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ON THE GEOMETRY OF THE MANIFOLD MEX_{2n}

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Abstract. A generalized even-dimensional Riemannian manifold defined by the ME-connection which is both Einstein and of the form (3.3) is called an even-dimensional ME-manifold and we denote it by MEX_{2n}. The purpose of this paper is to study a necessary and sufficient condition that there is an ME-connection, to derive the useful properties of some tensors, and to investigate a representation of the ME-vector in MEX_{2n}.

1. Introduction

In Appendix II to his last book Einstein [3] proposed a new unified field theory that would include both gravitation and electromagnetism. Although the intent of this theory is physical, its exposition is mainly geometrical. It may be characterized as a set of geometrical postulates for the space-time X_4. However, the geometrical consequences of these postulates were not developed very far by Einstein. Characterizing Einstein's unified field theory as a set of geometrical postulates for X_4, Hlavatý [4] gave its mathematical foundation for the first time. Generalizing X_4 to an n-dimensional generalized Riemannian manifold X_n was considered and studied by Hlavatý [4], Wrede [7], and Mishra [6].

Recently, Chung [2] introduced the concept of n-dimensional SE-manifold, imposing the semi-symmetric condition on X_n, which is similar to Yano [8] and Imai's [5] semi-symmetric metric connection, and found a unique representation of n-dimensional Einstein's connection in a beautiful and surveyable form.

In the present paper, we first introduce some preliminary notations, concepts and results which are needed in this paper. In the next we

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show that a necessary and sufficient condition for the existence of ME-connection, a representation of ME-vector, and some relations which hold in MEX_{2n}.

2. Preliminaries

This section is a brief collection of the basic concepts, notations, and results, which are needed in our further considerations in the present paper. It based on the results and symbolisms of Chung [2] and Hlavatý [4].

Let X_{2n} $(n > 1)$ be a generalized even-dimensional Riemannian manifold referred to a real coordinate system x^ν, which obeys coordinate transformation $x^\nu \longrightarrow \bar{x}^\nu$ for which

$$Det\left(\frac{\partial \bar{x}}{\partial x}\right) \neq 0, \tag{2.1}$$

where, here and in the sequel, Greek indices are used for the holonomics components of tensor in X_{2n}. They take the valves $1, 2, \cdots, n$ and follow the summation convention.

The manifold X_{2n} is endowed with a general real non-symmetric tensor $g_{\lambda\mu}$, which may be decomposed into its symmetric part $h_{\lambda\mu}$ and skew-symmetric part $k_{\lambda\mu}$:

$$g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu}, \tag{2.2a}$$

where

$$\mathfrak{g} = Det(g_{\lambda\mu}) \neq 0, \qquad \mathfrak{h} = Det(h_{\lambda\mu}) \neq 0, \qquad \mathfrak{k} = Det(k_{\lambda\mu}) \neq 0. \tag{2.2b}$$

Hence we may define a unique tensor $h^{\lambda\nu}$ by

$$h_{\lambda\mu}h^{\lambda\nu} = \delta^\nu_\mu. \tag{2.3}$$

The tensor $h_{\lambda\mu}$ and $h^{\lambda\nu}$ will serve for raising and/or lowering indices of tensor in X_{2n} in the usual manner.

The manifold X_{2n} is assumed to be connected by a general real connection $\Gamma^\nu_{\lambda\mu}$ with the following transformation rule:

$$\bar{\Gamma}^\nu_{\lambda\mu} = \frac{\partial \bar{x}^\nu}{\partial x^\alpha}\left(\frac{\partial x^\beta}{\partial \bar{x}^\lambda}\frac{\partial x^\gamma}{\partial \bar{x}^\mu}\Gamma^\alpha_{\beta\gamma} + \frac{\partial^2 x^\alpha}{\partial \bar{x}^\lambda \partial \bar{x}^\mu}\right). \tag{2.4}$$

The connection $\Gamma^{\nu}_{\lambda\mu}$ is called an *Einstein's connection* if it satisfied the following Einstein's equation:

$$\partial_\omega g_{\lambda\mu} - \Gamma^{\alpha}_{\lambda\omega} g_{\alpha\mu} - \Gamma^{\alpha}_{\omega\mu} g_{\lambda\alpha} = 0 \qquad \left(\partial_\omega = \frac{\partial}{\partial x^\omega}\right), \tag{2.5a}$$

or equivalently,

$$D_\omega g_{\lambda\mu} = 2 S_{\omega\mu}{}^{\alpha} g_{\lambda\alpha}, \tag{2.5b}$$

where D_ω denotes the symbolic vector of the covariant derivative with respect to $\Gamma^{\nu}_{\lambda\mu}$, and

$$S_{\omega\mu}{}^{\nu} = \Gamma^{\nu}_{[\omega\mu]} = \frac{1}{2}\left(\Gamma^{\nu}_{\omega\mu} - \Gamma^{\nu}_{\mu\omega}\right) \tag{2.6}$$

is a *torsion tensor* of $\Gamma^{\nu}_{\lambda\mu}$.

The following quantities will be frequently used in our subsequent considerations:

$$g = \frac{\mathfrak{g}}{\mathfrak{h}} \qquad k = \frac{\mathfrak{k}}{\mathfrak{h}}, \tag{2.7}$$

$$^{(0)}k_\lambda{}^\nu = \delta^\nu_\lambda, \quad ^{(p)}k_\lambda{}^\nu = {}^{(p-1)}k_\lambda{}^\alpha k_\alpha{}^\nu, \tag{2.8}$$

$$K_0 = 1, \quad K_p = k_{[\alpha_1}{}^{\alpha_1} k_{\alpha_2}{}^{\alpha_2} \cdots k_{\alpha_p]}{}^{\alpha_p}, \tag{2.9}$$

$$K_{\omega\mu\nu} = \nabla_\omega k_{\nu\mu} + \nabla_\mu k_{\omega\nu} + \nabla_\nu k_{\omega\mu}, \tag{2.10}$$

where ∇_ω is the symbolic vector of the covariant derivative with respect to the Christoffel symbol $\left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\}$ defined by $h_{\lambda\mu}$.

It has been shown that the following relations hold in X_{2n} [1].

$$Det(M h_{\lambda\mu} + k_{\lambda\mu}) = \mathfrak{h} \sum_{s=0}^{2n} K_s M^{2n-s}, \qquad (M \text{ is a real number}), \tag{2.11}$$

$$\sum_{s=0}^{2n} K_s {}^{(2n+p-s)}k_\lambda{}^\nu = 0, \qquad (p = 0, 1, 2, \cdots). \tag{2.12}$$

Here and in what follows, *the indices s and t are assumed to take the values* $0, 2, 4, 6, \cdots$ *in the specified range.*

It has been shown Hlavatý [4] that if the equations (2.5) admit a solution $\Gamma^{\nu}_{\lambda\mu}$, it must be of the form

$$\Gamma^{\nu}_{\lambda\mu} = \left\{ {\nu \atop \lambda\mu} \right\} + S_{\lambda\mu}{}^{\nu} + U^{\nu}{}_{\lambda\mu}, \tag{2.13}$$

where

$$U^{\nu}{}_{\lambda\mu} = 2h^{\nu\alpha} S_{\alpha(\lambda}{}^{\beta} k_{\mu)\beta}. \tag{2.14}$$

3. The ME-connection in MEX_{2n}

In this section, we first investigate the ME-connection $\Gamma^{\nu}_{\lambda\mu}$ and an even-dimensional ME-manifold defined by the ME-connection $\Gamma^{\nu}_{\lambda\mu}$. We also find a necessary and sufficient condition for the existence of ME-connection and some relations which hold in MEX_{2n}.

We use the following abbreviation for an arbitrary real vector A_λ and an arbitrary tensor $X^{\lambda\nu}$ defined by

$$^{(p)}A_\lambda = {}^{(p)}k_\lambda{}^\alpha A_\alpha \quad (p = 0, 1, 2, \cdots), \tag{3.1a}$$

$$^{(p)}A^\nu = (-1)^{p}\,{}^{(p)}k_\alpha{}^\nu A^\alpha \quad (p = 0, 1, 2, \cdots), \tag{3.1b}$$

$$^{(0)}X^{\lambda\nu} = X^{\lambda\nu}, \qquad {}^{(p)}X^{\lambda\nu} = {}^{(p)}k^\lambda{}_\alpha X^{\alpha\nu} \quad (p = 1, 2, 3, \cdots), \tag{3.2a}$$

$$X = X_\alpha X^\alpha. \tag{3.2b}$$

DEFINITION 3.1. The Einstein's connection $\Gamma^{\nu}_{\lambda\mu}$ which takes the form

$$\Gamma^{\nu}_{\lambda\mu} = \left\{ {\nu \atop \lambda\mu} \right\} + 2\delta^{\nu}_{\lambda} X_\mu - 2g_{\lambda\mu} X^\nu, \tag{3.3}$$

for a non-null vector X^ν, is called an ME-connection. In the representation of ME-connection, the vector X^ν will be called an ME-vector.

DEFINITION 3.2. A generalized even-dimensional Riemannian manifold X_{2n} connected by ME-connection is called an even-dimensional ME-manifold and denoted by MEX_{2n}.

LEMMA 3.3. *If there is a ME-connection in MEX_{2n}, the torsion tensor $S_{\lambda\mu}{}^{\nu}$ and the tensor $U^{\nu}{}_{\lambda\mu}$ are given by*

$$S_{\lambda\mu}{}^{\nu} = 2\delta^{\nu}_{[\lambda}X_{\mu]} - 2k_{\lambda\mu}X^{\nu}, \tag{3.4}$$

$$U^{\nu}{}_{\lambda\mu} = 2\delta^{\nu}_{(\lambda}X_{\mu)} - 2h_{\lambda\mu}X^{\nu}. \tag{3.5}$$

Proof. Substituting (3.3) into (2.6) and using (2.2a), we have the relation (3.4). In virtue of (2.2a), (2.13), (3.3), and (3.4), we obtain the relation (3.5). □

THEOREM 3.4. *If there is an ME-connection $\Gamma^{\nu}_{\lambda\mu}$ in MEX_{2n}, then it must be of the form:*

$$\Gamma^{\nu}_{\lambda\mu} = \left\{ {\nu \atop \lambda\mu} \right\} + 2\delta^{\nu}_{[\lambda}X_{\mu]} - 2k_{\lambda\mu}X^{\nu} + 2k_{(\lambda}{}^{\nu}\left(X_{\mu)} + 2{}^{(1)}X_{\mu)}\right). \tag{3.6}$$

Proof. Substituting (3.4) into (2.14), we have

$$U^{\nu}{}_{\lambda\mu} = 2k_{(\lambda}{}^{\nu}X_{\mu)} + 4k_{(\lambda}{}^{\nu}k_{\mu)}{}^{\alpha}X_{\alpha}. \tag{3.7}$$

Substituting (3.4) and (3.7) into (2.13), we obtain the relation (3.6). □

THEOREM 3.5. *A necessary and sufficient condition that there is an ME-connection $\Gamma^{\nu}_{\lambda\mu}$ on MEX_{2n} is that the following condition is satisfied:*

$$\nabla_{\omega}k_{\lambda\mu} = 2\left(h_{\omega[\mu}X_{\lambda]} + 2k_{\omega[\mu}X_{\lambda]} + {}^{(2)}k_{\omega[\lambda}X_{\mu]} + 2{}^{(2)}k_{\omega[\lambda}{}^{(1)}X_{\mu]}\right). \tag{3.8}$$

Proof. Suppose that there exists an ME-connection. Then by Theorem 3.4, it is given by (3.6). Substituting (3.6) into (2.5a) and making use of (2.2a), (2.8), and (3.2a) we obtained (3.8) by a long computation.

Conversely, suppose that the statement (3.8) holds. Now, we define a connection by (3.6) with the vector X_{μ} satisfying (3.8). Then this connection is clearly Einstein since it satisfies (2.5a) in virtue of our assumption (3.8). On the other hand, the Einstein's connection is of the form (3.3) in virtue of (2.12), (3.4), and (3.5). Therefore, this connection is an ME-connection. □

LEMMA 3.6. *In MEX_{2n}, the following relations hold:*

$$K_{\omega[\mu\nu]} = \nabla_\omega k_{\nu\mu}, \tag{3.9a}$$

$$K_{[\omega\mu]\nu} = K_{\omega\mu\nu}. \tag{3.9b}$$

Proof. The relations (3.9) immediately follow from (2.10). □

LEMMA 3.7. *The following relation holds in MEX_{2n}:*

$$K_{\omega\mu\nu} = 4\left(h_{\nu[\omega}X_{\mu]} - k_{\omega\mu}X_\nu + {}^{(2)}k_{\nu[\mu}X_{\omega]} + 2^{(2)}k_{\nu[\mu}{}^{(1)}X_{\omega]}\right). \tag{3.10}$$

Proof. Substituting (3.8) into (2.10), we have the relation (3.10). □

THEOREM 3.8. *In MEX_{2n}, the following relations hold:*

$$S_{\lambda\mu}{}^{\nu}X_\nu = -2k_{\lambda\mu}X, \tag{3.11a}$$

$$S_{\lambda\mu}{}^{\nu}X^\lambda = X_\mu X^\nu - \delta^\nu_\mu X + 2^{(1)}X_\mu X^\nu, \tag{3.11b}$$

$$S_{\lambda\mu}{}^{\nu}k_\nu{}^\lambda = -{}^{(1)}X_\mu - {}^{(2)}X_\mu. \tag{3.11c}$$

Proof. Making use of (3.1a), (3.2), and (3.4), we have the relations (3.11). □

THEOREM 3.9. *The following relations hold in MEX_{2n}:*

$$U^\nu{}_{\lambda\mu}X_\nu = 2X_{(\lambda}{}^{(1)}X_{\mu)} + 4^{(1)}X_\lambda{}^{(1)}X_\mu, \tag{3.12a}$$

$$U^\nu{}_{\lambda\mu}k_{\omega\nu} = -2\left({}^{(2)}k_{\omega(\lambda}X_{\mu)} + 2^{(2)}k_{\omega(\lambda}{}^{(1)}X_{\mu)}\right). \tag{3.12b}$$

Proof. The relations (3.12) result from (2.8), (3.1), (3.2), and (3.7). □

THEOREM 3.10. *The torsion vector* $S_\lambda (= S_{\lambda\alpha}{}^{\alpha})$ *and the vector* $U_\lambda (= U^{\alpha}{}_{\alpha\lambda})$ *may be given by*

$$S_\lambda = (1-n)X_\lambda - 2{}^{(1)}X_\lambda, \tag{3.13a}$$

$$U_\lambda = {}^{(1)}X_\lambda + 2{}^{(2)}X_\lambda. \tag{3.13b}$$

Proof. The relations (3.13) follow from (3.4) and (3.7), putting $\mu = \nu = \alpha$ and making use of (2.8) and (3.1). □

LEMMA 3.11. *The following relations hold in* MEX_{2n}*:*

$$S_{[\omega\mu]\nu} = S_{\omega\mu\nu}, \tag{3.14a}$$

$$S_{\omega[\mu\nu]} = h_{\omega[\nu}X_{\mu]} + 2k_{\omega[\nu}X_{\mu]}, \tag{3.14b}$$

$$S_{[\omega\mu\nu]} = -2k_{[\omega\mu}X_{\nu]}. \tag{3.14c}$$

Proof. Multiplying $h_{\nu\alpha}$ to the torsion tensor $S_{\omega\mu}{}^{\alpha}$ and making use of (3.4), we have the following relation:

$$2S_{\omega\mu\nu} = 4h_{\nu[\omega}X_{\mu]} - 4k_{\omega\mu}X_\nu. \tag{3.15}$$

The relations (3.14) are a direct consequence of (3.15). □

LEMMA 3.12. *In* MEX_{2n}, *the tensor* $U^{\nu}{}_{\lambda\mu}$ *satisfies the following conditions:*

$$U_{[\omega\lambda]\mu} = k_{\lambda\omega}\left(X_\mu + 2{}^{(1)}X_\mu\right) + k_{\mu[\omega}\left(X_{\lambda]} + 2{}^{(1)}X_{\lambda]}\right), \tag{3.16a}$$

$$U_{\omega[\lambda\mu]} = 0, \tag{3.16b}$$

$$U_{(\omega\lambda\mu)} = 0. \tag{3.16c}$$

Proof. Multiplying $h_{\nu\omega}$ to both sides of (3.7) and using (3.1), we obtain the following relation:

$$U_{\omega\lambda\mu} = -k_{\omega(\lambda}X_{\mu)} - 2k_{\omega(\lambda}{}^{(1)}X_{\mu)}. \tag{3.17}$$

The relations (3.16) immediately follow from (3.17). □

4. The ME-vector in MEX_{2n}

In this section, we introduce a representation of the ME-vector X_λ which holds in an even-dimensional ME-manifold with a certain special condition imposed on $g_{\lambda\mu}$.

We need a tensor $F_{\lambda\mu}$ defined by

$$F_{\lambda\mu} = k_{\lambda\mu} - 2^{(2)}k_{\lambda\mu}. \tag{4.1}$$

LEMMA 4.1. *The tensor $F_{\lambda\mu}$ is of rank n if and only if the tensor field $g_{\lambda\mu}$ satisfied the following condition:*

$$\sum_{s=0}^{2n} 2^s K_s \neq 0. \tag{4.2}$$

Proof. The tensor $F_{\lambda\mu}$ may be written as

$$F_{\lambda\mu} = 2k_{\lambda\alpha}(\frac{1}{2}h_{\mu\beta} + k_{\mu\beta})h^{\alpha\beta}. \tag{4.3}$$

In virtue of (2.11) and (4.3), we obtain the following relation:

$$Det(F_{\lambda\mu}) = 2^{2n}\mathfrak{k}\left(\mathfrak{h}\sum_{s=0}^{2n} K_s(\frac{1}{2})^{2n-s}\right)\frac{1}{\mathfrak{h}} = \mathfrak{k}\sum_{s=0}^{2n} 2^s K_s. \tag{4.4}$$

Our assertion (4.2) follows from (2.2b) and (4.4). □

By Lemma 4.1, there exists a unique inverse tensor $G^{\lambda\nu}$ defined by

$$G^{\lambda\nu}F_{\lambda\mu} = G^{\nu\lambda}F_{\mu\lambda} = \delta^\nu_\mu. \tag{4.5}$$

THEOREM 4.2. *In MEX_{2n}, the ME-vector X_ω may be given by the following representation:*

$$X_\omega = -\frac{1}{2}G_\omega{}^\alpha\partial_\alpha(log\ g). \tag{4.6}$$

Proof. Multiplying ${}^*g^{\lambda\mu}$, defined by

$$(4.7) \qquad {}^*g^{\lambda\nu}g_{\lambda\mu} = {}^*g^{\nu\lambda}g_{\mu\lambda} = \delta^{\nu}_{\mu},$$

to both sides of (2.5b), we have

$$(4.8) \qquad \partial_\omega log\ \mathfrak{g} - 2\Gamma^{\alpha}_{\alpha\omega} = 2S_{\omega\alpha}{}^{\alpha}.$$

On the other hand, multiply $h^{\lambda\mu}$ to both sides of the symmetric part of (2.5b) and making use of (2.2), (2.8), and (3.4) to obtain

$$(4.9) \qquad \partial_\omega log\ \mathfrak{h} - 2\Gamma^{\alpha}_{\alpha\omega} = 2S_{\omega\alpha}{}^{\alpha} - 2\left(k_\omega{}^\alpha + 2^{(2)}k_\omega{}^\alpha\right)X_\alpha.$$

Subtraction of (4.9) from (4.8) and making use of (2.7) and (4.1) gives the following relation:

$$(4.10) \qquad \partial_\omega log\ g = 2\left(k_\omega{}^\alpha + 2^{(2)}k_\omega{}^\alpha\right)X_\alpha = -2F_{\nu\omega}X^\nu.$$

The representation (4.6) immediately follows by multiplying $G^{\lambda\omega}$ to both sides of (4.10) using (4.5) and by multiplying $h_{\omega\lambda}$ for the result again.□

REMARK 4.3. In virtue of Theorem 4.2, our investigation of the ME-vector in MEX_{2n} is reduced to the study of the tensor $G_\omega{}^\nu$. In order to know that the ME-vector it is necessary and sufficient to know an explicit representation of $G_\omega{}^\nu$ in terms of $g_{\lambda\mu}$.

In our further considerations, we need the abbreviation ${}^{(p)}X^{\lambda\nu}$ for an arbitrary tensor $X^{\lambda\nu}$ and notations $\overset{\dagger}{K}_s$

$$(4.11) \qquad {}^{(0)}X^{\lambda\nu} = X^{\lambda\nu}, \quad {}^{(p)}X^{\lambda\nu} = {}^{(p)}k^\lambda{}_\alpha X^{\alpha\nu} \quad (p = 1, 2, 3, \cdots),$$

$$(4.12) \qquad \overset{\dagger}{K}_s = \frac{1}{4}\sum_{t=0}^{s}\frac{1}{2^t}K_{s-t}.$$

The following relations are immediate consequences of (4.11) and (4.12):

$$(4.13a) \qquad {}^{(p)}k^\lambda{}_\mu{}^{(q)}X^{\mu\nu} = {}^{(p+q)}X^{\lambda\nu} \quad (q = 1, 2, 3, \cdots),$$

$$(4.13b) \qquad {}^{(p)}k_\lambda{}^{\omega(q)}X_\omega{}^\nu = {}^{(p+q)}X_\lambda{}^\nu,$$

(4.14a)

$$\overset{\dagger}{K}_0 = \frac{1}{4}, \quad \overset{\dagger}{K}_2 = \frac{1}{4}\left(K_2 + \frac{1}{4}\right), \quad \overset{\dagger}{K}_4 = \frac{1}{4}\left(K_4 + \frac{1}{4}K_2 + \frac{1}{16}\right), \cdots,$$

$$(4.14b) \qquad \overset{\dagger}{K}_s = \frac{1}{4}\left(K_s + \overset{\dagger}{K}_{s-2}\right).$$

THEOREM 4.4. *In MEX_{2n}, the tensor ${}^{(p)}G_\omega{}^\nu$ satisfies the following recurrence relation:*

$$(4.15)\quad {}^{(2n)}G_\omega{}^\nu + K_2{}^{(2n-2)}G_\omega{}^\nu + \cdots + K_{2n-2}{}^{(2)}G_\omega{}^\nu + K_{2n}G_\omega{}^\nu = 0.$$

Proof. Multiplying $G^{\lambda\mu}$ to both sides of (2.12) and using (4.11), we obtain the relation (4.15). □

LEMMA 4.5. *The following relation holds in MEX_{2n}:*

$$(4.16a)\quad {}^{(p+2)}G_\omega{}^\nu + \frac{1}{2}{}^{(p+1)}G_\omega{}^\nu + \frac{1}{2}{}^{(p)}k_\omega{}^\nu = 0 \quad (p = 0, 1, 2, \cdots),$$

(4.16b)
$${}^{(q)}G_\omega{}^\nu = \frac{1}{4}{}^{(q-2)}G_\omega{}^\nu - \frac{1}{2}{}^{(q-2)}k_\omega{}^\nu + \frac{1}{4}{}^{(q-3)}k_\omega{}^\nu \quad (q = 3, 4, 5, \cdots).$$

Proof. Substituting (4.1) into (4.5) and making use of (2.3) gives

$$(4.17)\quad 2{}^{(2)}G_\mu{}^\nu + {}^{(1)}G_\mu{}^\nu + \delta_\mu^\nu = 0.$$

The relation (4.16a) may be obtained by multiplying $\frac{1}{2}{}^{(p)}k_\omega{}^\mu$ to both sides of (4.17). Using (4.16a) twice, the relation (4.16b) follows as in the following way:

$$\begin{aligned}{}^{(q)}G_\omega{}^\nu &= -\frac{1}{2}{}^{(q-1)}G_\omega{}^\nu - \frac{1}{2}{}^{(q-2)}k_\omega{}^\nu \\ &= \frac{1}{4}\left({}^{(q-2)}G_\omega{}^\nu + {}^{(q-3)}k_\omega{}^\nu\right) - \frac{1}{2}{}^{(q-2)}k_\omega{}^\nu \\ &= \frac{1}{4}{}^{(q-2)}G_\omega{}^\nu - \frac{1}{2}{}^{(q-2)}k_\omega{}^\nu + \frac{1}{4}{}^{(q-3)}k_\omega{}^\nu.\end{aligned}$$

□

LEMMA 4.6. *If the tensor $G_\omega{}^\nu$ satisfies the following equation in MEX_{2n}*

$$(4.18)\quad A{}^{(2)}G_\omega{}^\nu + BG_\omega{}^\nu + \Lambda_\omega{}^\nu = 0,$$

then the tensor $G_\omega{}^\nu$ must be of the form

$$(4.19)\quad B(A+4B)G_\omega{}^\nu = 2AB\delta_\omega^\nu + A^2k_\omega{}^\nu - (A+4B)\Lambda_\omega{}^\nu - 2A{}^{(1)}\Lambda_\omega{}^\nu,$$

where $A, B,$ and $\Lambda_\omega{}^\nu$ are functions of $g_{\lambda\mu}$.

Proof. Substituting of (4.17) into (4.18) for ${}^{(2)}G_\omega{}^\nu$ gives

$$A^{(1)}G_\omega{}^\nu = 2BG_\omega{}^\nu - A\delta_\omega^\nu + 2\Lambda_\omega{}^\nu. \tag{4.20}$$

Multiplying $k_\lambda{}^\omega$ to both sides of (4.20) and making use of (4.11), we have

$$A^{(2)}G_\omega{}^\nu = 2B^{(1)}G_\omega{}^\nu - Ak_\omega{}^\nu + 2^{(1)}\Lambda_\omega{}^\nu. \tag{4.21}$$

Substitution (4.17) into (4.21) for ${}^{(2)}G_\omega{}^\nu$ again gives

$$(\frac{A}{2} + 2B)^{(1)}G_\omega{}^\nu = -\frac{A}{2}\delta_\omega^\nu + Ak_\omega{}^\nu - 2^{(1)}\Lambda_\omega{}^\nu. \tag{4.22}$$

Consequently, the relation (4.19) follows by eliminating the tensor ${}^{(1)}G_\omega{}^\nu$ from (4.20) and (4.22). □

Now, we are ready to prove the following main theorem in the present section, which present a representation of the tensor $G_\omega{}^\nu$.

THEOREM 4.7. *In an even-dimensional ME-manifold MEX_{2n}, the tensor $G_\omega{}^\nu$ may be given by*

$$G_\omega{}^\nu = \frac{1}{2k\overset{\dagger}{K}_{2n}}\left(k\delta_\omega^\nu + 2k_\omega{}^\nu - {}^{(1)}\Lambda_\omega{}^\nu\right)\overset{\dagger}{K}_{2n-2} - \frac{1}{k}\Lambda_\omega{}^\nu, \tag{4.23}$$

where

$$\Lambda_\omega{}^\nu = \sum_{s=0}^{2n-4}\overset{\dagger}{K}_s\left(-2^{(2n-2-s)}k_\omega{}^\nu + {}^{(2n-3-s)}k_\omega{}^\nu\right). \tag{4.24}$$

Proof. Substituting (4.16b) into (4.15) for ${}^{(2n)}G_\omega{}^\nu$ and using (4.14), we have

$$\begin{aligned}&\overset{\dagger}{K}_0\left(-2^{(2n-2)}k_\omega{}^\nu + {}^{(2n-3)}k_\omega{}^\nu\right) + 4\overset{\dagger}{K}_2{}^{(2n-2)}G_\omega{}^\nu \\ &+ \cdots + K_{2n-2}{}^{(2)}G_\omega{}^\nu + K_{2n}G_\omega{}^\nu = 0.\end{aligned} \tag{4.25a}$$

Substituting again for ${}^{(2n-2)}G_\omega{}^\nu$ into (4.25a) from (4.16b) gives
(4.25b)

$$\begin{aligned}&\overset{\dagger}{K}_0\left(-2^{(2n-2)}k_\omega{}^\nu + {}^{(2n-3)}k_\omega{}^\nu\right) + \overset{\dagger}{K}_2\left(-2^{(2n-4)}k_\omega{}^\nu + {}^{(2n-5)}k_\omega{}^\nu\right) \\ &+ 4\overset{\dagger}{K}_4{}^{(2n-4)}G_\omega{}^\nu + \cdots + K_{2n-2}{}^{(2)}G_\omega{}^\nu + K_{2n}G_\omega{}^\nu = 0.\end{aligned}$$

After $\frac{n-2}{2}$ steps of successive repeated substitution for ${}^{(q)}G_\omega{}^\nu$, we have in virtue of (4.24)

$$4\overset{\dagger}{K}_{2n-2}{}^{(2)}G_\omega{}^\nu + K_{2n}G_\omega{}^\nu + \Lambda_\omega{}^\nu = 0. \tag{4.25c}$$

Comparison of (4.18) with (4.25c) gives

$$A = 4\overset{\dagger}{K}_{2n-2}, \qquad B = K_{2n} = k. \tag{4.26}$$

Consequently, the relation (4.23) follows by substituting (4.26) into (4.19) and making use of (4.14b). □

Now that we have obtained a representation of $G_\omega{}^\nu$ in Theorem 4.7, it is possible for us to represent the ME-vector X_ω in terms of $g_{\lambda\mu}$ by only substituting (4.23) into (4.6).

THEOREM 4.8. *In MEX_{2n}, the ME-vector X_ω may be given by*
(4.27)

$$X_\omega = -\frac{1}{4k\overset{\dagger}{K}_{2n}}\left((k\delta_\omega^\alpha + 2k_\omega{}^\alpha - {}^{(1)}\Lambda_\omega{}^\alpha)\overset{\dagger}{K}_{2n-2} - \Lambda_\omega{}^\alpha\overset{\dagger}{K}_{2n}\right)\partial_\alpha(log\ \ g).$$

REMARK 4.9. In virtue of (2.8), (4.11), (4.13), (4.14b), and (4.24), we may represent the last two terms on the right-hand side of (4.27) as follows:

$$\begin{aligned} &- {}^{(1)}\Lambda_\omega{}^\alpha\overset{\dagger}{K}_{2n-2} - 2\Lambda_\omega{}^\alpha\overset{\dagger}{K}_{2n} \\ &= \sum_{s=0}^{2n-4}\overset{\dagger}{K}_s\left(2\overset{\dagger}{K}_{2n-2}{}^{(2n-1-s)}k_\omega{}^\alpha + k^{(2n-2-s)}k_\omega{}^\alpha - 2\overset{\dagger}{K}_{2n}{}^{(2n-3-s)}k_\omega{}^\alpha\right).\end{aligned}$$

Therefore, we know that the ME-vector X_ω representation in terms of $g_{\lambda\mu}$.

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