.25) $$z(t) = y_0 + \int_0^t \bar{z}(\sigma)d\sigma \quad \text{for } t \in [0,T).$$

This shows that $z'(t)$ exists a.e. in the V_2 sense and $\bar{z} = z' \in L^2(0,T;V_2)$, $z(0) = y_0$, hence we from (3.18)-(3.22) that $z_1 = A_1(\cdot,q)z, z_2 = A'_1(\cdot,q)z$,

$z_3 = A_2(\cdot, q)z', z_4 = C(\cdot)z'$ and $z_5 = C'(\cdot)z$ (cf. [6]). Let j be fixed. Multiply both sides of (3.2) by the scalar function $\xi(t)$ with

$$\xi \in C^1([0,T]), \ \ \xi(T) = 0, \tag{3.26}$$

and put $\phi_j = \xi(t)w_j$. Integrating these over $[0,T]$ for $m_l > j$ and using integration by parts, we have

$$\begin{aligned} &\int_0^T [-(C(t)y'_{m_l}(t), \phi'_j(t))_H + a_2(t,q;y'_{m_l}(t), \phi_j(t)) \\ &+a_1(t,q;y_{m_l}(t), \phi_j(t))]dt \\ = &\int_0^T \langle f(t, y_{m_l}(t)), \phi_j(t)\rangle_{V_2^*,V_2} dt + (C(0)y'_{m_l}, \phi_j(0))_H. \end{aligned} \tag{3.27}$$

If we take $l \to \infty$ in (3.27) and use (3.16), (3.17) and (3.23), then we have

$$\begin{aligned} &\int_0^T [-(C(t)z'(t), \phi'_j(t))_H + a_2(t,q;z'(t), \phi_j(t)) \\ &+a_1(t,q;z(t), \phi_j(t))]dt \\ = &\int_0^T \langle Y(t), \phi_j(t)\rangle_{V_2^*,V_2} dt + (C(0)y_1, \phi_j(0))_H, \end{aligned} \tag{3.28}$$

so that

$$\begin{aligned} &\int_0^T \xi'(t)(-C(t)z'(t), w_j)_H dt \\ &+\int_0^T \xi(t)\{a_2(t,q;z'(t), w_j) + a_1(t,q;z(t), w_j) \\ &-\langle Y(t), w_j\rangle_{V_2^*,V_2}\}dt \\ = &\ \xi(0)(C(0)y_1, w_j)_H. \end{aligned} \tag{3.29}$$

It we take $\xi \in \mathcal{D}(0,T)$ in (3.29), then

$$\begin{aligned} &\frac{d}{dt}(C(\cdot)z'(\cdot), w_j) + a_2(\cdot, q; z'(\cdot), w_j) + a_1(\cdot, q; z(\cdot), w_j) \\ = &\ \langle Y(\cdot), w_j\rangle_{V_2^*,V_2} \end{aligned} \tag{3.30}$$

in the sense of distribution $\mathcal{D}'(0,T)$. Since $\{\sum_{j=1}^m \xi_j w_j | \xi_j \in \mathcal{R}, m \in \mathcal{N}\}$ is dense in V, we conclude by (3.30) that $(C(t)z'(t))' = -A_1(t,q)z(t) - A_2(t,q)z'(t) + Y(t) \in L^2(0,T;V^*)$ and for all $\phi \in V$

$$\begin{aligned} &\langle (C(\cdot)z'(\cdot))', \phi\rangle_{V^*,V} + a_2(\cdot, q; z'(\cdot), \phi) + a_1(\cdot, q; z(\cdot), \phi) \\ = &\ \langle Y(\cdot), \phi\rangle_{V_2^*,V_2} \end{aligned} \tag{3.31}$$

in the sense of $\mathcal{D}'(0,T)$. Multiplying both sides of (3.30) by ξ in (3.26) and using integration by parts, we have from (3.28)

$$(C(0)z'(0), w_j)_H \xi(0) = (C(0)y_1, w_j)_H \xi(0),$$

and that $(C(0)z'(0), w_j)_H = (C(0)y_1, w_j)_H$. Since $\{w_j\}_{j=1}^{\infty}$ is dense in H, we obtain $C(0)z'(0) = C(0)y_1$. From (HC)(3) it is easily verified that $C(t)$ is invertible for all $t \in [0,T]$, and thus we have $z'(0) = y_1$ in H. This also proves that z is a weak solution of the linear problem (3.1) in which $f(t,y)$ is replaced by $Y(t)$.

Step 4. Strong convergence of approximate solutions.
In this step we show that $Y(\cdot) = f(\cdot, z(\cdot))$ in (3.31). In order to prove this we must show $y_{m_l} \to z$ strongly in $L^2(0,T;V_2)$. In what follows we write $y_{m_l} = y_m$ for simplicity. Since z is a weak solution of with $f(t) = Y(t)$, we can prove the following energy equality (see Ha [11] for more detailed proof),

$$\begin{aligned}(3.32)\ & c(t; z'(t), z'(t)) + a_1(t,q; z(t), z(t)) + 2\int_0^t a_2(\sigma, q; z'(\sigma), z'(\sigma))d\sigma \\ = \ & c(0,q; z'(0), z'(0)) + a_1(0,q; z(0), z(0)) + \int_0^t a_1'(\sigma, q; z(\sigma), z(\sigma))d\sigma \\ & - \int_0^t c'(\sigma; z'(\sigma), z'(\sigma))d\sigma + 2\int_0^t \langle Y(\sigma), z'(\sigma)\rangle_{V_2^*,V_2} d\sigma.\end{aligned}$$

For each $t \in [0,T]$, the following equalities hold ;

$$\begin{aligned} & a_1(t,q; y_m, y_m) + a_1(t,q; z, z) \\ = \ & a_1(t,q; y_m - z, y_m - z) + 2a_1(t,q; y_m, z); \\ & a_2(t,q; y_m', y_m') + a_2(t,q; z', z') \\ = \ & a_2(t,q; y_m' - z', y_m' - z') + 2a_2(t,q; y_m', z'); \\ & a_1'(t,q; y_m, y_m) + a_1'(t,q; z, z) \\ = \ & a_1'(t,q; y_m - z, y_m - z) + 2a_1'(t,q; y_m, z); \\ & c(t; y_m', y_m') + c(t; z', z') \\ = \ & c(t; y_m' - z', y_m' - z') + 2c(t; y_m', z'); \\ & c'(t; y_m', y_m') + c'(t; z', z') \\ = \ & c'(t; y_m' - z', y_m' - z') + 2c'(t; y_m', z'); \\ & \langle f(t,y_m), y_m'\rangle_{V_2^*,V_2} + \langle Y(t), z'\rangle_{V_2^*,V_2} \\ = \ & \langle f(t,y_m) - f(t,z), y_m' - z'\rangle_{V_2^*,V_2} + \langle f(t,z) - Y(t), y_m' - z'\rangle_{V_2^*,V_2} \\ & + \langle f(t,y_m), z'\rangle_{V_2^*,V_2} + \langle Y(t), y_m'\rangle_{V_2^*,V_2}. \end{aligned}$$

Adding (3.8) to (3.32) and using the above equalities, we have

$$
\begin{aligned}
(3.33)\qquad & c(t; y_m'(t)-z'(t), y_m'(t)-z'(t)) \\
& +a_1(t,q; y_m(t)-z(t), y_m(t)-z(t)) \\
& +2\int_0^t a_2(\sigma,q; y_m'(\sigma)-z'(\sigma), y_m'(\sigma)-z'(\sigma))d\sigma \\
= \; & Y_m^0 + \sum_{i=0}^{3} Y_m^i(t) + \int_0^t a_1'(\sigma,q; y_m(\sigma)-z(\sigma), y_m(\sigma)-z(\sigma))d\sigma \\
& -\int_0^t c'(\sigma; y_m'(\sigma)-z'(\sigma), y_m'(\sigma)-z'(\sigma))d\sigma \\
& +2\int_0^t \langle f(\sigma, y_m(\sigma)) - f(\sigma, z(\sigma)), y_m'(\sigma)-z'(\sigma)\rangle_{V_2^*,V_2} d\sigma,
\end{aligned}
$$

where

$$
\begin{aligned}
(3.34)\qquad Y_m^0 &= c(0; y_{1m}, y_{1m}) + c(0; y_1, y_1) \\
&\quad +a_1(0,q; y_{0m}, y_{0m}) + a_1(0,q; y_0, y_0), \\
(3.35)\; Y_m^1(t) &= -2c(t; y_m'(t), z'(t)) - 2a_1(t,q; y_m(t), z(t)), \\
(3.36)\; Y_m^2(t) &= -4\int_0^t a_2(\sigma,q; y_m'(\sigma), z'(\sigma))d\sigma \\
&\quad +2\int_0^t a_1'(\sigma,q; y_m(\sigma), z(\sigma))d\sigma \\
&\quad -2\int_0^t c'(\sigma; y_m'(\sigma), z'(\sigma))d\sigma, \\
(3.37)\; Y_m^3(t) &= 2\int_0^t \langle f(\sigma, z(\sigma)) - Y(\sigma), y_m'(\sigma)-z'(\sigma)\rangle_{V_2^*,V_2} d\sigma \\
&\quad +2\int_0^t \langle f(\sigma, y_m(\sigma)), z'(\sigma)\rangle_{V_2^*,V_2} d\sigma \\
&\quad +2\int_0^t \langle Y(\sigma), y_m'(\sigma)\rangle_{V_2^*,V_2} d\sigma.
\end{aligned}
$$

We set

$$
(3.38)\qquad Y_m(t) = Y_m^0 + \sum_{i=1}^{3} Y_m^i(t).
$$

By the similar calculations as in the step 2, the equalities (3.33) and (3.38) imply

$$
\begin{aligned}
(3.39)\qquad & \alpha_1||y_m(t)-z(t)||_V^2+\alpha_3|y_m'(t)-z'(t)|_H^2 \\
& +(2\alpha_2-\epsilon)\int_0^t ||y_m(\sigma)-z(\sigma)||_{V_2}^2 d\sigma \\
\leq\ & Y_m(t)+2\lambda_1 k_1^2||y_{0m}-y_0||_V^2 \\
& +(2\lambda_2+2\lambda_1 T+c_{32})\int_0^t |y_m'(\sigma)-z'(\sigma)|_H^2 d\sigma \\
& +\int_0^t (c_{12}+\frac{k_2^2}{\epsilon}\beta^2(\sigma))||y_m(\sigma)-z(\sigma)||_V^2 d\sigma \quad \text{for any } \epsilon>0.
\end{aligned}
$$

We divide (3.39) by $\alpha=\min\{\alpha_1,\alpha_3\}$ and choose ϵ sufficiently small that $\gamma^{-1}(2\alpha_2-\epsilon)>0$. If we set

$$(3.40)\qquad \Phi_m(t)=||y_m(t)-z(t)||_V^2+|y_m'(t)-z'(t)|_H^2,$$

$$(3.41)\qquad Z_m(t)=\alpha^{-1}(Y_m(t)+2\lambda_1 k_1^2||y_{0m}-y_0||_V^2) \quad \text{and}$$

$$(3.42)\qquad h(t)=\alpha^{-1}(c_{12}+c_{32}+2\lambda_2+2\lambda_1 T+\frac{k_2^2}{\epsilon}\beta^(t)),$$

then the equality (3.39) implies

$$(3.43)\qquad \Phi_m(t)\leq Z_m(t)+\int_0^t h(s)\Phi_m(s)ds.$$

Since $Z_m(t)$ is continuous and $h(s)>0$, we apply the extended Bellman-Gronwall inequality

$$(3.44)\qquad \Phi_m(t)\leq Z_m(t)+\int_0^t \exp(\int_s^t h(\tau)d\tau)h(s)Z_m(s)ds.$$

Let $K(t,s)=\exp(\int_0^t h(\tau)d\tau)h(s)$ and $M_m(t)=\int_0^t K(t,s)Z_m(s)ds$. Then we see that

$$|K(t,s)|\leq \exp(||h||_{L^1(0,T;\mathcal{R}^+)})h(s)$$

and $M_m(t)$ is uniformly bounded on $[0,T]$. We shall show that

(3.45)

$$\lim_{m\to\infty}\int_0^t Z_m(s)ds=0 \quad \text{and} \quad \lim_{m\to\infty} M_m(t)=0 \quad \text{for each } t\in[0,T].$$

By (3.41) and $y_{0m}\to y_0$ strongly in V, it is sufficient to prove that

$$(3.46)\qquad \lim_{m\to\infty}\int_0^t Y_m(s)ds=0 \quad \text{and} \quad \lim_{m\to\infty}\int_0^t K(t,s)Y_m(s)ds=0.$$

Consider the integral

$$I_m(t) = \int_0^t K(t,s)Y_m(s)ds = \int_0^t K(t,s)dsY_m^0 + \sum_{i=1}^{3}\int_0^t K(t,s)Y_m^i(s)ds.$$

Since $y_{0m} \to y_0$ strongly in V and $y_{1m} \to y_1$ strongly in H, we see

$$Y_m^0 \to 2c(0;y_1,y_1) + 2a_1(0,q;y_0,y_0). \tag{3.47}$$

For each $t \in [0,T]$, we have by (3.17)-(3.20) and (3.22) that

$$Y_m^2(t) \to \int_0^t (-4a_2(\sigma,q;z',z') + 2a_1'(\sigma,q;z,z) - 2c'(\sigma;z',z'))d\sigma, \tag{3.48}$$

$$Y_m^3(t) \to 4\int_0^t \langle Y(\sigma), z'\rangle_{V_2^*,V_2} d\sigma. \tag{3.49}$$

We note that $Y_m^2(t)$ and $Y_m^3(t)$ are uniformly bounded on $[0,T]$. Since $K(t,\cdot) \in L^1(0,T;\mathcal{R}^+)$, it follows from (3.16) and (3.17) that

$$\begin{aligned} &\int_0^t K(t,\sigma)Y_m^1(\sigma)d\sigma \\ =\ &\int_0^t (-2\langle A_1(\sigma,q)y_m, K(t,\sigma)z\rangle_{V^*,V} - 2(C(\sigma)y_m', K(t,\sigma)z')_H)d\sigma \\ \to\ &-2\int_0^t K(t,\sigma)\{a_1(\sigma,q;z,z) + c(\sigma;z',z')\}d\sigma. \end{aligned} \tag{3.50}$$

We also note that the integrals in (3.50) are uniformly bounded on $[0,T]$. Hence by using (3.47)-(3.50) and the Lebesgue dominated convergence theorem, we have

$$\begin{aligned} I_m \to\ & 2\int_0^t K(t,s)\{a_1(0,q;y_0,y_0) + c(0;y_1,y_1)\}ds \\ &+2\int_0^t K(t,s)\{-a_1(s,q;z(s),z(s)) - c(s;z'(s),z'(s))\}ds \\ &+2\int_0^t K(t,s)\{-2\int_0^s a_2(\sigma,q;z',z')d\sigma \\ &+\int_0^s a_1'(\sigma,q;z,z)d\sigma - \int_0^s c'(\sigma;z',z')d\sigma\}ds \\ &+2\int_0^t K(t,s)\{2\int_0^s \langle Y(\sigma),z'\rangle_{V_2^*,V_2}d\sigma\}ds = 0, \end{aligned} \tag{3.51}$$

because of (3.32). This shows the second part of (3.46). Similarly we can prove the first part of (3.46). By integrating (3.44) on $[0,T]$ and using

(3.45) and the Lebeque dominated convergence theorem, we verify that

$$\lim_{m\to\infty}\int_0^T \Phi(s)ds = 0. \tag{3.52}$$

This implies that y_m converges strongly to z in $L^2(0,T;V)\subset L^2(0,T;V_2)$. Then, it follows from (Hf)(2) and (3.23) that $Y(\cdot)=f(\cdot,z(\cdot))$ in $L^2(0,T;V_2^*)$. Therefore, we prove the existence of a weak solution z of (2.1). □

Uniqueness proof of Theorem 3.1. Let y_1 and y_2 be the solutions of (2.1) and $z=y_1-y_2$. Then by the energy equality we have

$$\begin{aligned} &c(t;z'(t),z'(t))+a_1(t,q;z(t),z(t))+2\int_0^t a_2(\sigma,q;z',z')d\sigma \\ &= \int_0^t a_1'(\sigma,q;z,z)d\sigma - \int_0^t c'(\sigma;z',z')d\sigma \\ &\quad +2\int_0^t \langle f(\sigma,y_1)-f(\sigma,y_2),z'\rangle_{V_2^*,V_2}d\sigma. \end{aligned} \tag{3.53}$$

Now by the similar calculations as in step 4(see (3.33)) and noting that $Y_m(t)=0$ in this case, we have

$$||z(t)||_V^2+|z'(t)|_H^2=0 \quad \text{for all } t\in[0,T]. \tag{3.54}$$

Therefore the uniqueness is proved. □

4. Sufficient conditions

In this section, we consider the case where all the parameters q related to the diffusion operator $A_1(t,q)$ has already known, i.e., the damping operator $A_2(t,q)$ contains unknown parameters only. Hence letting $A_1(t,q)=A_1(t)$, the system (2.1) is written as

$$\begin{cases} (C(t)y')'+A_2(t,q)y'+A_1(t)y=f(t,q,y,u) & \text{in } (0,T), \\ y(0,q,u)=y_0\in V, y'(0,q,u)=y_1\in H. \end{cases} \tag{4.1}$$

Let us consider a cost functional attached to (4.1) as

$$J(q,u)=\int_0^T g(t,y(t),u(t))dt, \quad q\in Q_\tau, u\in\mathcal{U}_{ad}, \tag{4.2}$$

where y is a solution of (4.1) for given $q\in Q_\tau$ and $u\in\mathcal{U}_{ad}$. Our main aim is to find $(\bar q,\bar u)\in Q_\tau\times\mathcal{U}_{ad}$ satisfying

$$J(\bar q,\bar u)=\inf_{u\in\mathcal{U}_{ad}}\sup_{q\in Q_\tau} J(q,u). \tag{4.3}$$

For our purpose we need the following conditions;

(Hg) $g : [0,T] \times H \times Y \to \mathcal{R}$ is an integrand such that
(1) $(t, y, u) \to g(t, y, u)$ is measurable,
(2) $u \to g(t, y, u)$ is convex and lower semicontinuous(l.s.c) for all $t \in [0, T], y \in H$,
(3) $y \to g(t, y, u)$ is continuous for all $t \in [0, T], u \in \mathcal{U}_{ad}$,
(4) $\phi(t) - \lambda(|y|_H + ||u||_Y) \le g(t, y, u)$ a.e. with $\phi \in L^1(0, T; R)$, $\lambda \ge 0$,
(5) for every $M > 0$ there exists $\eta_M \in L^1(0, T; R^+)$ such that $|g(t, y, u)| \le \eta_M(t)$ a.e. $t \in [0, T]$, all $u \in U(t), |y|_H \le M$.

Furthermore, we give assumptions to $a_2(t, q; \phi, \varphi)$ and $f(t, q, y, u)$;

(4.4) $q \to a_2(t, q; \phi, \varphi)$ is continuous for all $t \in [0, T], \phi, \varphi \in V_2$,

(4.5) $q \to f(t, q, y, u)$ is continuous for all $t \in [0, T], y \in H, u \in Y$,

(4.6) $u \to f(t, q, y, u)$ is continuous for all $t \in [0, T], q \in Q_\tau, y \in H$.

Note that for each $q \in Q_\tau, \phi, \varphi \in V_2$ the following equalities hold:

$$\sup_{||\varphi||_{V_2}=1} |a_2(t, q; \phi, \varphi)| = \sup_{||\varphi||_{V_2}=1} |\langle A_2(t, q)\phi, , \varphi\rangle_{V_2^*, V_2}| = ||A_2(t, q)\phi||_{V_2^*},$$

whence the assumption (4.4) and the above equality imply that $||A_2(t, q) \phi||_{V_2^*}$ is continuous on q.

Lemma 4.1. *Assume that the conditions in Theorem 3.1,* (4.4) *and* (4.5) *hold. Then* $y(q, u) \in C(Q_\tau, W(0, T))$ *for every* $u \in \mathcal{U}_{ad}$.

Proof. Let $u \in \mathcal{U}_{ad}$ and suppose that $q_n \to q$ in Q_τ. Let $y_n = y(q_n, u)$ and $y = y(q, u)$ be the solutions corresponding to q_n and q, respectively. Then by letting $z_n = y_n - y$ we obtain equation

$$\begin{aligned} (4.7) \qquad & (C(t)z_n')' + A_2(t, q_n)z_n' + A_1(t)z_n \\ = \ & f(t, q_n, y_n, u) - f(t, q, y, u) + (A_2(t, q) - A_2(t, q_n))y'. \end{aligned}$$

Since $f(t, q_n, y_n, u) - f(t, q, y, u) + [A_2(t, q) - A_2(t, q_n)]y' \in L^2(0, T; V_2^*)$, we can apply (4.7) to (3.8). Hence we have from $z_n(0) = z_n'(0) = 0$,

$$\begin{aligned} (4.8) \qquad & c(t; z_n', z_n') + a_1(t, q; z_n, z_n) + 2\int_0^t a_2(\sigma, q_n; z_n', z_n')d\sigma \\ = \ & \int_0^t a_1'(\sigma; z_n, z_n)d\sigma - \int_0^t c'(\sigma; z_n', z_n')d\sigma \\ & + 2\int_0^t \langle f(t, q_n, y_n, u) - f(t, q, y, u), z_n'\rangle_{V_2^*, V_2} d\sigma \\ & + 2\int \langle (A_2(\sigma, q) - A_2(\sigma, q_n))y', z_n'\rangle_{V_2^*, V_2} d\sigma. \end{aligned}$$

Denote $\lambda_i = |\lambda_i|, i = 1, 2$ for notational convenience. If we estimate the above equality by using (HA)(2)-(4), (HC)(3), (4), (Hf)(2) and Cauchy-Schwartz inequality, we have

$$
\begin{aligned}
(4.9) \quad & \alpha_3|z_n'|_H^2 + \alpha_1||z_n||_V^2 + \alpha_2\int_0^t ||z_n'||_{V_2}^2 d\sigma \\
\le \quad & \lambda_1|z_n|_H^2 + 2\lambda_2\int_0^t |z_n'|_H^2 d\sigma + c_{12}\int_0^t ||z_n||_V^2 d\sigma + c_{32}\int_0^t |z_n'|_H^2 d\sigma \\
& + \frac{3}{\alpha_2}\int_0^t ||A_2(\sigma, q) - A_2(\sigma, q_n))y'||_{V_2^*}^2 d\sigma + \frac{3}{\alpha_2}\int_0^t \beta^2(\sigma)||z_n||_{V_2}^2 d\sigma \\
& + \frac{3}{\alpha_2}\int_0^t ||f(t, q_n, y_n, u) - f(t, q, y_n, u)||_{V_2^*}^2 d\sigma.
\end{aligned}
$$

Note that $|z_n(t)|_H^2 \le 2T\int_0^t |z_n'(\sigma)|_H^2 d\sigma$. It follows from (4.9) that

$$
\begin{aligned}
(4.10) \quad & \alpha_3|z_n'|_H^2 + \alpha_1||z_n||_V^2 + \alpha_2\int_0^t ||z_n'||_{V_2}^2 d\sigma \\
\le \quad & (2\lambda_1 T + 2\lambda_2 + c_{32})\int_0^t |z_n'|_H^2 d\sigma + \int_0^t (c_{12} + k_2^2\beta^2(\sigma)\frac{3}{\alpha_2})||z_n||_V^2 d\sigma \\
& + \frac{3}{\alpha_2}\int_0^t ||(A_2(\sigma, q) - A_2(\sigma, q_n))y'||_{V_2^*}^2 d\sigma \\
& + \frac{3}{\alpha_2}\int_0^t ||f(t, q_n, y_n, u) - f(t, q, y_n, u)||_{V_2^*}^2 d\sigma.
\end{aligned}
$$

Put $\alpha = \min\{\alpha_1, \alpha_2, \alpha_3\} > 0$ and $\Phi_n(t) = ||z_n(t)||_V^2 + |z_n'(t)|_H^2$. Then we have from (4.10)

$$
\begin{aligned}
(4.11) \quad & \Phi_n(t) + \int_0^t ||z_n'(\sigma)||_V^2 d\sigma \\
\le \quad & \int_0^t \tilde{\beta}(\sigma)\Phi_n(\sigma)d\sigma + \frac{3}{\alpha\alpha_2}\int_0^T ||(A_2(\sigma, q) - A_2(\sigma, q_n))y'(\sigma)||_{V_2^*}^2 d\sigma \\
& + \frac{3}{\alpha\alpha_2}\int_0^T ||f(t, q_n, y_n, u) - f(t, q, y_n, u)||_{V_2^*}^2 d\sigma,
\end{aligned}
$$

where $\tilde{\beta}(\sigma) = \frac{1}{\alpha}(2\lambda_1 T + 2\lambda_2 + c_{32} + c_{12} + k_2^2\beta^2(\sigma)\frac{3}{\alpha_2})$. By Bellman-Gronwall's lemma, we have

$$
\begin{aligned}
(4.12)\, \Phi_n(t) \quad \le \quad & \frac{3}{\alpha\alpha_2}\int_0^T (||(A_2(\sigma, q) - A_2(\sigma, q_n))y'||_{V_2^*}^2 \\
& + ||f(t, q_n, y_n, u) - f(t, q, y_n, u)||_{V_2^*}^2)d\sigma \exp(BT),
\end{aligned}
$$

where $B = ||\tilde{\beta}||_{L^1(0,T;\mathcal{R}^+)}$. Since $a_2(t, q; \phi, \varphi)$ and $f(t, q, y, u)$ are continuous on q and the right hand side of (4.12) goes to zero, $\Phi_n(t) \to 0$ for all $t \in [0, T]$. Applying this fact to (4.11), we conclude that $y_n \to y$ in $C([0, T], V), y'_n \to y'$ in $C([0, T], H)$ and $y'_n \to y'$ in $L^2(0, T; V_2)$. In particular, we also have $y_n \to y$ in $W(0, T)$. □

LEMMA 4.2. *Assume that the conditions of Theorem 3.1,* (4.4) *and* (4.6) *hold. Then* $y(q, u) \in C(\mathcal{U}_{ad}; W(0, T))$ *for every* q *in* Q_τ.

Proof. The proof is similar to that of Lemma 4.1. □

LEMMA 4.3. *Consider the functional* $J(q, u)$ *as in* (4.2) *and assume that* (Hg) *and* (4.4)-(4.6) *hold. Then the mapping* $(q, u) \to J(q, u)$ *is lower semicontinuous on* $Q_\tau \times \mathcal{U}_{ad}$.

Proof. By virtue of the assumption (Hg) and Lemma 4.1 and Lemma 4.2, it is easy to verify the lower semicontinuity of $J(q, u)$. □

THEOREM 4.1. *Assume that the conditions in Theorem 3.1 and Lemma 4.3 hold. Then* $J(q, u)$ *admits its optimal control if* Q_τ *is compact.*

Proof. Let $m_0 = \inf_{u \in \mathcal{U}_{ad}} \sup_{q \in Q_\tau} J(q, u)$. Since $g(t, y, u) < \infty$ for all $(t, y, u) \in [0, T] \times H \times Y$ and by (Hg)(4), this is well-defined and m_0 is finite. Define

$$J_0(u) = \sup_{q \in Q_\tau} J(q, u),$$

then, by Theorem 4, p.122 of Berge [9], $u \to J_0(u)$ is lower semicontinous from $\mathcal{U}_{ad}$ to $\mathcal{R}$. Let $\{u_n\} \subset \mathcal{U}_{ad}$ be a minimizing sequence, that is,

$$\lim_{m \to \infty} J_0(u_n) = m_0.$$

Since $\mathcal{U}_{ad}$ is compact, there exist a subsequence, relabeled as u_n, and an $\bar{u} \in \mathcal{U}_{ad}$ such that $u_n \to \bar{u}$. Then we have

$$m_0 \leq J_0(\bar{u}) \leq \liminf_{n \to \infty} J_0(u_n) = m_0.$$

Consider the functional $J_0(\bar{u}) = \sup_{q \in Q_\tau} \int_0^T g(t, y(q, \bar{u}), \bar{u}) dt$. By Lemma 4.1 and (Hg), there exists $\bar{q} \in Q_\tau$ such that

(4.13)

$$J_0(\bar{u}) = \sup_{q \in Q_\tau} \int_0^T f(t, y(q, \bar{u}), \bar{u}) dt = \int_0^T f(t, y(\bar{q}, \bar{u}), \bar{u}) dt = J(\bar{q}, \bar{u})$$

because Q_τ is compact. This completes the theorem. □

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