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OSCILLATION OF SECOND ORDER NONLINEAR DELAY DIFFERENCE EQUATIONS

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ABSTRACT. In this paper we shall consider the nonlinear delay difference equation

$$\Delta(p_n \Delta x_n) + q_n f(x_{n-\sigma}) = 0, \qquad n = 0, 1, 2, \ldots$$

when $\sum_{n=n_0}^{\infty} \frac{1}{p_n} < \infty$. We will establish some sufficient conditions which guarantee that every solution is oscillatory or converges to zero.

1. Introduction

Recently, there has been an increasing interest in the study of oscillation and asymptotic behavior of solutions of second order difference equations, see, e.g., [1]-[16], [18]-[25]. Following this trend, in this paper we shall consider the nonlinear delay difference equation

$$\Delta(p_n \Delta x_n) + q_n f(x_{n-\sigma}) = 0, \qquad n = 0, 1, 2, \ldots, \tag{1.1}$$

where Δ denotes the forward difference operator $\Delta x_n = x_{n+1} - x_n$ for any sequence $\{x_n\}$ of real numbers, σ is nonnegative integer, $\{p_n\}_{n=0}^{\infty}$ and $\{q_n\}_{n=0}^{\infty}$ are sequences of real numbers such that $p_n > 0$, $q_n \geq 0$ and $\{q_n\}$ has a positive subsequence, and f is a continuous, nondecreasing real valued function which satisfies

$$uf(u) > 0 \text{ for } u \neq 0 \text{ and } f(u)/u \geq \gamma > 0. \tag{1.2}$$

By a *solution* of (1.1) we mean a nontrivial sequence $\{x_n\}$ which is defined for $n \geq -\sigma$ and satisfies equation (1.1) for $n = 0, 1, 2 \ldots$. Clearly if

$$x_n = A_n \quad \text{for} \quad n = -\sigma, \ldots, -1, 0, \tag{1.3}$$

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are given, then equation (1.1) has a unique solution satisfying the initial conditions (1.3). A solution $\{x_n\}$ of (1.1) is said to be *oscillatory* if for every $n_1 > 0$ there exists an $n \geq n_1$ such that $x_n x_{n+1} \leq 0$, otherwise it is *nonoscillatory*. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

In most of the above mentioned papers, the authors considered the linear or nonlinear difference equations and gave some sufficient conditions for oscillation when $p_n > 0$, and

$$\sum_{n=0}^{\infty} \frac{1}{p_n} = \infty. \tag{1.4}$$

In [21], Zhang, considered the equation

$$\Delta(p_n \Delta x_n) + q_n x_{n+1}^{\gamma} = 0, \quad n = 0, 1, 2, \ldots, \tag{1.5}$$

when

$$\sum_{n=0}^{\infty} \frac{1}{p_n} < \infty, \tag{1.6}$$

and proved the following: Every solution of superlinear equation (1.5) oscillates if and only if

$$\sum_{n=n_0}^{\infty} q_n \left(\rho_{n+1}\right)^{\gamma} = \infty, \qquad \text{where } \rho_n = \sum_{i=n}^{\infty} \frac{1}{p_i}, \quad \gamma > 1, \tag{1.7}$$

and every solution of sublinear equation (1.5) oscillates if and only if

$$\sum_{n=n_0}^{\infty} q_n \left(\rho_{n+1}\right) = \infty \qquad \text{where } \rho_n = \sum_{i=n}^{\infty} \frac{1}{p_i}, \quad 0 < \gamma < 1. \tag{1.8}$$

In [4], Arul and Thandapani considered the equation

$$\Delta(p_n \phi(\Delta x_n)) + f(n, x_{n+1}) = 0, \quad n = 0, 1, 2, \ldots, \tag{1.9}$$

when (1.6) holds and gave some sufficient conditions for the existence of positive solutions.

In this paper we intend to use the Riccati transformation technique for obtaining several new sufficient conditions which guarantee that every solution of equation (1.1) oscillates or converges to zero when (1.6) holds. Our results in this paper are different from those in [1]-[16], [18], [22]-[25].

2. Main results

THEOREM 2.1. *Assume that* (1.2) *and* (1.6) *hold. Furthermore, we assume that there exists a positive sequence* $\{\beta_n\}_{n=0}^{\infty}$ *such that*

(h_1) $\Delta\beta_n \le 0,\ \Delta(p_n\Delta\beta_n) \ge 0,\ \sum_{n=n_0}^{\infty} \beta_{n+1}q_n = \infty$

and $\sum_{n=n_0}^{\infty} \frac{1}{p_n\beta_n} \sum_{i=n_0}^{n-1} \beta_{i+1}q_i = \infty$, for some $n_0 > 0$, and

(h_2) $\sum_{i=n+1}^{n+\sigma} Q_i > 0$ and $\sum_{n=n_0}^{\infty} Q_n \left[\left(\sum_{i=n+1}^{n+\sigma} Q_i \right)^{\frac{1}{1+\sigma}} (\sigma+1) - \sigma \right] = \infty,$

where

$$Q_n = \gamma \frac{n-\sigma}{2(p_{n-\sigma})} q_n.$$

Then every solution of equation (1.1) oscillates or converges to zero.

Proof. Suppose to the contrary that $\{x_n\}$ is an eventually positive solution of (1.1) such that $x_n > 0$ and $x_{n-\sigma} > 0$ for all $n \ge n_0$. We shall consider only this case, since the substitution $y_n = -x_n$ transforms equation (1.1) into an equation of the same form subject to the assumptions of Theorem. From equation (1.1) we have

$$\Delta(p_n\Delta x_n) = -q_n f(x_{n-\sigma}) \le 0, \qquad n \ge n_0, \tag{2.1}$$

and so $\{p_n\Delta x_n\}$ is an eventually nonincreasing sequence. Since $\{q_n\}$ has a positive subsequence, the nondecreasing sequence $\{p_n(\Delta x_n)\}$ is either eventually positive or eventually negative and then from (2.1) there exist two possible cases of Δx_n.

Case (I). Suppose that $\Delta x_n < 0$ for $n \ge n_1 > n_0$. It follows that $\lim_{n\to\infty} x_n = b \ge 0$. We assert that b=0. If not, then $f(x_{n-\sigma}) \to f(b) > 0$ as $n \to \infty$. Since $f(x)$ in nondecreasing there exists $n_2 > n_1$ such that $f(x_{n-\sigma}) \ge f(b)$ for $n \ge n_2$. Therefore from (2.1) we have

$$\Delta(p_n\Delta x_n) \le -q_n f(b).$$

Define the sequence $u_n = \beta_n(p_n\Delta x_n)$ for $n \ge n_2$. Then we have

$$\Delta u_n \le -f(b)\beta_{n+1}q_n + \Delta\beta_n(p_n\Delta x_n). \tag{2.2}$$

Summing (2.2) from n_2 to $n-1$, we have

$$u_n \le u_{n_2} - f(b) \sum_{s=n_2}^{n-1} \beta_{s+1}q_s + \sum_{s=n_2}^{n-1} (p_s\Delta\beta_s)\Delta x_s, \tag{2.3}$$

and then

$$u_n \leq u_{n_2} - f(b) \sum_{s=n_2}^{n-1} \beta_{s+1} q_s + p_s \Delta\beta_s \Delta x_s |_{s=n_2}^{n} - \sum_{s=n_2}^{n-1} \Delta(p_s \Delta\beta_s) x_{s+1}.$$

In view of (h_1) we have

$$u_n \leq M - f(b) \sum_{s=n_2}^{n-1} \beta_{s+1} q_s,$$

where $M = u_{n_2} - p_{n_2} \Delta\beta_{n_2} \Delta x_{n_2}$. In view of (h_1), since $\sum_{n=n_0}^{\infty} \beta_{n+1} q_n = \infty$ it is possible to choose integer n_3 sufficiently large such that for all $n \geq n_3$

$$u_n \leq -\frac{f(b)}{2} \sum_{n=n_2}^{n-1} \beta_{s+1} q_s.$$

Summing the last inequality from n_3 to n we obtain

$$x_{n+1} \leq x_{n_3} - \frac{f(b)}{2} \sum_{s=n_3}^{n} \frac{1}{p_s \beta_s} \sum_{i=n_2}^{s-1} \beta_{i+1} q_i.$$

Condition (h_1) implies that $\{x_n\}$ is eventually negative, which is a contradiction. Thus $\{x_n\}$ converges to zero.

Case (II). Suppose that $\Delta x_n > 0$ for $n \geq n_1$. Then from equation (1.1) we have $\Delta^2 x_n \leq 0$ for $n \geq n_1$, and then $\{\Delta x_n\}$ is nonincreasing sequence, and $x_n - x_{n_1} = \sum_{k=n_1}^{n-1} \Delta x_k \geq (n - n_1)\Delta x_n$ which implies that $x_n \geq \frac{n}{2} \Delta x_n$ for $n \geq n_2 \geq 2n_1 + 1$. Then

$$x_{n-\sigma} \geq \frac{n-\sigma}{2} \Delta x_{n-\sigma}, \quad n \geq n_3 = n_2 + \sigma. \tag{2.4}$$

From equation (1.1) and (1.2) we have

$$\Delta(p_n \Delta x_n) + \gamma q_n x_{n-\sigma} \leq 0. \tag{2.5}$$

Then by using (2.4) in (2.5) we have

$$\Delta(p_n \Delta x_n) + \gamma q_n \frac{n-\sigma}{2} \Delta x_{n-\sigma} \leq 0, \quad n \geq n_3, \tag{2.6}$$

Setting $y_n = p_n \Delta x_n$, $y_n > 0$ and satisfies

$$\Delta y_n + Q_n y_{n-\sigma} \leq 0, \quad n \geq n_3, \tag{2.7}$$

where $Q_n = \gamma q_n \frac{n-\sigma}{2(p_{n-\sigma})}$. Let

$$\lambda_n = -\frac{\Delta y_n}{y_n}. \tag{2.8}$$

Since $\{y_n\}$ is a nonincreasing sequence, then we have $0 \leq \lambda_n < 1$ for large n. From (2.8) we have $\frac{y_{n+1}}{y_n} = 1 - \lambda_n$ and $\frac{y_{n-\sigma}}{y_n} = \prod_{i=n-\sigma}^{n-1} (1-\lambda_i)^{-1}$. Then by (2.7) and (2.8) and employing the arithmetic mean-geometric inequality, we have

$$\lambda_n \geq Q_n \prod_{i=n-\sigma}^{n-1} (1-\lambda_i)^{-1} \geq Q_n \left(1 - \frac{1}{\sigma} \sum_{i=n-\sigma}^{n-1} \lambda_i\right)^{-\sigma}. \tag{2.9}$$

Let $b_n = \sum_{i=n+1}^{n+\sigma} Q_i$. Then (2.9) can be rewritten as

$$\lambda_n \geq Q_n \left(1 - \frac{1}{\sigma b_n} b_n \sum_{i=n-\sigma}^{n-1} \lambda_i\right)^{-\sigma}. \tag{2.10}$$

Then from (2.10) by using the inequality

$$\left[1 - \frac{1}{\sigma} r x\right]^{-\sigma} \geq x + \frac{\left[r^{\frac{1}{\sigma+1}}(\sigma+1) - \sigma\right]}{r} \quad \text{for } r > 0 \text{ and } x < \frac{\sigma}{r}, \tag{2.11}$$

we have

$$\lambda_n \geq Q_n \left[\frac{1}{b_n} \sum_{i=n-\sigma}^{n-1} \lambda_i + \frac{1}{b_n}\left((b_n)^{\frac{1}{\sigma+1}}(\sigma+1) - \sigma\right)\right]. \tag{2.12}$$

It follows that

$$\lambda_n b_n - Q_n \sum_{i=n-\sigma}^{n-1} \lambda_i \geq Q_n \left(\left(\sum_{i=n+1}^{n+\sigma} Q_i\right)^{\frac{1}{\sigma+1}} (\sigma+1) - \sigma\right).$$

Then, for $N > n_3$,

(2.13)

$$\sum_{n=n_3}^{N} \lambda_n b_n - \sum_{n=n_3}^{N} Q_n \sum_{i=n-\sigma}^{n-1} \lambda_i \geq \sum_{n=n_3}^{N} Q_n \left(\left(\sum_{i=n+1}^{n+\sigma} Q_i\right)^{\frac{1}{\sigma+1}} (\sigma+1) - \sigma\right).$$

Interchanging the bounds of summation, we find

$$(2.14)\qquad \begin{aligned}\sum_{n=n_3}^{N} Q_n \sum_{i=n-\sigma}^{n-1} \lambda_i &\geq \sum_{n=n_3}^{N-\sigma-1} \sum_{n=i+1}^{i+\sigma} \lambda_i Q_n \\ &= \sum_{i=n_3}^{N-\sigma-1} \lambda_i \sum_{n=i+1}^{i+\sigma} Q_n = \sum_{n=n_3}^{N-\sigma-1} \lambda_n \sum_{i=n+1}^{n+\sigma} Q_i.\end{aligned}$$

Combining (2.13) and (2.14), it follows that

$$(2.15)\qquad \sum_{n=N-\sigma}^{N} \lambda_n \sum_{i=n+1}^{n+\sigma} Q_i \geq \sum_{n=n_3}^{N} Q_n \left(\left(\sum_{i=n+1}^{n+\sigma} Q_i \right)^{\frac{1}{\sigma+1}} (\sigma+1) - \sigma \right).$$

Summing (2.7) from $n+1$ to $n+\sigma$, we get

$$y_{n+1+\sigma} - y_{n+1} + \sum_{i=n+1}^{n+\sigma} Q_i y_{i-\sigma} \leq 0.$$

Using the fact that $\{y_n\}$ is a positive nonincreasing function, we have

$$y_{n+1} > y_n \sum_{i=n+1}^{n+\sigma} Q_i,$$

and so

$$(2.16)\qquad \sum_{i=n+1}^{n+\sigma} Q_i < 1,$$

eventually. Then, from (2.15) and (2.16) we have

(2.17)

$$\sum_{n=N-\sigma}^{N} \lambda_n \geq \sum_{n=n_3}^{N} Q_n \left(\left(\sum_{i=n+1}^{n+\sigma} Q_i \right)^{\frac{1}{\sigma+1}} (\sigma+1) - \sigma \right) \to \infty \text{ as } N \to \infty,$$

by (h_2). But, from the definition of λ_n we have

$$(2.18)\qquad \lambda_n = \left(1 - \frac{y_{n+1}}{y_n}\right).$$

Hence,

$$(2.19)\qquad \sum_{n=N-\sigma}^{N} \lambda_n = \sum_{n=N-\sigma}^{N} \left(1 - \frac{y_{n+1}}{y_n}\right) < \sigma + 1.$$

and this contradicts (2.17). Then every solution of (1.1) oscillates. The proof is complete. □

Note that Theorem 2.1 can not be applied to equation (1.1) when $\sigma = 0$. Then the retarded arguments σ appearing in the nonlinear term plays an important role in the generating qualitative behavior for equation (1.1) different from that for the corresponding equations with $\sigma = 0$. It is of interest to find some new oscillation criteria different from the results in Theorem 2.1.

THEOREM 2.2. *Assume that* (1.2) *and* (1.6) *hold. Furthermore, we assume that there exist positive sequences* $\{\beta_n\}_{n=0}^{\infty}$ *and* $\{\rho_n\}_{n=0}^{\infty}$ *such that* (h_1) *holds, and*

$$\lim_{n\to\infty} \sup \sum_{l=n_0}^{n} \left[\gamma \rho_l q_l - \frac{p_{l-\sigma}(\Delta \rho_l)^2}{4\rho_l}\right] = \infty. \tag{2.20}$$

Then every solution of equation (1.1) *oscillates or converges to zero.*

Proof. We proceed as in Theorem 2.1. We may assume that equation (1.1) has a positive solution $\{x_n\}$ such that $x_n > 0$ and $x_{n-\sigma} > 0$ for all $n \geq n_0$. Then we have $\{\Delta x_n\}$ is of one sign. If $\{\Delta x_n\}$ is eventually negative, then we may follow the proof of Theorem 2.1 to show that $\{x_n\}$ converges to zero. Next we consider the second case when $\{\Delta x_n\}$ is positive for all $n \geq n_1 \geq n_0$. From (1.2) and (1.1) we have

$$\Delta(p_n \Delta x_n) + \gamma q_n x_{n-\sigma} \leq 0. \tag{2.21}$$

Define the sequence $\{w_n\}$ by

$$w_n = \rho_n \frac{p_n \Delta x_n}{x_{n-\sigma}}. \tag{2.22}$$

Then $w_n > 0$ and

$$\Delta w_n = p_{n+1}\Delta x_{n+1} \Delta\left[\frac{\rho_n}{x_{n-\sigma}}\right] + \frac{\rho_n \Delta(p_n \Delta x_n)}{x_{n-\sigma}}. \tag{2.23}$$

From (2.21) we have $\Delta(p_n \Delta x_n) \leq 0$, and since $\Delta x_n > 0$, then we conclude that

$$p_{n-\sigma}\Delta x_{n-\sigma} \geq p_{n+1}\Delta x_{n+1}, \text{ and } x_{n+1-\sigma} \geq x_{n-\sigma}. \tag{2.24}$$

From (2.21)-(2.24), we have
(2.25)

$$\begin{aligned}\Delta w_n \le & -\gamma\rho_n q_n + \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - \frac{\rho_n}{p_{n-\sigma}\left(\rho_{n+1}\right)^2} w_{n+1}^2 \\ = & -\gamma\rho_n q_n + \frac{p_{n-\sigma}(\Delta\rho_n)^2}{4\rho_n} - \left[\frac{\sqrt{\rho_n/p_{n-\sigma}}}{\rho_{n+1}} w_{n+1} - \frac{\Delta\rho_n}{2\sqrt{\rho_n/p_{n-\sigma}}}\right]^2 \\ < & -\left[\gamma\rho_n q_n - \frac{p_{n-\sigma}(\Delta\rho_n)^2}{4\rho_n}\right].\end{aligned}$$

Then, we have

$$\Delta w_n < -\left[\gamma\rho_n q_n - \frac{p_{n-\sigma}(\Delta\rho_n)^2}{4\rho_n}\right]. \tag{2.26}$$

Summing (2.26) from n_1 to n, we obtain

$$-w_{n_1} < w_{n+1} - w_{n_1} < -\sum_{l=n_1}^{n}\left[\gamma\rho_l q_l - \frac{p_{l-\sigma}(\Delta\rho_l)^2}{4\rho_l}\right], \tag{2.27}$$

which yields

$$\sum_{l=n_1}^{n}\left[\gamma\rho_l q_l - \frac{p_{l-\sigma}(\Delta\rho_l)^2}{4\rho_l}\right] \le c_1, \tag{2.28}$$

for all large n, which is contrary to (2.20). The proof is complete. □

From Theorem 2.2, we can obtain different conditions for oscillation of all solutions of equation (1.1) by different choices of $\{\rho_n\}$. Let $\rho_n = n^\lambda$, $n \ge n_0$ and $\lambda \ge 1$ is a constant. By theorem 2.2 we have the following result.

COROLLARY 2.1. *Assume that all the assumptions of Theorem 2.2 hold, except the condition* (2.20) *is replaced by*

$$\limsup_{n\to\infty} \sum_{s=n_0}^{n}\left[\gamma s^\lambda q_s - \frac{p_{s-\sigma}((s+1)^\lambda - s^\lambda)^2}{4s^\lambda}\right] = \infty. \tag{2.29}$$

Then, every solution of equation (1.1) *oscillates or converges to zero.*
The following example is illustrative

EXAMPLE 2.1. Consider the difference equation

$$\Delta(n^2\Delta x_n) + \mu x_n = 0, \quad n \ge 1 \tag{2.30}$$

where $\mu > \frac{1}{4}$. Then, $p_n = n^2$, $\gamma = 1$. If we take $\lambda = 1$, then we have

$$\sum_{s=n_0}^{n}\left[\gamma s^{\lambda} q_s - \frac{p_{s-\sigma}((s+1)^{\lambda} - s^{\lambda})^2}{4s^{\lambda}}\right] = \sum_{s=1}^{n}\left[\mu s - \frac{s^2}{4s}\right]$$

$$= \sum_{s=1}^{n} \frac{(4\mu - 1)}{4} s \to \infty.$$

as $n \to \infty$. By Corollary 2.1, every solution of (2.30) oscillates or converges to zero. Note that none of the above mentioned papers can be applied to (2.30). Hence, Theorem 2.2 and Corollary 2.1 are sharp.

As a variant of the Riccati transformation technique used above, we will derive a Kamenev type oscillation criteria which can be considered as a discrete analogy of Philos's condition for oscillation of second order differential equations [17].

THEOREM 2.3. *Assume that* (1.2) *and* (1.5) *hold, and let* $\{\beta_n\}_{n=0}^{\infty}$ *and* $\{\rho_n\}_{n=0}^{\infty}$ *be two positive sequences such that* (h_1) *holds. Furthermore, we assume that there exists a double sequence* $\{H_{m,n} : m \geq n \geq 0\}$ *such that* (i) $H_{m,m} = 0$ *for* $m \geq 0$, (ii) $H_{m,n} > 0$ *for* $m > n > 0$, (iii) $\Delta_2 H_{m,n} = H_{m,n+1} - H_{m,n}$. *If*

(2.31)

$$\lim_{m\to\infty} \sup \frac{1}{H_{m,0}} \sum_{n=n_0}^{m-1}\left[\gamma H_{m,n}\rho_n q_n - \frac{\rho_{n+1}^2}{4\,\bar{\rho}_n}\left(h_{m,n} - \frac{\Delta\rho_n}{\rho_{n+1}}\sqrt{H_{m,n}}\right)^2\right] = \infty,$$

where

$$h_{m,n} = -\frac{\Delta_2 H_{m,n}}{\sqrt{H_{m,n}}}, \qquad \bar{\rho}_n = \rho_n / p_{n-\sigma}.$$

Then every solution of equation (1.1) *oscillates or converges to zero.*

Proof. We proceed as in the proof of Theorem 2.1. We may assume that (1.1) has a nonoscillatory solution $\{x_n\}_{n=0}^{\infty}$. Then we have $\{\Delta x_n\}$ is of one sign. If $\{\Delta x_n\}$ is eventually negative, then we may follow the proof of Theorem 2.1 to show that $\{x_n\}$ converges to zero. Next we consider the case when $\Delta x_n \geq 0$ for $n \geq n_1$. Define $\{w_n\}$ by (2.22) as before. Then we have $w_n > 0$ and (2.25) holds. For the sake of convenience, let us set

$$\bar{\rho}_n = \rho_n / p_{n-\sigma}.$$

Then,

$$\gamma\rho_n q_n \leq -\Delta w_n + \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - \frac{\bar{\rho}_n}{(\rho_{n+1})^2} w_{n+1}^2.$$

Therefore, we have

(2.32)

$$\sum_{n=n_1}^{m-1} \gamma H_{m,n}\rho_n q_n$$
$$\leq -\sum_{n=n_1}^{m-1} H_{m,n}\Delta w_n + \sum_{n=n_1}^{m-1} H_{m,n}\frac{\Delta\rho_n}{\rho_{n+1}}w_{n+1} - \sum_{n=n_1}^{m-1} H_{m,n}\frac{\bar{\rho}_n}{(\rho_{n+1})^2}w_{n+1}^2$$

which yields, after summing by parts

$$\begin{aligned}
&\sum_{n=n_1}^{m-1} \gamma H_{m,n}\rho_n q_n \\
\leq\ & H_{m,n_1}w_{n_1} + \sum_{n=n_1}^{m-1} w_{n+1}\Delta_2 H_{m,n} + \sum_{n=n_1}^{m-1} H_{m,n}\frac{\Delta\rho_n}{\rho_{n+1}}w_{n+1} \\
& - \sum_{n=n_1}^{m-1} H_{m,n}\frac{\bar{\rho}_n}{(\rho_{n+1})^2}w_{n+1}^2 \\
=\ & H_{m,n_1}w_{n_1} - \sum_{n=n_1}^{m-1} h_{m,n}\sqrt{H_{m,n}}w_{n+1} + \sum_{n=n_1}^{m-1} H_{m,n}\frac{\Delta\rho_n}{\rho_{n+1}}w_{n+1} \\
& - \sum_{n=n_1}^{m-1} H_{m,n}\frac{\bar{\rho}_n}{(\rho_{n+1})^2}w_{n+1}^2 = H_{m,n_1}w_{n_1} \\
& - \sum_{n=n_1}^{m-1}\left[\frac{\sqrt{H_{m,n}\,\bar{\rho}_n}}{\rho_{n+1}}w_{n+1}\right. \\
& \qquad \left. + \frac{\rho_{n+1}}{2\sqrt{H_{m,n}\,\bar{\rho}_n}}\left(h_{m,n}\sqrt{H_{m,n}} - \frac{\Delta\rho_n}{\rho_{n+1}}H_{m,n}\right)\right]^2 \\
& + \frac{1}{4}\sum_{n=n_1}^{m-1}\frac{(\rho_{n+1})^2}{\bar{\rho}_n}\left(h_{m,n} - \frac{\Delta\rho_n}{\rho_{n+1}}\sqrt{H_{m,n}}\right)^2.
\end{aligned}$$

Then,

$$\sum_{n=n_1}^{m-1}\left[\gamma H_{m,n}\rho_n q_n - \frac{(\rho_{n+1})^2}{4\bar{\rho}_n}\left(h_{m,n} - \frac{\Delta\rho_n}{\rho_{n+1}}\sqrt{H_{m,n}}\right)^2\right] < H_{m,n_1}w_{n_1}$$
$$\leq H_{m,0}w_{n_1}$$

which implies that

$$\sum_{n=0}^{m-1}\left[\gamma H_{m,n}\rho_n q_n - \frac{(\rho_{n+1})^2}{4\,\bar{\rho}_n}\left(h_{m,n} - \frac{\Delta\rho_n}{\rho_{n+1}}\sqrt{H_{m,n}}\right)^2\right] < H_{m,0}\left(w_{n_1} + \sum_{n=0}^{n_1-1}\gamma\rho_n q_n\right).$$

Hence

$$\lim_{m\to\infty}\sup\frac{1}{H_{m,0}}\sum_{n=0}^{m-1}\left[\gamma H_{m,n}\rho_n q_n - \frac{(\rho_{n+1})^2}{4\,\bar{\rho}_n}\left(h_{m,n} - \frac{\Delta\rho_n}{\rho_{n+1}}\sqrt{H_{m,n}}\right)^2\right] < \left(w_{n_1} + \sum_{n=0}^{n_1-1}\gamma\rho_n q_n\right) < \infty,$$

which is contrary to (2.31). The proof is complete. □

By choosing the sequence $\{H_{m,n}\}$ in appropriate manners, we can derive several oscillation criteria for (1.1). For instance, let us consider the double sequence $\{H_{m,n}\}$ defined by

$$\left.\begin{aligned} &H_{m,n} = (m-n)^{\lambda}, \quad \lambda \geq 1, m \geq n \geq 0,\\ &H_{m,n} = \left(\log\tfrac{m+1}{n+1}\right)^{\lambda}, \lambda \geq 1, m \geq n \geq 0,\\ &H_{m,n} = (m-n)^{(\lambda)} \quad \lambda > 2,\ m \geq n \geq 0, \end{aligned}\right\} \tag{2.33}$$

where $(m-n)^{(\lambda)} = (m-n)(m-n+1)\cdots(m-n+\lambda-1)$, and

$$\Delta_2(m-n)^{(\lambda)} = (m-n-1)^{(\lambda)} - (m-n)^{(\lambda)} = -\lambda(m-n)^{(\lambda-1)}.$$

Then $H_{m,m} = 0$ for $m \geq 0$ and $H_{m,n} > 0$ and $\Delta_2 H_{m,n} \leq 0$ for $m > n >\geq 0$. Hence we have the following results.

COROLLARY 2.2. *Assume that all the assumptions of Theorem 2.3 hold, except the condition* (2.31) *is replaced by*

$$\begin{aligned} \lim_{m\to\infty}\sup\frac{1}{m^{\lambda}}\sum_{n=0}^{m-1}&\left[\gamma(m-n)^{\lambda}\rho_n q_n\right.\\ &\left.-\frac{p_{n-\sigma}\rho_{n+1}^2}{4\rho_n}\left(\lambda(m-n)^{\frac{\lambda-2}{2}} - \frac{\Delta\rho_n}{\rho_{n+1}}\sqrt{(m-n)^{\lambda}}\right)^2\right] = \infty. \end{aligned} \tag{2.34}$$

Then every solution of equation (1.1) *oscillates or converges to zero.*

Corollary 2.3. *Assume that all the assumptions of Theorem 2.3 hold, except the condition* (2.31) *is replaced by*

(2.35)

$$\lim_{m\to\infty}\sup\frac{1}{(\log(m+1))^{\lambda}}\sum_{n=0}^{m-1}\left[\gamma\left(\log\frac{m+1}{n+1}\right)^{\lambda}\rho_n q_n - \frac{p_{n-\sigma}\rho_{n+1}^2}{4\rho_n}\left(\frac{\lambda}{n+1}\left(\log\frac{m+1}{n+1}\right)^{\frac{\lambda-2}{2}}-\frac{\Delta\rho_n}{\rho_{n+1}}\sqrt{\left(\log\frac{m+1}{n+1}\right)^{\lambda}}\right)^2\right] = \infty.$$

Then, every solution of equation (1.1) *oscillates or converges to zero.*

Corollary 2.4. *Assume that all the assumptions of Theorem 2.3 hold, except the condition* (2.31) *is replaced by*

$$\lim_{m\to\infty}\sup\frac{1}{m^{(\lambda)}}\sum_{n=0}^{m-1}(m-n)^{(\lambda)}\left[\gamma\rho_n q_n - \frac{p_{n-\sigma}\rho_{n+1}^2}{4\rho_n}\left(\frac{\lambda}{m-n+\lambda-1}-\frac{\Delta\rho_n}{\rho_{n+1}}\right)^2\right]=\infty. \tag{2.36}$$

Then, every solution of equation (1.1) *oscillates or converges to zero.*

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