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FIXED POINTS OF SEQUENTIALLY CONDENSING OPERATORS

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ABSTRACT. Introducing the concept of a sequentially condensing operator in a more general framework, we give a new fixed point theorem for sequentially condensing operators in Banach spaces, with the aid of attractors.

1. Introduction

A fixed point theorem for condensing operators goes back to B. N. Sadovskii [8]; see also [1, 2]. S. J. Daher [3] gave the following definition of a sequentially condensing operator which generalizes the one of a condensing operator in [8].

DEFINITION 1. Let E be a Banach space and α the Hausdorff measure of noncompactness on E. A continuous operator $f : E \to E$ is said to be *sequentially condensing* if for every bounded $Kf \subset E$ with $\alpha(Kf) > 0$ and $f(Kf)$ bounded, f satisfies the condition:

$$\alpha(f(Kf)) < \alpha(Kf).$$

For the notation of Kf, see Definition 3 below. Here $\alpha(A)$ of a set A is the infimum of the numbers $\varepsilon > 0$ such that A can be covered by a finite number of closed balls in E with radius ε.

A fixed point theorem was established in [3] as follows:

THEOREM A. *Suppose that $K \subset B \subset S \subset E$ are convex sets in a Banach space E such that K is nonempty and compact, B is relatively*

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open in S, and S is bounded and closed in E. Let $f : S \to E$ be a sequentially condensing operator such that $f^j(B) \subset S$ for all $j \geq 0$ and K attracts compact sets of B. Then f has a fixed point.

In this paper, we introduce the concept of a sequentially γ-condensing operator in a more general framework to extend Theorem A. For this, we mainly follow the basic idea of the proofs of related results in [3] and [5], although we salvage the proofs in [3], for instance, see Lemma 1 and Lemma 4 below.

DEFINITION 2. Let E be a topological vector space and $\mathcal{M}$ a collection of nonempty subsets of E containing all precompact subsets of E with the property that for any $M \in \mathcal{M}$, the closure $\overline{M}$ and the convex hull $\operatorname{co} M$ belong to $\mathcal{M}$. A nonnegative real-valued function $\gamma : \mathcal{M} \to [0, \infty)$ is called *a measure of noncompactness* on E if the following conditions hold for any $M \in \mathcal{M}$:

(1) $\gamma(\overline{M}) = \gamma(M) = \gamma(\operatorname{co} M)$; and
(2) if $M \subset E$ is precompact, then $\gamma(M) = 0$.

The measure γ of noncompactness on E is said to be *regular* provided that $\gamma(M) = 0$ if and only if M is precompact; see [7].

DEFINITION 3. Let X be a nonempty subset of a locally convex topological vector space E and $f : X \to E$ an operator. For a nonempty compact subset K of X let

$$Kf := \bigcup_{n \geq 0} \{K_n : K_0 = \operatorname{co} K,\ K_n = \operatorname{co} f(X \cap K_{n-1})\}.$$

Given a measure γ of noncompactness on E, a continuous operator $f : X \to E$ is said to be *sequentially γ-condensing* provided that if Kf is any subset of X such that $\gamma(Kf) \leq \gamma(f(Kf))$, then $f(Kf)$ is relatively compact.

In case where α is the Hausdorff measure of noncompactness on a Banach space E, a sequentially condensing operator $f : E \to E$ is sequentially α-condensing. Also some examples of sequentially α-condensing operator can be found in [3].

2. Main result

We begin with the following lemma which is a corrected version of [3, Lemma 3].

LEMMA 1. *Let X be a nonempty convex subset of a locally convex topological vector space, $f : X \to X$ an operator, and K a nonempty compact subset of X. Then $\overline{co}\, Kf = \overline{co}\,(co\, K \cup \overline{co}\, f(Kf))$.*

Proof. Since $Kf = \bigcup_{n\geq 0} K_n$, where $K_0 = \mathrm{co}\, K$ and $K_n = \mathrm{co}\, f(K_{n-1})$, we have $f(Kf) \subset Kf$ and $\mathrm{co}\, f(Kf) \supset \bigcup_{n\geq 1} K_n$. From $\mathrm{co}K \cup \overline{\mathrm{co}}\, f(Kf) \supset Kf \supset f(Kf)$ it follows that

$$\overline{\mathrm{co}}\,(\mathrm{co}K \cup \overline{\mathrm{co}}\, f(Kf)) \supset \overline{\mathrm{co}}\, Kf \supset \overline{\mathrm{co}}\, f(Kf) \quad \text{and}$$

$$\overline{\mathrm{co}}\, Kf \supset \overline{\mathrm{co}}\,(\mathrm{co}\, K \cup \overline{\mathrm{co}}\, f(Kf)).$$

Therefore we conclude that $\overline{\mathrm{co}}\, Kf = \overline{\mathrm{co}}\,(\mathrm{co}\, K \cup \overline{\mathrm{co}}\, f(Kf))$. This completes the proof. □

For our aim, we need several auxiliary facts one of which is given in [3, Lemma 6], with the aid of attractors.

DEFINITION 4. Let E be a metrizable locally convex topological vector space. For a given continuous operator $f : E \to E$, we say that a set $K \subset E$ *attracts a set* $H \subset E$ if for any $\varepsilon > 0$, there is an integer $N(H, \varepsilon)$ such that $f^n(H) \subset B_\varepsilon(K)$ for all $n \geq N(H, \varepsilon)$, where $B_\varepsilon(K)$ is the ε-neighborhood of K. If K attracts each compact set $H \subset E$, we say that K *attracts compact sets* of E; see [5]. In particular, K *attracts points* of $B \subset E$ if for every $x \in B$ and for every $\varepsilon > 0$, there is an integer $N(x, \varepsilon)$ such that $f^n(x) \in K + B_\varepsilon(0)$ for all $n \geq N(x, \varepsilon)$.

LEMMA 2. *Suppose that $K \subset B \subset S \subset E$ are subsets of a metrizable locally convex topological vector space (E, d) such that K is nonempty and compact, B is convex and relatively open in S, and S is closed in E. Let $f : S \to E$ be a continuous operator such that $f^j(B) \subset S$ for all $j \geq 0$ and K attracts points of B. Then there is a nonempty compact set C with $C \subset K$ such that $f(C) = C$.*

Proof. Fix $x \in B$ and let

$$C = \bigcap_{i\geq 1} \overline{\bigcup_{n\geq i} f^n(x)}.$$

Since K attracts points of B, for every $\varepsilon > 0$, there is a positive integer $N(\varepsilon)$ such that

$$\{f^n(x) : n \geq N(\varepsilon)\} \subset K + B_{\frac{\varepsilon}{2}}(0).$$

Hence it follows from the closedness of K that

$$C = \bigcap_{i \geq 1} \overline{\bigcup_{n \geq i} f^n(x)} \subset \bigcap_{\varepsilon > 0} \overline{K + B_{\frac{\varepsilon}{2}}(0)} \subset \bigcap_{\varepsilon > 0} K + B_\varepsilon(0) = \overline{K} = K.$$

Thus, $C \subset K$ and C is compact because C is a closed subset of the compact set K. Moreover, since f is continuous, we have

$$f(C) \subset \bigcap_{i \geq 1} f(\overline{\bigcup_{n \geq i} f^n(x)}) \subset \bigcap_{i \geq 1} \overline{\bigcup_{n \geq i} f^{n+1}(x)} = C.$$

Now it remains to show that $C \subset f(C)$. Let $z \in C$ be arbitrary and set $A_i = \overline{\bigcup_{n \geq i} f^n(x)}$ for $i \geq 1$. Then there exists a subsequence $\{f^{n_i}(x)\}$ of $\{f^n(x)\}$ with $f^{n_i}(x) \in A_i$ such that

$$z = \lim_{i \to \infty} f^{n_i}(x).$$

In fact, since $z \in A_i$ for all $i \in \mathbb{N}$, if $i = 1$, there is an integer $n_1 \geq 1$ such that

$$d\,(f^{n_1}(x), z) < 1;$$

if $i = n_1$, there is an integer n_2 with $n_2 > n_1$ such that

$$d\,(f^{n_2}(x), z) < \frac{1}{2}.$$

By induction, there exists a subsequence $\{f^{n_i}(x)\}$ with $f^{n_i}(x) \in A_i$ such that

$$d\,(f^{n_i}(x), z) < \frac{1}{i} \quad \text{for every } i \in \mathbb{N}$$

and so $\lim_{i \to \infty} f^{n_i}(x) = z$.

Since K attracts points of B, we have

$$d\,(f^{n_i - 1}(x), K) \to 0 \quad \text{as } i \to \infty,$$

where $d(y, K) = \inf\{d(y, p) : p \in K\}$. The compactness of K implies that

$$K \cap \overline{\{f^{n_i - 1}(x) : i \in \mathbb{N}\}} \neq \emptyset;$$

see [5, Lemma 4]. Hence there exists a subsequence of $\{f^{n_i - 1}(x)\}$ which converges to some point $u \in K$. Without loss of generality we may suppose that the subsequence of $\{f^{n_i - 1}(x)\}$ is again denoted by $\{f^{n_i - 1}(x)\}$.

Since f is continuous, we have $f(u) = z$. From $f^{n_i-1}(x) \in A_{i-1}$ it follows that

$$u = \lim_{i\to\infty} f^{n_i-1}(x) \in \bigcap_{i\geq 2} A_{i-1} = C$$

and hence $z = f(u) \in f(C)$. It has been proved that $C \subset f(C)$. This completes the proof. □

Now the proof of the next lemma is exactly the same as the one given for sequentially condensing operators in Banach spaces; see [3, Lemma 5].

LEMMA 3. *Let S be a nonempty, convex and closed subset of a locally convex topological vector space E. If $f : S \to E$ is a sequentially γ-condensing operator and $r : E \to S$ is a retraction of E on S, then the composition $r \circ f$ is sequentially γ-condensing.*

The following result is a modification of [5, Theorem 4]. Recall that a topological vector space E is *quasi-complete* if every bounded, closed subset of E is complete.

LEMMA 4. *Suppose that $K \subset B \subset S \subset E$ are convex sets in a quasi-complete metrizable locally convex topological vector space E such that K is nonempty and compact, B is relatively open in S, and S is closed in E. Let $f : S \to E$ be a sequentially γ-condensing operator such that $f^j(B) \subset S$ for all $j \geq 0$ and K attracts points of B, where γ is a regular measure of noncompactness on E. Then there exists a compact convex set A with $A \subset S$ and $A \cap B \neq \emptyset$ such that $f^j(A \cap B) \subset A$ for all $j \geq 0$.*

Proof. Let C be a nonempty compact set in K such that $f(C) = C$, as in Lemma 2. Define

$$\begin{aligned}\mathcal{F}(S) := \{L \subset S : C \subset L,\ L \text{ is convex and closed,} \\ f^j(L \cap B) \subset L \text{ for all } j \geq 0\}.\end{aligned}$$

Then $S \in \mathcal{F}(S)$. Take $A = \bigcap\{L : L \in \mathcal{F}(S)\}$. Note that $C \subset A \subset S$, A is a nonempty, convex and closed set, and

$$f^j(A \cap B) \subset f^j(L \cap B) \subset L \quad \text{for all } j \geq 0 \text{ and for all } L \in \mathcal{F}(S).$$

Since $f^j(A \cap B) \subset A$ for all $j \geq 0$, we have $A \in \mathcal{F}(S)$.

Consider $C \subset K \subset B \subset S$ and an operator $g := r \circ f : S \to E \to S$, where r is a retraction of E on S. The existence of such a retraction

r is assured since S is a convex closed subset of a metrizable locally convex topological vector space; see [4]. By Lemma 3, g is sequentially γ-condensing. Define

$$\mathcal{G}(S) := \{G \subset S : C \subset G,\ G \text{ is convex and closed},\ g(G) \subset G\}.$$

We can take $H = \bigcap\{G : G \in \mathcal{G}(S)\}$. Then H is convex and closed, $C \subset H \subset S$, and $g(H) \subset H$. Thus, $H \in \mathcal{G}(S)$. Observe that g is continuous and $g(C) = C$. Then $Cg = \bigcup_{n\geq 0} C_n$ and $C_{i-1} \subset C_i$ for all $i \geq 1$, where $C_0 = \operatorname{co} C$ and $C_n = \operatorname{co} g(C_{n-1})$. Furthermore, Cg is convex, $C \subset \overline{Cg} \subset H \subset S$, and $g(\overline{Cg}) \subset \overline{g(Cg)} \subset \overline{Cg}$ by Lemma 1. Hence $\overline{Cg} \in \mathcal{G}(S)$ and therefore $\overline{Cg} = H$.

Now we will prove that H is compact. $\operatorname{co} C = C_0 \subset C_1 = \operatorname{co} g(C_0) \subset \operatorname{co} g(Cg)$ implies that $\overline{Cg} = \overline{\operatorname{co}}\,(\operatorname{co} C \cup \overline{\operatorname{co}}\, g(Cg)) = \overline{\operatorname{co}}\, g(Cg)$ by Lemma 1. The definition of a measure of noncompactness on E implies that

$$\gamma(Cg) = \gamma(\overline{Cg}) = \gamma(\overline{\operatorname{co}}\, g(Cg)) = \gamma(g(Cg)).$$

Since g is sequentially γ-condensing, $g(Cg)$ is relatively compact. Notice that in a quasi-complete Hausdorff topological vector space the notions of precompactness and relative compactness coincide; see [9]. From the regularity of γ it follows that $\gamma(\overline{Cg}) = \gamma(g(Cg)) = 0$ and hence $\overline{Cg}$ is compact. Thus, H is compact.

Finally, since $f(H \cap B) \subset f(H) = g(H) \subset H$ and so inductively $f^j(H \cap B) \subset H$ for all $j \geq 0$, we have $H \in \mathcal{F}(S)$ and so $A \subset H$, by definition of A. Since H is compact, the closed set A is compact. In particular, $A \in \mathcal{F}(S)$ implies that A is convex and $f^j(A \cap B) \subset A$ for all $j \geq 0$. This completes the proof. □

REMARK 1. In the proof of Lemma 4, to show that H is compact, we mainly follow the basic line of the one of [3, Lemma 7] in case of Banach spaces, with minor correction, say, $\overline{Cg} = \overline{\operatorname{co}}\, g(Cg)$ instead of $\overline{Cg} = \overline{\operatorname{co}}\, g(\overline{Cg})$, although we use a more direct method to prove that $A \in \mathcal{F}(S)$.

The following lemma which is originally a result of W.A. Horn holds also for metrizable locally convex topological vector spaces, observing that every convex closed subset of a metrizable locally convex topological vector space is a retract; see [6, Theorem 6] and [4, Theorem 4.1].

LEMMA 5. *Suppose that $S_0 \subset S_1 \subset S_2$ are convex subsets of a metrizable locally convex topological vector space E such that S_0 and*

S_2 are compact and S_1 is relatively open in S_2. Let $f : S_2 \to E$ be a continuous operator such that there exists an integer $m > 0$ with $f^j(S_1) \subset S_2$ for $0 \le j \le m-1$ and $f^j(S_1) \subset S_0$ for $m \le j \le 2m-1$. Then f has a fixed point.

Using the previous results, we prove a new fixed point theorem for sequentially γ-condensing operators in an accurate process, based on the proof of [5, Theorem 5].

THEOREM 1. *Suppose that $K \subset B \subset S \subset E$ are convex sets in a Banach space $(E, \|\cdot\|)$ such that K is nonempty and compact, B is relatively open in S, and S is closed in E. Let $f : S \to E$ be a sequentially γ-condensing operator such that $f^j(B) \subset S$ for all $j \ge 0$ and K attracts compact sets of B, where γ is a regular measure of noncompactness on E. Then f has a fixed point.*

Proof. For every $\varepsilon > 0$, let

$$B_\varepsilon(K) = \{\, y \in E : \|y - x\| < \varepsilon \quad \text{for some } x \in K \,\}.$$

Then $B_\varepsilon(K)$ is an open neighborhood of K and convex, because of the convexity of K. Since B is relatively open in S and K is compact, there is an $\varepsilon > 0$ such that

$$\overline{B_\varepsilon(K)} \cap S \subset B.$$

In fact, for every $x \in B$, there exists an $\varepsilon_x > 0$ such that $B_{\varepsilon_x}(x) \cap S \subset B$. By the compactness of K, there are $x_i \in K$ and $\varepsilon_i = \varepsilon_{x_i}$ for $i = 1, \dots, n$ such that $K \subset \bigcup_{i=1}^n (B_{\frac{\varepsilon_i}{4}}(x_i) \cap S)$. Taking $\varepsilon := \min\{\frac{\varepsilon_1}{4}, \dots, \frac{\varepsilon_n}{4}\}$, it is easy to verify that $\overline{B_\varepsilon(K)} \cap S \subset \bigcup_{i=1}^n (B_{\varepsilon_i}(x_i) \cap S) \subset B$.

Put $B_0 := B_\varepsilon(K) \cap S$. Then B_0 is relatively open in S and $K \subset B_0 \subset S$. Since $f^j(B_0) \subset S$ for all $j \ge 0$ and K attracts points of B_0, Lemma 4 implies that there is a compact convex set A with $A \subset S$ such that $f^j(A \cap B_0) \subset A$ for all $j \ge 0$. Set

$$S_0 := \overline{B_{\frac{\varepsilon}{2}}(K)} \cap A, \quad S_1 := B_\varepsilon(K) \cap A, \quad \text{and} \quad S_2 := S \cap A.$$

Then $S_0 \subset S_1 \subset S_2$ are convex sets, the sets S_0, S_2 are compact, and S_1 is relatively open in S_2. Since K attracts compact sets of B and $H := \overline{B_\varepsilon(K)} \cap A$ is a compact set of B, there exists a positive integer $N(H, \varepsilon)$ such that $f^j(\overline{B_\varepsilon(K)} \cap A) \subset B_{\frac{\varepsilon}{2}}(K)$ for all $j \ge N(H, \varepsilon)$ and so $f^j(S_1) \subset \overline{B_{\frac{\varepsilon}{2}}(K)}$. Hence it follows from $f^j(S_1) = f^j(A \cap B_0) \subset A$ that $f^j(S_1) \subset S_0$ for all $j \ge N(H, \varepsilon)$. Moreover, we have $f^j(S_1) =$

$f^j(A \cap B_0) \subset A = S_2$ for $0 \leq j \leq N(H, \varepsilon)$. By Lemma 5, the restriction $f|_{S_2}$ has a fixed point and so does f. This completes the proof. □

The following is a result of J. K. Hale and O. Lopes on condensing operators when γ is the Kuratowski measure of noncompactness on E; see [5, Corollary 1].

COROLLARY 1. *Let E, K, B, and S be as in Theorem 1. Let $f : S \to E$ be a continuous operator that satisfies the condition:*

$$\gamma(f(A)) < \gamma(A) \quad \text{for every bounded } A \subset E \text{ with } \gamma(A) > 0 \text{ and } f(A) \text{ bounded},$$

where γ is the Hausdorff or Kuratowski measure of noncompactness on E. If $f^j(B) \subset S$ for all $j \geq 0$ and K attracts compact sets of B, then f has a fixed point.

Proof. This is an immediate consequence of Theorem 1 since f is clearly sequentially γ-condensing and γ is regular; see [1]. □

COROLLARY 2. *Let E, K, B, and S be as in Theorem 1. Let $f : S \to E$ be a continuous operator such that $f^j(B) \subset S$ for all $j \geq 0$ and K attracts compact sets of B. If $f(S)$ is relatively compact, then f has a fixed point.*

Proof. Apply Theorem 1 because f is sequentially γ-condensing, where γ is a regular measure of noncompactness on E. □

Thus, Theorem 1 includes many of known results, as well as Theorem A. For related results and its applications to flows, we refer to [5, 6].

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